Analysis and Topology Course Notes

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December 2023

Preface

Personalities and Stakes

Differential calculus is a collection of algebraic tools that enable the exact resolution of certain geometrical problems posed by the ancients: determining the **length of a curve**, the **area of a geometric figure**, or the **volume of a solid**, for example, or finding the **tangent line to any geometric shape**. By the 17^{th} century, it was already possible to solve these problems.

Isaac Newton and his British contemporaries relied on velocity and rates of change in a **theory of fluxions**, whereas Leibniz and European mathematicians used infinitesimal increments, or **differentials**, mysterious quantities larger than 0, but smaller than any other number.[1] The results obtained were valid in both frameworks, but the methods used in either case were far from satisfactory. It was not necessary to understand *why* the methods were valid for them to work, but the question often recurred, scratching the collective subconscious of mathematicians, philosophers, and theologians of the time: mathematics were considered “divine” or “heavenly”, so why was there so much ambiguity?[2]

Both methods used the notion of **infinity** without ever defining the concept; unfortunately, infinity has the troublesome habit of defying intuition when one least expects it.

The French mathematician d’Alembert then attempted to provide a certain formalism by introducing the notion of a **limit**,

> [...] the number which one can approach as closely as one wishes by using a sequence of secant approximations [...],

but his definition hardly proved more precise. What is meant by “approaching”? Do we ever reach this number?

It was in response to this lack of formalism that **mathematical analysis** was established. Its foundations are owed, among others, to Cauchy, Gauss, and Weierstrass; calculus is an “intuitive” version of their analysis.

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[1]Mathematicians of the era engaged in furious academic battles over the priority of discovery; the British insisted that Newton was the inventor of differential calculus since he had used it to calculate the orbits of planets; but Leibniz published his results on the derivative of a product before Newton. Several collaborations, as well as many friendships, became casualties of the conflict.

[2]Bishop Berkeley, in a (now famous) treatise published in 1734, attacked the approaches of both camps: if velocity is the first derivative (the first fluxion) of a particle’s position, what corresponds to the second and third derivative? How can a quantity be smaller than any other quantity? Are infinitesimals the ghosts of departed quantities?

References and Influences

These notes are mostly based on courses I taught at the University of Ottawa between 2004 and 2021, but also on courses that I took as a student between 1994 and 2002. I was blessed with fantastic calculus, analysis, differential systems, and topology instructors and mentors:

- **Richard Blute**, **Luc Demers**, **Marcel Déruaz**, **Benoit Dionne**, **Thierry Giordano**, **Barry Jessup**, **Victor Leblanc**, and **Rémi Vaillancourt** at the University of Ottawa, and
- **Wojciech Jaworski** and **Michael Moore** at Carleton University.

It is no exaggeration to say that I would not be a professional mathematician without their guidance, for which I thank them heartily.

More pragmatically, these notes could not exist without their influence and hard work, in particular that of B. Dionne (chapters 1-6), T. Giordano (chapters 7-14, 21-24), and M. Moore (chapters 15-20). I should also mention Aaron Smith with whom I co-taught MAT 2125 (Elementary Real Analysis) online during the COVID-19 pandemic, who contributed some material and solved problems to chapters 1-6.

I have consulted and borrowed from a whole slew of references over the years, of which the following are the most prominent:

- Bartle, R.G., Sherbert, D.R. [1992], *Introduction to Real Analysis*, 2nd edition, Wiley.
- Brown, J.W., Churchill, R.V. [1996], *Complex Variables and Applications*, 7th edition, McGraw-Hill.
- Gourdon, X. [2000], *Les maths en tête: analyse*, 2e édition, Ellipses.
- Hirsch, M.W., Smale, S. [1974], *Differential Equations, Dynamical Systems, and Linear Algebra*, Academic Press.
- Marsden, J.E., Hoffman, M.J. [1993], *Elementary Classical Analysis*, W.H. Freeman.
- Marsden, J.E., Tromba, A.J. [1988], *Vector Calculus*, W.H. Freeman.
- Munkres, J.R. [1974], *Topology: a First Course*, Prentice-Hall.
- Royden, H.L. [1968], *Real Analysis*, Macmillan.
- Rudin, W.R. [1991], *Functional Analysis*, McGraw-Hill.
- Rudin, W.R. [1987], *Real and Complex Analysis*, McGraw-Hill.
- Savage, A. [2017], *Elementary Real Analysis*, course notes (these also form the basis of Section 6.3).
- Spivak, M. [1965], *Calculus on Manifolds*, Addison-Wesley.

Be sure to give these masterful works the attention they deserve.

Pre-Requisites and Course Notes Overview

Readers are assumed to have taken three semesters of calculus, two semesters of linear algebra, and one course in mathematical reasoning and proofs at the university level (MAT 1320, MAT 1322, MAT1341, MAT 1362, MAT 2122, and MAT 2141 at the University of Ottawa), and more importantly, to have **mastered their contents**.

Each of the first four parts correspond roughly to a course offered (or previously offered) at the University of Ottawa:

- Part I: MAT 2125 (*Elementary Real Analysis*, formerly *Real Analysis I*);
- Part II: MAT 3120 (formerly *Real Analysis III*, currently *Real Analysis*);
- Part III: MAT 2121 (formerly *Real Analysis II*, not in the course catalogue anymore, except as a special topics course), and
- Part IV: MAT 4153 (*General Topology*),

whereas Part V contains tidbits that could easily be found in MAT 3121 (*Complex Analysis I*), MAT 3130 (*Introduction to Dynamical Systems*), MAT 4121 (*Measure and Integration I*), and/or MAT 4124 (*Introduction to Functional Analysis*).

Any analyst and any topologist worth their salt will have to tackle all the topics listed above in their training (and more besides, depending on their individual research interests), but there is no substitute for taking courses and learning from specific instructors.

I make these notes available mainly to help students bridge gaps caused by scheduling issues and to whet their appetites by giving them a chance to look ahead.

No matter how we swing it, there is a **lot of material to cover**, and there is no denying that some of it can be scary the first time it is encountered ... but analysis is mostly **fun** once we get the hang of it.

So roll up your sleeves, and happy learning!

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December 2023

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