Chapter 10

Normed Vector Spaces

> The main objective of this chapter is to show that linear transformations of **finite-dimensional** normed vector spaces over $\mathbb{K}$ are **continuous**.

Norms were introduced in chapter 8; we provided a family of examples, the $p-$norms on $\mathbb{K}^n$. Let $p \geq 1$ and $A \in \mathbb{M}_{m,n}(\mathbb{K})$, the set $\mathbb{M}_{m,n}(\mathbb{K})$ of matrices of size $m \times n$ with entries in $\mathbb{K}$. The **induced** $p-$**norm on** $\mathbb{M}_{m,n}(\mathbb{K})$ is given by

$$\|A\|_p = \sup_{\|\mathbf{x}\|_p \leq 1} \{\|A\mathbf{x}\|_p\} .$$

It is easy to show that:

$$\|A\|_\infty = \max_{1\leq i\leq m} \left\{ \sum_{j=1}^{n} |a_{ij}| \right\}, \quad \|A\|_1 = \max_{1\leq j\leq n} \left\{ \sum_{i=1}^{m} |a_{ij}| \right\}, \quad \|A\|_2 = \text{ largest singular value of } A.$$

A **normed vector space** $(E, \|\cdot\|_E)$ is a vector space $(E, +, \cdot, \mathbf{0}_E)$ over $\mathbb{K}$ endowed with a norm $\|\cdot\|_E$; with matrix addition and multiplication by a scalar, the set $\mathbb{M}_{m,n}(\mathbb{K})$ is such a space for any of the induced $p-$norms. A normed vector space's operations behave as well as they could be hoped to, under the circumstances.

> **Proposition 139**
> *Let E be a normed vector space over $\mathbb{K}$. The maps $+ : E \times E \to E$ and $\cdot : \mathbb{K} \times E \to E$ are continuous.*
>
> **Proof:** left as an exercise. ■

In what follows, let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed vector spaces over $\mathbb{K}$. A map $T : E \to F$ is **linear** if

$$T(\mathbf{0}_E) = \mathbf{0}_F \quad \text{and} \quad T(a\mathbf{x} + b\mathbf{y}) = aT(\mathbf{x}) + bT(\mathbf{y}), \quad \forall a, b \in \mathbb{K}, \mathbf{x}, \mathbf{y} \in E.$$

The set of **all linear maps from** E **to** F is denoted by $L(E, F)$. For instance, if $E = \mathbb{K}^n$ and $F = \mathbb{K}^m$, then $L(E, F) \simeq \mathbb{M}_{m,n}(\mathbb{K})$.

Theorem 140
Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be two normed vector spaces over $\mathbb{K}$ and let $f \in L(E, F)$. The following conditions are equivalent:

1. *f is continuous over E*
2. *f is continuous at $\mathbf{0} \in E$*
3. *f is bounded over $\overline{B(\mathbf{0}, 1)}$*
4. *f is bounded over $S(\mathbf{0}, 1)$*
5. *$\exists M > 0$ such that $\|f(\mathbf{x})\|_F \leq M\|\mathbf{x}\|_E$ for all $\mathbf{x} \in E$.*
6. *f is Lipschitz continuous*
7. *f is uniformly continuous*

Proof: the implications $1. \implies 2.$, $3. \implies 4.$, $5. \implies 6. \implies 7. \implies 1.$ are clear.

$2. \implies 3.$: Let $\varepsilon = 1$. By continuity at $\mathbf{0}$, $\exists \delta > 0$ such that

$$\|f(\mathbf{x}) - f(\mathbf{0})\|_F = \|f(\mathbf{x})\|_F \leq 1$$

whenever $\|\mathbf{x} - \mathbf{0}\|_E = \|\mathbf{x}\|_E \leq \delta$. Now, let $\mathbf{y} \in \overline{B(\mathbf{0}, 1)}$. Since f is linear, we have

$$\|f(\mathbf{y})\|_F = \|f(\tfrac{1}{\delta}\delta\mathbf{y})\|_F = \tfrac{1}{\delta}\|f(\delta\mathbf{y})\|_F.$$

Since $\|\delta\mathbf{y}\|_E \leq \delta\|\mathbf{y}\|_E \leq \delta$. Consequently, $\|f(\delta\mathbf{y})\|_F \leq 1$ and

$$\|f(\mathbf{y})\|_F = \tfrac{1}{\delta}\|f(\delta\mathbf{y})\|_F \leq \tfrac{1}{\delta}.$$

But $\mathbf{y} \in \overline{B(\mathbf{0}, 1)}$ is arbitrary, so that f is bounded by $\frac{1}{\delta}$ over $\overline{B(\mathbf{0}, 1)}$.

