Chapter 11

Sequences of Functions in Metric Spaces

In this chapter, we study properties of sequences and series of functions, extending Chapters 5 and 6 to **general metric spaces** and provide important **Fourier analysis** results.

The symbol $\mathbb{K}$ is used to denote either $\mathbb{R}$ or $\mathbb{C}$; $\mathcal{C}^\ell(X, \mathbb{K})$ represents the $\mathbb{K}-$vector space of ℓ times continuously differentiable functions $X \to \mathbb{K}$; $\mathcal{F}(X, \mathbb{K})$, the $\mathbb{K}-$vector space of functions $X \to \mathbb{K}$; $\mathcal{R}(X, \mathbb{K})$, the $\mathbb{K}-$vector space of Riemann-integrable functions $X \to \mathbb{K}$, $\mathcal{C}_c(\mathbb{R}, \mathbb{C})$ is the set of continuous functions $\mathbb{R} \to \mathbb{C}$ with **compact support**,[1] and $\mathcal{B}(X, \mathbb{K})$, the $\mathbb{K}-$vector space of bounded functions $X \to \mathbb{K}$.

11.1 Uniform Convergence

Let X be a set and let (E, d) be a metric space. A sequence $(f_n)_{n\in\mathbb{N}}$ of functions $f_n : X \to E$ is said to **converge pointwise** to a function $f : X \to E$ (denoted by $f_n \to f$ on X) if $f_n(\mathbf{x}) \to f(\mathbf{x})$ for all $\mathbf{x} \in X$.

Symbolically, $f_n \to f$ on X if

$$\forall \varepsilon > 0, \forall \mathbf{x} \in X, \exists N = N_{\varepsilon,\mathbf{x}} \text{ such that } n > N \implies d(f_n(\mathbf{x}), f(\mathbf{x})) < \varepsilon$$

(note the **explicit dependence** of N on $\mathbf{x}$).

As we have discussed in Chapters 5 and 6, pointwise convergence is quite often **not strong enough** of a property for our needs. Consequently, we introduce a second kind of convergence: the sequence (f_n) is said to **converge uniformly** to a function $f : X \to E$ (denoted by $f_n \rightrightarrows f$ on X) if we can remove the explicit dependence of N on **x**.

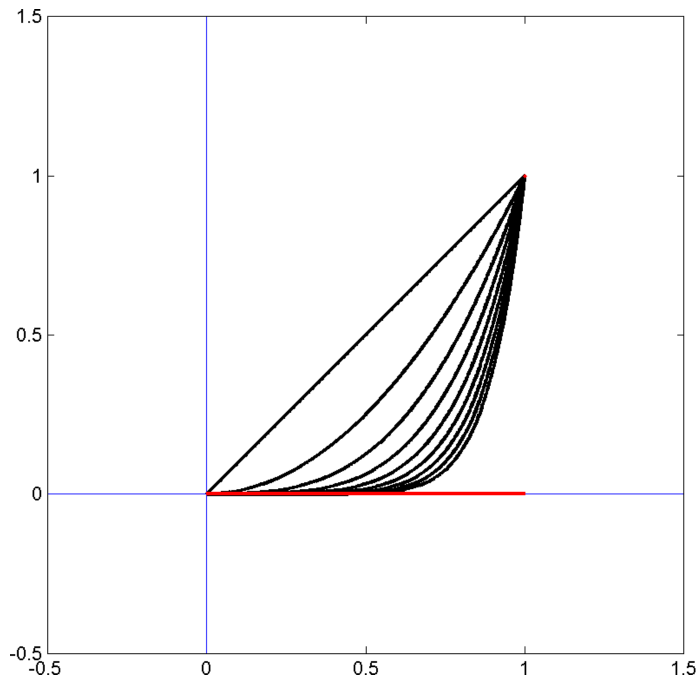
Symbolically, $f_n \rightrightarrows f$ on X if

$$\forall \varepsilon > 0, \exists N = N_\varepsilon \text{ such that } n > N \implies \sup_{\mathbf{x}\in X}\{d(f_n(\mathbf{x}), f(\mathbf{x}))\} < \varepsilon.$$

[1] That is, functions taking on the value 0 outside of some compact subset $K \subseteq \mathbb{R}$.

