

Analysis and Topology Course Notes

Patrick Boily
University of Ottawa
pboily@uottawa.ca

December 2023

Preface

Personalities and Stakes

Differential calculus is a collection of algebraic tools that enable the exact resolution of certain geometrical problems posed by the ancients: determining the **length of a curve**, the **area of a geometric figure**, or the **volume of a solid**, for example, or finding the **tangent line to any geometric shape**. By the 17th century, it was already possible to solve these problems.

Isaac Newton and his British contemporaries relied on velocity and rates of change in a **theory of fluxions**, whereas Leibniz and European mathematicians used infinitesimal increments, or **differentials**, mysterious quantities larger than 0, but smaller than any other number.¹ The results obtained were valid in both frameworks, but the methods used in either case were far from satisfactory. It was not necessary to understand *why* the methods were valid for them to work, but the question often recurred, scratching the collective subconscious of mathematicians, philosophers, and theologians of the time: mathematics were considered “divine” or “heavenly”, so why was there so much ambiguity?²

Both methods used the notion of **infinity** without ever defining the concept; unfortunately, infinity has the troublesome habit of defying intuition when one least expects it.

The French mathematician d’Alembert then attempted to provide a certain formalism by introducing the notion of a **limit**,

[...] the number which one can approach as closely as one wishes by using a sequence of secant approximations [...],

but his definition hardly proved more precise. What is meant by “approaching”? Do we ever reach this number?

It was in response to this lack of formalism that **mathematical analysis** was established. Its foundations are owed, among others, to Cauchy, Gauss, and Weierstrass; calculus is an “intuitive” version of their analysis.

¹Mathematicians of the era engaged in furious academic battles over the priority of discovery; the British insisted that Newton was the inventor of differential calculus since he had used it to calculate the orbits of planets; but Leibniz published his results on the derivative of a product before Newton. Several collaborations, as well as many friendships, became casualties of the conflict.

²Bishop Berkeley, in a (now famous) treatise published in 1734, attacked the approaches of both camps: if velocity is the first derivative (the first fluxion) of a particle’s position, what corresponds to the second and third derivative? How can a quantity be smaller than any other quantity? Are infinitesimals the ghosts of departed quantities?

References and Influences

These notes are mostly based on courses I taught at the University of Ottawa between 2004 and 2021, but also on courses that I took as a student between 1994 and 2002. I was blessed with fantastic calculus, analysis, differential systems, and topology instructors and mentors:

- **Richard Blute, Luc Demers, Marcel Déruaz, Benoit Dionne, Thierry Giordano, Barry Jessup, Victor Leblanc, and Rémi Vaillancourt** at the University of Ottawa, and
- **Wojciech Jaworski and Michael Moore** at Carleton University.

It is no exaggeration to say that I would not be a professional mathematician without their guidance, for which I thank them heartily.

More pragmatically, these notes could not exist without their influence and hard work, in particular that of B. Dionne (chapters 1-6), T. Giordano (chapters 7-14, 21-24), and M. Moore (chapters 15-20). I should also mention Aaron Smith with whom I co-taught MAT 2125 (Elementary Real Analysis) online during the COVID-19 pandemic, who contributed some material and solved problems to chapters 1-6.

I have consulted and borrowed from a whole slew of references over the years, of which the following are the most prominent:

- Bartle, R.G., Sherbert, D.R. [1992], *Introduction to Real Analysis*, 2nd edition, Wiley.
- Brown, J.W., Churchill, R.V. [1996], *Complex Variables and Applications*, 7th edition, McGraw-Hill.
- Gourdon, X. [2000], *Les maths en tête: analyse*, 2e édition, Ellipses.
- Hirsch, M.W., Smale, S. [1974], *Differential Equations, Dynamical Systems, and Linear Algebra*, Academic Press.
- Marsden, J.E., Hoffman, M.J. [1993], *Elementary Classical Analysis*, W.H. Freeman.
- Marsden, J.E., Tromba, A.J. [1988], *Vector Calculus*, W.H. Freeman.
- Munkres, J.R. [1974], *Topology: a First Course*, Prentice-Hall.
- Royden, H.L. [1968], *Real Analysis*, Macmillan.
- Rudin, W.R. [1991], *Functional Analysis*, McGraw-Hill.
- Rudin, W.R. [1987], *Real and Complex Analysis*, McGraw-Hill.
- Savage, A. [2017], *Elementary Real Analysis*, course notes (these also form the basis of Section 6.3).
- Spivak, M. [1965], *Calculus on Manifolds*, Addison-Wesley.