$4. \implies 5.$: Since f is bounded over $S(\mathbf{0}, 1)$, $\exists N > 0$ such that $\|f(\mathbf{x})\|_F \leq N$ whenever $\|\mathbf{x}\|_E = 1$. Suppose $\mathbf{y} \neq 0_E \in E$. Then, since f is linear we have

$$\|f(\mathbf{y})\|_F = \left\| f\left(\|\mathbf{y}\|_E \frac{\mathbf{y}}{\|\mathbf{y}\|_E}\right)\right\|_F = \|\mathbf{y}\|_E \left\| f\left(\frac{\mathbf{y}}{\|\mathbf{y}\|_E}\right)\right\|_F. \tag{10.1}$$

However, $\left\|\frac{\mathbf{y}}{\|\mathbf{y}\|_E}\right\|_E = 1$ so that $\left\| f\left(\frac{\mathbf{y}}{\|\mathbf{y}\|_E}\right)\right\|_F \leq N$.

Substituting this last result in (10.1), we get that $\|f(\mathbf{y})\|_F \leq N\|\mathbf{y}\|_E$ for all $\mathbf{0} \neq \mathbf{y} \in E$. When $\mathbf{y} = 0$, the inequality remains valid since $f(\mathbf{0}_E) = \mathbf{0}_F$ and $0 = \|\mathbf{0}_F\|_F \leq N\|\mathbf{0}_E\|_E = 0$. This completes the proof. ■

If $f \in L(E,F)$ is also **continuous** (that is, if $f \in L_c(E,F)$), it then makes sense to define

$$\|f\| = \sup_{\|\mathbf{x}\|_E=1} \{\|f(\mathbf{x})\|_F\} = \sup_{\|\mathbf{x}\|_E\leq 1} \{\|f(\mathbf{x})\|_F\}.$$

With this definition, $(L_c(E,F), \|\cdot\|)$ is a normed vector space. Furthermore, if $f \in L_c(E,F)$ and $g \in L_c(F,G)$ then $g \circ f \in L_c(E,G)$ and we have

$$\|(g \circ f)(\mathbf{x})\| = \|g(f(\mathbf{x}))\| \leq \|g\|\|f(\mathbf{x})\| \leq \|g\|\|f\|\|\mathbf{x}\| \leq M\|\mathbf{x}\|$$

for some $M > 0$ and for all $\mathbf{x} \in E$. In particular, $\|f \circ g\| \leq \|f\|\|g\|$. The composition thus defines a kind of multiplication on $L_c(E,E)$; together with this multiplication, $L_c(E,E)$ is a **normed algebra**.

Theorem 141
If F is a Banach space over $\mathbb{K}$, then so is $L_c(E,F)$.

Proof: let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $L_c(E,F)$. For all $\mathbf{x} \in E$, $(f_n(\mathbf{x}))_{n\in\mathbb{N}}$ is a sequence in F. Fix such an $\mathbf{x}$. Thus, for all $p,q \in \mathbb{N}$,

$$\|f_p(\mathbf{x}) - f_q(\mathbf{x})\|_F = \|(f_p - f_q)(\mathbf{x})\|_F \leq \|f_p - f_q\|\|\mathbf{x}\|_E.$$

Let $\varepsilon > 0$. Since (f_n) is a Cauchy sequence in $L_c(E,F)$, $\exists M \in \mathbb{N}$ such that $\|f_p - f_q\|_F \leq \varepsilon$ whenever $p,q > M$. As a result, $\|f_p(\mathbf{x}) - f_q(\mathbf{x})\|_F < \varepsilon\|\mathbf{x}\|_E$ whenever $p,q > M$, so that $(f_n(\mathbf{x}))_{n\in\mathbb{N}}$ is a Cauchy sequence in F.