Examples

1. Let $(E, d) = (\mathbb{R}, |\cdot|)$, $X = [0, 1]$ and $f_n : X \to E$ be defined by $f_n(x) = x^n$. Then $f_n \to f$ on X, where $f : X \to E$ is given by $f(x) = 0$ if $x \neq 1$ and $f(1) = 1$. Note that f is not continuous on X, even though each of the f_n is continuous.



The sequence (f_n) in black, the limit f in red.

2. With the definitions as in the last example, $f_n \not\rightrightarrows f$ on X. Indeed,

$$\sup_{x\in[0,1]} \{d(f_n(x), f(x))\} = \sup_{x\in[0,1]} \{|x^n|\} = 1^n = 1,$$

which can never be smaller than any $1 > \varepsilon > 0$.

However, $f_n \rightrightarrows f$ on $[0, a]$ for all $a \in [0, 1)$ (see Chapter 5).

Theorem 66 generalizes to metric spaces as one would expect.

Proposition 147 (Cauchy's Criterion for Sequences of Functions)
Let (E, d) be a complete metric space and (f_n) be a sequence of functions $f_n : X \to E$. Then, $f_n \rightrightarrows f$ on X if and only if

$$\forall\varepsilon > 0, \exists N = N_\varepsilon > 0 \textit{ s.t. } n, m > N \implies \sup_{\mathbf{x}\in X}\{d(f_n(\mathbf{x}), f_m(\mathbf{x}))\} < \varepsilon.$$

Proof: suppose that $f_n \rightrightarrows f$ on X and let $\varepsilon > 0$. By hypothesis, $\exists N_1, N_2$ such that

$$\sup_{\mathbf{x}\in X}\{d(f_n(\mathbf{x}), f(\mathbf{x}))\} < \frac{\varepsilon}{2}, \quad \sup_{\mathbf{x}\in X}\{d(f_m(\mathbf{x}), f(\mathbf{x}))\} < \frac{\varepsilon}{2}$$

whenever $n > N_1$ and $n > N_2$. Set $N = \max\{N_1, N_2\}$.

Then, whenever $n, m > N$, we have

$$\begin{aligned}\sup_{\mathbf{x}\in X}\{d(f_n(\mathbf{x}), f_m(\mathbf{x}))\} &\leq \sup_{\mathbf{x}\in X}\{d(f_n(\mathbf{x}), f(\mathbf{x})) + d(f_m(\mathbf{x}), f(\mathbf{x}))\} \\ &\leq \sup_{\mathbf{x}\in X}\{d(f_n(\mathbf{x}), f(\mathbf{x}))\} + \sup_{\mathbf{x}\in X}\{d(f_m(\mathbf{x}), f(\mathbf{x}))\} < \varepsilon.\end{aligned}$$

Conversely, suppose that the $\varepsilon-$statement holds. Then, for any $\mathbf{x} \in X$, $(f_n(\mathbf{x}))$ is a Cauchy sequence in E and thus converges to a $f(\mathbf{x}) \in E$, as E is complete. As a result, $f_n \to f$ on X. It remains to show that $f_n \rightrightarrows f$ on X.