Be sure to give these masterful works the attention they deserve.

Pre-Requisites and Course Notes Overview

Readers are assumed to have taken three semesters of calculus, two semesters of linear algebra, and one course in mathematical reasoning and proofs at the university level (MAT 1320, MAT 1322, MAT1341, MAT 1362, MAT 2122, and MAT 2141 at the University of Ottawa), and more importantly, to have **mastered their contents**.

Each of the first four parts correspond roughly to a course offered (or previously offered) at the University of Ottawa:

- Part I: MAT 2125 (*Elementary Real Analysis*, formerly *Real Analysis I*);
- Part II: MAT 3120 (formerly *Real Analysis III*, currently *Real Analysis*);
- Part III: MAT 2121 (formerly *Real Analysis II*, not in the course catalogue anymore, except as a special topics course), and
- Part IV: MAT 4153 (*General Topology*),

whereas Part V contains tidbits that could easily be found in MAT 3121 (*Complex Analysis I*), MAT 3130 (*Introduction to Dynamical Systems*), MAT 4121 (*Measure and Integration I*), and/or MAT 4124 (*Introduction to Functional Analysis*).

Any analyst and any topologist worth their salt will have to tackle all the topics listed above in their training (and more besides, depending on their individual research interests), but there is no substitute for taking courses and learning from specific instructors.

I make these notes available mainly to help students bridge gaps caused by scheduling issues and to whet their appetites by giving them a chance to look ahead.

No matter how we swing it, there is a **lot of material to cover**, and there is no denying that some of it can be scary the first time it is encountered ... but analysis is mostly **fun** once we get the hang of it.

So roll up your sleeves, and happy learning!

Patrick Boily
Wakefield, Canada
December 2023

Contents

Preface	iii
Personalities and Stakes	iii
References and Influences	iv
Pre-Requisites and Course Notes Overview	v
I Elementary Real Analysis	1
1 The Real Numbers	3
1.1 Hierarchy of Number Systems	3
1.2 Cardinality of Sets	14
1.3 Nested Intervals Theorem	17
1.4 Solved Problems	19
1.5 Exercises	32
2 Sequences of Real Numbers	33
2.1 Infinity vs. Intuition	33
2.2 Limit of a Sequence	35
2.3 Operations on Sequences and Basic Theorems	41
2.4 Bounded Monotone Convergence Theorem	47
2.5 Bolazano-Weierstrass Theorem	48
2.6 Cauchy Sequences	52
2.7 Solved Problems	54
2.8 Exercises	66
3 Limits and Continuity	67
3.1 Limit of a Function	67
3.2 Properties of Limits	75
3.3 Continuous Functions	77
3.4 Max/Min Theorem	82
3.5 Intermediate Value Theorem	84
3.6 Uniform Continuity	86
3.7 Solved Problems	89
3.8 Exercises	100

4	Differential and Integral Calculus	101
4.1	Differentiation	101
4.1.1	Mean Value Theorem	107
4.1.2	Taylor Theorem	109
4.1.3	Relative Extrema	111
4.2	Riemann Integral	114
4.2.1	Riemann's Criterion	118
4.2.2	Properties of the Riemann Integral	121
4.2.3	Fundamental Theorem of Calculus	128
4.2.4	Evaluation of Integrals	130
4.3	Solved Problems	132
4.4	Exercises	144
5	Sequences of Functions	145
5.1	Pointwise and Uniform Convergence	145
5.2	Limit Interchange Theorems	149
5.3	Solved Problems	154
5.4	Exercises	156
6	Series of Functions	157
6.1	Series of Numbers	157
6.2	Series of Functions	163
6.3	Power Series	165
6.4	Solved Problems	171
6.5	Exercises	178
II	Real Analysis and Metric Spaces	179
7	The Real Numbers (Reprise)	181
7.1	Cauchy Sequences in \mathbb{Q}	181
7.2	Building \mathbb{R} by Completing \mathbb{Q}	182
7.3	An Order Relation on \mathbb{R}	183
7.4	Exercises	186
8	Metric Spaces and Sequences	187
8.1	Preliminaries	187
8.1.1	Norms, Metrics, and Topology	187
8.1.2	Continuity	203
8.2	Sequence in a Metric Space	208
8.2.1	Closure, Closed Subsets, and Continuity	209
8.2.2	Complete Spaces and Cauchy Sequences	211
8.3	Solved Problems	216
8.4	Exercises	233