But F is complete so that $f_n(\mathbf{x}) \to f(\mathbf{x}) \in F$ for all $\mathbf{x} \in E$, which defines a map $f : E \to F$. It remains only to show that $f \in L_c(E,F)$ and that $f_n \to f$ in $(L_c(E,F), \|\cdot\|)$. The map f is linear as

$$f(a\mathbf{x} + b\mathbf{y}) = \lim_{n\to\infty} f_n(a\mathbf{x} + b\mathbf{y}) = \lim_{n\to\infty} [af_n(\mathbf{x}) + bf_n(\mathbf{y})] = af(\mathbf{x}) + bf(\mathbf{y})$$

for all $\mathbf{x},\mathbf{y} \in E$, $a,b \in \mathbb{K}$. Furthermore, f is continuous since, as the Cauchy sequence (f_n) is necessarily bounded, $\exists N > 0$ such that $\|f_n\| \leq N$. Fix $\mathbf{x} \in E$ to get $\|f_n(\mathbf{x})\|_F \leq N\|\mathbf{x}\|_E$ for all n. As $n \to \infty$, we see that $\|f(\mathbf{x})\|_F \leq N\|\mathbf{x}\|_E$.

Finally, we need to show that $f_n \to f$ in $L_c(E,F)$. Let $\varepsilon > 0$. Since (f_n) is a Cauchy sequence in $L_c(E,F)$, $\exists K > 0$ such that $\|f_p - f_q\| < \varepsilon$ whenever $p,q > K$. Now, fix $\mathbf{x} \in E$. Then,

$$\|f_p(\mathbf{x}) - f_q(\mathbf{x})\|_F \leq \|f_p - f_q\|\|\mathbf{x}\|_E < \varepsilon\|\mathbf{x}\|_E$$

whenever $p,q > N$. If we fix p and let $q \to \infty$, then we have

$$\|f_p(\mathbf{x}) - f(\mathbf{x})\|_F < \varepsilon\|\mathbf{x}\|_E$$

whenever $p > N$. Since this holds for all $\mathbf{x} \in E$, we have $\|f_p - f\| \leq \varepsilon$ for all $p > N$, i.e. $f_n \to f$ in $L_c(E,F)$. ■

We have seen that the metrics d_p are equivalent in $\mathbb{K}^n$, for $p \geq 1$. Can the same be said about the norms? In fact, we can say even more: not only are the $p-$norms equivalent, but *all* norms on $\mathbb{K}^n$ are equivalent.

Proposition 142
Let E be a finite dimensional vector space over $\mathbb{K}$. All norms on E are equivalent.

Proof: suppose $\dim_{\mathbb{K}}(E) = n < \infty$. If $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a basis of E, any $\mathbf{x} \in E$ can be written uniquely as a linear combination $\mathbf{x} = \sum_{i=1}^n x_i \mathbf{e}_i$. It is easy to see that the function $N_0 : E \to \mathbb{R}$, where

$$N_0(\mathbf{x}) = \|\varphi(\mathbf{x})\|_\infty = \|(x_1, \ldots, x_n)\|_\infty = \sup\{|x_i| \mid i = 1, \ldots, n\},$$

defines a norm on E. Let $N : E \to \mathbb{R}$ be any norm on E and set $a = \sum_{i=1}^n N(\mathbf{e}_i)$. If $\mathbf{x} \in E$, we have:

$$N(\mathbf{x}) = N\left(\sum_{i=1}^n x_i \mathbf{e}_i\right) \leq \sum_{i=1}^n N(x_i \mathbf{e}_i) \leq \sum_{i=1}^n |x_i| N(\mathbf{e}_i) \leq \sup_{i=1,\ldots,n}\{|x_i|\} \sum_{i=1}^n N(\mathbf{e}_i) = N_0(\mathbf{x}) \cdot a$$

so that $N(\mathbf{x}) \leq a N_0(\mathbf{x})$ for all $\mathbf{x} \in E$.

But the map $\varphi : (E, N_0) \to (\mathbb{K}^n, \|\cdot\|_\infty)$ is an **isometry** since $N_0(\mathbf{x}) = \|\mathbf{x}\|_\infty$ for all $\mathbf{x} \in E$, which means that it must be continuous (why?). Since

$$\tilde{S} = \{(x_1, \ldots, x_n) \in \mathbb{K}^n \mid \|(x_1, \ldots, x_n)\|_\infty = 1\} \subseteq_K \mathbb{K}^n,$$

then $S = \varphi^{-1}(\tilde{S}) = \{\mathbf{x} \in E | N_0(\mathbf{x}) = 1\} \subseteq_K E$; the norm $N : (E, N_0) \to (\mathbb{R}, |\cdot|)$ is also a continuous function – according to the max/min theorem, $\exists \mathbf{x}^* \in S$ such that $N(\mathbf{x}^*) = \inf_{\mathbf{x} \in S}\{N(\mathbf{x})\}$. Clearly, $N(\mathbf{x}^*) \neq 0$; otherwise we have $\mathbf{x}^* = \mathbf{0}$, which contradicts the fact that $\mathbf{x} \in S$ as $N_0(\mathbf{x}^*) = N_0(\mathbf{0}) = 0 \neq 1$. Hence $\inf_{\mathbf{x} \in S}\{N(\mathbf{x})\} > 0$.