Let $\varepsilon > 0$. By hypothesis, $\exists N > 0$ such that $\sup_{\mathbf{x}\in X}\{d(f_n(\mathbf{x}), f_m(\mathbf{x}))\} < \frac{\varepsilon}{2}$ whenever $n, m > N$. Now, fix $n > N$ and let

$$a_m(\mathbf{x}) = d(f_n(\mathbf{x}), f_m(\mathbf{x})) \quad \text{and} \quad a(\mathbf{x}) = d(f_n(\mathbf{x}), f(\mathbf{x})).$$

Then $a_m(\mathbf{x}) \to a(\mathbf{x})$ Since $a_m(\mathbf{x}) < \frac{\varepsilon}{2}$ for all $\mathbf{x} \in X$, then $a(\mathbf{x}) \leq \frac{\varepsilon}{2}$ for all $\mathbf{x} \in X$. Hence,

$$\sup_{\mathbf{x}\in X}\{d(f_n(\mathbf{x}), f(\mathbf{x}))\} \leq \sup_{\mathbf{x}\in X}\{a(\mathbf{x})\} \leq \frac{\varepsilon}{2} < \varepsilon.$$

As such, $f_n \rightrightarrows f$ on X. ■

Series of Functions

Similar notions exist for **series** of functions. Let (E, d) be a metric space and let (u_n) be a sequence of functions $u_n : X \to E$. For any $m \in \mathbb{N}$, define the **partial sum** $f_m : X \to E$ by

$$f_m(\mathbf{x}) = u_1(\mathbf{x}) + \cdots + u_m(\mathbf{x}) = \sum_{n=1}^{m} u_n(\mathbf{x}).$$

The sequence (f_m) is the **series generated** by (u_n), and it is usually denoted by $\sum_{n\in\mathbb{N}} u_n$.

If $f_m \to f$ on X, we say that the series **converges (pointwise)** on X; if $f_m \rightrightarrows f$ on X, we say that the series **converges uniformly** on X. In both cases, f is said to be the **sum** of the series. If (f_m) does not converge, we say that the series **diverges**.

Finally, let E be a Banach space and let (g_n) be a sequence of functions $g_n \in \mathcal{B}(X, E)$. The series $\sum g_n$ **converges absolutely** on X if $\sum \|g_n\|_\infty$ converges.[2]

[2]There is no need to stipulate the type of convergence in the latter case, since that is a numerical series.

Proposition 148
If $\sum g_n$ converges absolutely on X, then $\sum g_n$ converges uniformly on X.

Proof: according to the Cauchy criterion, it suffices to show that $\forall\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$\Big\| \sum_{k=n}^{m} g_k \Big\|_\infty < \varepsilon.$$

But according to the triangle inequality,

$$\Big\| \sum_{k=n}^{m} g_k \Big\|_\infty \leq \sum_{k=n}^{m} \|g_k\|_\infty.$$

Since $\sum g_k$ converges absolutely, $\forall\varepsilon > 0$, $\exists N > 0$ such that

$$\sum_{k=n}^{m} \|g_k\|_\infty < \varepsilon$$

whenever $n > N$. ■

11.1.1 Properties

The two main types of convergence are not created equal, however. We establish the superiority of uniform convergence over pointwise convergence in a series of well-known theorems (which all have counterparts in Chapter 5).

Theorem 149
Let (E, d) and $(F, \tilde{d})$ be metric spaces. If $(f_n) \subseteq \mathcal{C}(E, F)$ is such that $f_n \rightrightarrows f$ on E, then $f \in \mathcal{C}(E, F)$.

Proof: let $\varepsilon > 0$ and $\mathbf{x}_0 \in E$.

Since $f_n \rightrightarrows f$ on E, then $\exists n > N$ for which $\sup_{\mathbf{x}\in E}\{d(f_n(\mathbf{x}), f(\mathbf{x}))\} < \frac{\varepsilon}{3}$. Furthermore, since f_n is continuous at $\mathbf{x}_0$, $\exists\delta > 0$ such that

$$\tilde{d}(f_n(\mathbf{x}), f_n(\mathbf{x}_0)) < \frac{\varepsilon}{3} \quad \text{whenever } d(\mathbf{x}, \mathbf{x}_0) < \delta.$$

Then

$$\begin{aligned}\tilde{d}(f(\mathbf{x}), f(\mathbf{x}_0)) &= \tilde{d}(f(\mathbf{x}), f_n(\mathbf{x})) + \tilde{d}(f_n(\mathbf{x}), f_n(\mathbf{x}_0)) + \tilde{d}(f_n(\mathbf{x}_0), f(\mathbf{x})) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\end{aligned}$$

whenever $d(\mathbf{x}, \mathbf{x}_0) < \delta$, hence f is continuous at $\mathbf{x}_0$. ■

We have already seen an example showing that this may not hold for pointwise convergence.