9	Metric Spaces and Topology	237
9.1	Compact Spaces	237
9.1.1	The Borel-Lebesgue Property	238
9.1.2	The Bolzano-Weierstrass Property	240
9.2	Connected Spaces	245
9.2.1	Characterization of Connected Spaces	247
9.2.2	Path-Connected Spaces	250
9.3	Solved Problems	252
9.4	Exercises	257
10	Normed Vector Spaces	259
10.1	Solved Problems	263
10.2	Exercises	266
11	Sequences of Functions in Metric Spaces	267
11.1	Uniform Convergence	267
11.1.1	Properties	270
11.1.2	Abel's Criterion	274
11.2	Fourier Series	276
11.2.1	Trigonometric Series and Periodic Functions	277
11.2.2	Again, Abel's Criterion	279
11.2.3	Convergence of Fourier Series	283
11.2.4	Dirichlet's Convergence Theorem	285
11.2.5	Quadratic Mean Convergence	289
11.3	Exercises	292
III	Vector Analysis and Differential Forms	295
12	Alternating Multilinear Forms	297
12.1	Linear Algebra Notions	297
12.2	Anti-Symmetric Forms	299
12.3	Wedge Product of Alternating Forms	302
12.4	Solved Problems	305
12.5	Exercises	310
13	Differential Forms	311
13.1	Differential p -Forms	311
13.2	Exterior Derivative	314
13.3	Antiderivative	317
13.4	Pullback of a Differential Form	323
13.5	Vector Fields	327
13.6	Solved Problems	331
13.7	Exercises	332

14 Integrating Differential Forms	333
14.1 Line Integral of a Differential 1–Form	333
14.2 Integral of a Differential p –Form	341
14.3 Green’s Theorem	342
14.4 Surfaces and Orientable Surfaces in \mathbb{R}^3	347
14.5 Integral of a Differential Form on an Orientable Surface	350
14.6 Area of a Surface and Flux Integral	354
14.7 Stokes’ Theorem	355
14.8 Solved Problems	356
14.9 Exercises	366
IV Topology	367
15 General Topology Concepts	369
15.1 Basic Definitions	369
15.2 Box and Subspace Topologies	373
15.3 Dual Definitions and Separation Axioms	375
15.4 Continuity and Homeomorphisms	379
15.5 Product Topology	384
15.6 Quotient Topology	385
15.7 Solved Problems	387
15.8 Exercises	394
16 Connected Spaces	395
16.1 Connected Sets	395
16.2 Path-Connectedness	399
16.3 Local (Path) Connectedness	401
16.4 Solved Problems	402
16.5 Exercises	404
17 Compact Spaces	405
17.1 Compactness	405
17.2 Limit Point and Sequential Compactness	415
17.3 Local Compactness and One-Point Compactification	418
17.4 Solved Problems	420
17.5 Exercises	426
18 Countability and Separation	427
18.1 Countability Axioms	427
18.2 Separation Axioms	431
18.3 Results of Urysohn and Tietze	435
18.4 Solved Problems	441
18.5 Exercises	445

19 Advanced Topics	447
19.1 Tychonoff's Theorem	447
19.2 Stone-Čech Compactification	451
19.3 Solved Problems	454
19.4 Exercises	456
20 Introduction to Algebraic Topology	457
20.1 Fundamental Groups	457
20.2 Covering Spaces	462
20.3 Fundamental Groups of S^1 and $\mathbb{R}^2 \setminus \{0\}$	465
20.4 Special Seifert-Van Kampen Theorem	471
20.5 Solved Problems	472
20.6 Exercises	478
V Special Topics in Analysis and Topology	479
21 Borel-Lebesgue Integration	481
21.1 Borel Sets and Borel Functions	483
21.2 Integral of Simple Functions	491
21.3 Integral of Positive Borel Functions	493
21.4 Integral of Borel Functions	501
21.5 Integration Over a Subset	503
21.6 Multiple Integrals	504
21.7 Change of Variables and/or Coordinates	508
21.7.1 Polar Coordinates	510
21.7.2 Spherical Coordinates	511
21.8 Solved Problems	512
21.8.1 Borel-Lebesgue Integral on \mathbb{R}^n	512
21.8.2 Multivariate Calculus	520
21.9 Exercises	539
22 Complex Analysis Fundamentals	543
23 Stone-Weierstrass Theorem	545
24 Baire's Theorem	547
25 Hale's Theorem	549
26 Basics of Functional Analysis	551
27 A Classical Hilbert Space Example	553