Write $\inf_{\mathbf{x} \in S}\{N(\mathbf{x})\} = 1/b$ for the appropriate $b > 0$. If $\mathbf{x} = \mathbf{0} \in E$, then

$$N(\mathbf{x}) = N(\mathbf{0}) = 0 \geq 0 = \frac{1}{b} N_0(\mathbf{0}) = \frac{1}{b} N_0(\mathbf{x}).$$

If $\mathbf{x} \neq \mathbf{0} \in E$, then $\frac{\mathbf{x}}{N_0(\mathbf{x})} \in S$ and

$$N(\mathbf{x}) = N\left(N_0(\mathbf{x}) \frac{\mathbf{x}}{N_0(\mathbf{x})}\right) = N_0(\mathbf{x}) N\left(\frac{\mathbf{x}}{N_0(\mathbf{x})}\right) \geq N_0(\mathbf{x}) \cdot \frac{1}{b}.$$

In both cases, $N_0(\mathbf{x}) \leq b N(\mathbf{x})$ for all $\mathbf{x} \in E$, and so any norm N on E is equivalent to the norm N_0. By transitivity, any such norms are then equivalent to one another. ■

In general, this result is not valid if E is infinite-dimensional.

Corollary 143
Let E be a finite-dimensional vector space over $\mathbb{K}$ and let $(F, \|\cdot\|_F)$ be any normed vector space over $\mathbb{K}$. If $f : E \to F$ is a linear mapping, then f is continuous.

Proof: Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be a basis of E. For any $\mathbf{x} \in E$, we have

$$\begin{aligned} \|f(\mathbf{x})\|_F &= \|f(\textstyle\sum x_i \mathbf{e}_i)\|_F = \|\textstyle\sum x_i f(\mathbf{e}_i)\|_F \\ &\leq \textstyle\sum |x_i| \|f(\mathbf{e}_i)\|_F \leq N_0(\mathbf{x}) \cdot \textstyle\sum \|f(\mathbf{e}_i)\|_F := aN_0(\mathbf{x}). \end{aligned}$$

Then for any $\varepsilon > 0$, $\exists \delta = \frac{\varepsilon}{a}$ such that

$$\|f(\mathbf{x}) - f(\mathbf{y})\|_F = \|f(\mathbf{x} - \mathbf{y})\|_F \leq aN_0(\mathbf{x} - \mathbf{y}) < a\delta = \varepsilon$$

whenever $N_0(\mathbf{x} - \mathbf{y}) < \delta$, and so f is continuous. ■

This leads to a series of useful results.

Corollary 144
Any finite-dimensional vector space over $\mathbb{K}$ is a Banach space.

Proof: this is an easy consequence of the facts that the map

$$\varphi : (E, N_0) \to (\mathbb{K}^n, \|\cdot\|_\infty)$$

is an isometry and that $(\mathbb{K}^n, \|\cdot\|_\infty)$ is a Banach space. ■

Corollary 145
Any finite-dimensional subspace of a normed vector space over $\mathbb{K}$ is closed.

Corollary 146
The compact subsets of a finite-dimensional normed vector are the subsets that are both closed and bounded under the norm.

10.1 Solved Problems

1. Let E be a normed vector space over $\mathbb{R}$ and $A, B \subseteq E$. Denote

$$A + B = \{\mathbf{a} + \mathbf{b} \mid (\mathbf{a}, \mathbf{b}) \in A \times B\}.$$

   a) If $A \subseteq_O E$, show that $A + B \subseteq_O E$.

   b) If $A \subseteq_K E$ and $B \subseteq_C E$, show that $A + B \subseteq_C E$. Is the result still true if A is only assumed to be closed in E?

Proof:

a) We can write
$$A+B=\bigcup_{\mathbf{b}\in B}(A+\{\mathbf{b}\}).$$
If $A\subseteq_O E$, we obviously have $A+\{\mathbf{b}\}\subseteq_O E$ for any $\mathbf{b}\in B$.

Indeed, if $B(\mathbf{x},\rho)\subseteq A$, then $B(\mathbf{x}+\mathbf{b},\rho)\subseteq A+\{\mathbf{b}\}$. Thus $A+B$ is a union of open sets: as a result, $A+B\subseteq_O E$.

b) Let $(\mathbf{z}_n)=(\mathbf{x}_n+\mathbf{y}_n)\subseteq A+B$ be such that $\mathbf{z}_n\to\mathbf{z}$ where $(\mathbf{x}_n)\subseteq A$ and $(\mathbf{y}_n)\subseteq B$. Since $A\subseteq_K E$, there is a convergent subsequence $(\mathbf{x}_{\varphi(n)})$ with $\mathbf{x}_{\varphi(n)}\to\mathbf{x}\in A$.

Since $(\mathbf{z}_{\varphi(n)})$ converges to $\mathbf{z}$, the sequence $(\mathbf{y}_{\varphi(n)})\subseteq B$ converges to $\mathbf{y}=\mathbf{z}-\mathbf{x}$. But $B\subseteq_C E$ so that $\mathbf{y}\in B$. Thus, $\mathbf{z}=\mathbf{x}+\mathbf{y}\in A+B$, which proves the desired result. If A is only closed (and not compact), the result is false in general. Let $E=\mathbb{R}^2$, $A=\{(x,e^x)\mid x\in\mathbb{R}\}$ and $B=\mathbb{R}\times\{0\}$. Both $A,B\subseteq_C\mathbb{R}^2$ but $A+B=\mathbb{R}\times(0,\infty)$ is not closed in $\mathbb{R}^2$. ■ ■

2. Let E be a normed vector space over $\mathbb{R}$ and $\varphi:E\to\mathbb{R}$ be a linear functional on E.

   a) Show directly that φ is continuous on E if and only if $\ker\varphi\subseteq_C E$.

   b) i. Let F be a subspace of E. Show that the map $N:E/F\to\mathbb{R}$ defined by
   $$N([\mathbf{x}])=\inf_{\mathbf{y}\in[\mathbf{x}]}\{\|\mathbf{y}\|\}$$
   is a **semi-norm** on the quotient space E/F. What can you say if $F\subseteq_C E$?

   ii. Show part a) again, this time using part b)i.

Proof:

a) If φ is continuous, then $\ker\varphi=\varphi^{-1}(\{0\})\subseteq_C E$ since $\{0\}\subseteq_C\mathbb{R}$.

Conversely, suppose that $\ker\varphi\subseteq_C E$. If φ is not continuous, φ is unbounded on the unit sphere $S(\mathbf{0},1)$. Thus, $\exists(\mathbf{x}_n)\subseteq E$ such that $\|\mathbf{x}_n\|=1$ for all $n\in\mathbb{N}$ and for which $|\varphi(\mathbf{x}_n)|\to\infty$. Let $\mathbf{u}\in E$ be such that $\varphi(\mathbf{u})=1$: such a $\mathbf{u}\in E$ necessarily exists because φ is linear. Indeed, if $0\neq\varphi(\mathbf{w})=r\in\mathbb{R}$, then $\mathbf{w}\neq\mathbf{0}$. Set $\mathbf{u}=\frac{\mathbf{w}}{\varphi(\mathbf{w})}$. Then
$$\varphi(\mathbf{u})=\varphi\left(\frac{\mathbf{w}}{\varphi(\mathbf{w})}\right)=\frac{1}{\varphi(\mathbf{w})}\varphi(\mathbf{w})=1.$$
For any $n\in\mathbb{N}$, set $\mathbf{u}_n=\mathbf{u}-\frac{\mathbf{x}_n}{\varphi(\mathbf{x}_n)}$. Then
$$\varphi(\mathbf{u}_n)=\varphi(\mathbf{u})-\varphi\left(\frac{\mathbf{x}_n}{\varphi(\mathbf{x}_n)}\right)=\varphi(\mathbf{u})-\frac{\varphi(\mathbf{x}_n)}{\varphi(\mathbf{x}_n)}=\varphi(\mathbf{u}_n)-1=0,$$

whence $\mathbf{u}_n \in \ker\varphi$ for all $n \in \mathbb{N}$. Note that $\mathbf{u}_n = \mathbf{u} - \frac{\mathbf{x}_n}{\varphi(\mathbf{x}_n)} \to \mathbf{u}$ since $|\varphi(\mathbf{x}_n)| \to \infty$ and $\|\mathbf{x}_n\| = 1$ for all n. Since $\ker\varphi$, the limit $\mathbf{u} \in \ker\varphi$, *i.e.* $\varphi(\mathbf{u}) = 0$. But this contradicts the fact that $\varphi(\mathbf{u}) = 1$. Hence φ is continuous.

b) i. Let $\mathbf{x} \in E$ and $\lambda \in \mathbb{R}$. Recall that $[\mathbf{x}] = \mathbf{x} + F$. Since $[\lambda\mathbf{x}] = \lambda[\mathbf{x}]$, we have

$$N(\lambda[\mathbf{x}]) = |\lambda| N([\mathbf{x}]).$$

It remains only to show that N satisfies the triangle inequality.
Let $\mathbf{x}, \mathbf{y} \in E$. For any $\mathbf{u}, \mathbf{v} \in F$, we have

$$N([\mathbf{x}+\mathbf{y}]) \leq \|(\mathbf{x}+\mathbf{y}) + (\mathbf{u}+\mathbf{v})\| \leq \|\mathbf{x}+\mathbf{u}\| + \|\mathbf{y}+\mathbf{v}\|.$$

Thus

$$\begin{aligned} N([\mathbf{x}+\mathbf{y}]) &\leq \inf_{\mathbf{u},\mathbf{v}\in F}\{\|\mathbf{x}+\mathbf{u}\| + \|\mathbf{y}+\mathbf{v}\|\} \\ &\leq \inf_{\mathbf{u}\in F}\{\|\mathbf{x}+\mathbf{u}\|\} + \inf_{\mathbf{v}\in F}\{\|\mathbf{y}+\mathbf{v}\|\} = N([\mathbf{x}]) + N([\mathbf{y}]). \end{aligned}$$

As such, N is a semi-norm on E/F. Since $[\mathbf{x}] = \mathbf{x}+F$ for all $\mathbf{x} \in E$, $N([\mathbf{x}]) = \inf_{\mathbf{y}\in F}\{\|\mathbf{x}-\mathbf{y}\|\} = d(\mathbf{x}, F)$. As a result, if $F \subseteq_C E$, $N([\mathbf{x}]) = 0$ if and only if $\mathbf{x} \in F$, i.e. $[\mathbf{x}] = \mathbf{0}$. Consequently, if $F \subseteq_C E$, N is a norm on E/F.

ii. Let $\varphi : E \to \mathbb{R}$ be a linear functional for which $\ker\varphi \subseteq_C E$. If $\varphi \equiv 0$, φ is clearly continuous. Otherwise, $\varphi(E) = \mathbb{R}$. Indeed, let $x \in \mathbb{R}$. If $\varphi(\mathbf{u}) = 1$ for some $\mathbf{u} \in E$, then $x\mathbf{u} \in E$, $\varphi(x\mathbf{u}) = x$ and φ is onto. Let $\eta : E \to E/\ker\varphi$ be the canonical surjection $\eta(\mathbf{u}) = \mathbf{u} + \ker\varphi$. By the Isomorphism Theorem for vector spaces, it is possible to factor $\varphi = \psi \circ \eta$, where $\psi : E/\ker\varphi \to \mathbb{R}$ is linear.

According to Corollary 143, ψ is thus continuous, being linear. If N is the norm defined in (b)i. with $F = \ker\varphi$, we have

$$N([\mathbf{x}] - [\mathbf{y}]) = N([\mathbf{x}-\mathbf{y}]) \leq \|\mathbf{x}-\mathbf{y}\| \quad \forall \mathbf{x}, \mathbf{y} \in E$$

and so η is continuous Thus, ϕ is continuous being the composition of two continuous functions. ■

3. If $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, define $\|\mathbf{x}\|_\infty = \sup\{|x_1|, \ldots, |x_n|\}$. Show that $\mathbf{x} \mapsto \|\mathbf{x}\|_\infty$ defines a norm on $\mathbb{R}^n$.

Proof: There are 4 conditions to verify:

a) $\|\mathbf{x}\|_\infty = \sup\{|x_1|, \ldots, |x_n|\} \geq 0$ is clear since $|x_i| \geq 0$ for all i.

b) $\|\mathbf{x}\|_\infty = 0 \iff \sup\{|x_1|, \ldots, |x_n|\} = 0 \iff |x_i| = 0,\ \forall i \iff$
$x_i = 0,\ \forall i \iff \mathbf{x} = \mathbf{0}$.

c) If $a \in \mathbb{R}$, then

$$\|a\mathbf{x}\|_\infty = \sup\{|ax_1|, \ldots, |ax_n|\} = |a| \sup\{|x_1|, \ldots, |x_n|\} = |a|\|\mathbf{x}\|_\infty.$$

d) Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then

$$\begin{aligned}\|\mathbf{x}+\mathbf{y}\|_\infty &= \sup\{|x_1+y_1|, \ldots, |x_n+y_n|\} \leq \sup\{|x_1|+|y_1|, \ldots, |x_n|+|y_n|\} \\ &\leq \sup\{|x_1|, \ldots, |x_n|\} + \sup\{|y_1|, \ldots, |y_n|\} = \|\mathbf{x}\|_\infty + \|\mathbf{y}\|_\infty.\end{aligned}$$

Thus, $\mathbf{x} \to \|\mathbf{x}\|_\infty$ defines a norm on $\mathbb{R}^n$. ■

4. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and define the inner product $(\mathbf{x} \mid \mathbf{y}) = x_1y_1 + \cdots + x_ny_n$. As seen in the notes, this inner product defines a norm $\|\mathbf{x}\| = \sqrt{(\mathbf{x} \mid \mathbf{x})}$. Show the **Parallelogram Identity**: $\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = 2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2), \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

   Proof: We have

$$\begin{aligned}\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 &= (\mathbf{x}+\mathbf{y} \mid \mathbf{x}+\mathbf{y}) + (\mathbf{x}-\mathbf{y} \mid \mathbf{x}-\mathbf{y}) \\ &= (\mathbf{x} \mid \mathbf{x}) + 2(\mathbf{x} \mid \mathbf{y}) + (\mathbf{y} \mid \mathbf{y}) + (\mathbf{x} \mid \mathbf{x}) - 2(\mathbf{x} \mid \mathbf{y}) + (\mathbf{y} \mid \mathbf{y}) \\ &= 2(\mathbf{x} \mid \mathbf{x}) + 2(\mathbf{y} \mid \mathbf{y}) = 2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2)\end{aligned}$$

   Now, consider a parallelogram with vertices $\mathbf{0}, \mathbf{x}, \mathbf{y}, \mathbf{x}+\mathbf{y}$. Then the sum of squares of the lengths of the four sides is $2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2)$, while the sum of squares of the diagonals is $\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2$. ■

5. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Is it true that $\|\mathbf{x}+\mathbf{y}\|_\infty = \|\mathbf{x}\|_\infty + \|\mathbf{y}\|_\infty$ if and only if $\mathbf{x} = c\mathbf{y}$ or $\mathbf{y} = c\mathbf{x}$ for some $c \geq 0$?

   Proof: No. Consider the following example in $\mathbb{R}^2$: let $\mathbf{x} = (1,0)$ and $\mathbf{y} = (1,1)$. Then $\mathbf{x}+\mathbf{y} = (2,1)$ and $\|\mathbf{x}\|_\infty + \|\mathbf{y}\|_\infty = \|\mathbf{x}+\mathbf{y}\|_\infty = 2$, but $\mathbf{x} \neq c\mathbf{y}$ for any $c \in \mathbb{R}$. ■

10.2 Exercises

1. Prepare a 2-page summary of this chapter, with important definitions and results.
2. Show that $\|A\|_\infty$, $\|A\|_1$, and $\|A\|_2$ (from the first page of this chapter) define norms over $\mathbb{M}_{m,n}(\mathbb{K})$.
3. Show that the induced $p-$norm is a norm on $\mathbb{M}_{m,n}(\mathbb{K})$ for all $p \geq 1$.
4. Prove Proposition 139.
5. Show that all isometries are continuous.
6. Prove Corollary 145.
7. Prove Corollary 146.
8. Let E be a normed vector space with a countably infinite basis. Show that E cannot be complete.
9. Let E be an infinite-dimensional normed vector space over $\mathbb{R}$. Show that $D(\mathbf{0}, 1)$ is not compact in E by showing that it is not pre-compact in E (by what name is this result usually known?).