Theorem 150 (LIMIT INTERCHANGE; RIEMANN-INTEGRABLE FUNCTIONS)
Let $(E, \|\cdot\|)$ be a Banach space. If $(f_n) \subseteq \mathcal{F}([a,b],E)$ is such that $f_n \rightrightarrows f$ on $[a,b]$, and if f_n is Riemann-integrable over $[a,b]$ for all n, then f is Riemann-integrable and $\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx$.

Proof: left as an exercise (see Chapter 5). ■

Although, the fact that the limit interchange is not necessarily valid if $f_n \to f$ instead of $f_n \rightrightarrows f$ on $[a,b]$ could be seen as an indictment of the **Riemann integral** rather than as an indictment of pointwise convergence. In chapter 21, we will take the former position and introduce the **Lebesgue (Borel) integral** to circumvent this difficulty.

The next result is a companion to Theorem 150.

Theorem 151 (LIMIT INTERCHANGE; FUNDAMENTAL THEOREM)
Let $(E, \|\cdot\|)$ be a Banach space. If $(f_n) \subseteq \mathcal{F}([a,b],E)$ is such that $f_n \rightrightarrows f$ on $[a,b]$, and if f_n is Riemann-integrable over $[a,b]$ for all n, then f is Riemann-integrable according to Theorem 150. Define $F_n, F : [a,b] \to E$ by $F_n(x) = \int_a^x f_n(t)\,dt$ and $F(x) = \int_a^x f(t)\,dt$. Then $F_n \rightrightarrows F$ on $[a,b]$.

Proof: let $\varepsilon > 0$.

Since $f_n \rightrightarrows f$ on $[a,b]$, $\exists N \in \mathbb{N}$ such that $\|f - f_n\|_\infty < \frac{\varepsilon}{2(b-a)}$ whenever $n > N$. Now,

$$\begin{aligned}\|F_n(\mathbf{x}) - F(\mathbf{x})\| &= \left\|\int_a^x (f_n(t) - f(t))\,dt\right\| \leq \int_a^x \|f_n(t) - f(t)\|\,dt \\ &\leq \int_a^x \|f_n - f\|_\infty\,dt < \frac{\varepsilon}{2(b-a)}(x-a) \leq \frac{\varepsilon}{2(b-a)}(b-a) = \frac{\varepsilon}{2}.\end{aligned}$$

Since this is true for all $x \in [a,b]$, then $\|F_n - F\|_\infty \leq \frac{\varepsilon}{2} < \varepsilon$. By the Cauchy criterion, $F_n \rightrightarrows F$ on $[a,b]$. ■

Theorem 151 has an interesting corollary when applied to series, which is often assumed to hold (without proof) when solving differential equations.

Theorem 152 Let $(E, \|\cdot\|)$ be a Banach space and let $\sum g_n$ be a series of functions in $\mathcal{R}([a,b],E)$. If $\sum g_n$ is uniformly convergent, then

$$\int_a^b \left(\sum_{n\in\mathbb{N}} g_n(t)\right)\,dt = \sum_{n\in\mathbb{N}} \left(\int_a^b g_n(t)\,dt\right).$$

Proof: this is a direct consequence of Theorem 151. ■

We have not defined differentiability of functions $\mathbb{R} \to E$ in a general normed vector space E, but we can use functions $\mathbb{R} \to \mathbb{K}^n$ as a template: a function $f : \mathbb{R} \to \mathbb{K}^n$ is **differentiable at** t if

$$f'(t) = \lim_{h\to 0} \frac{f(t+h) - f(t)}{h}$$

exists; it is **differentiable over** $\mathbb{R}$ if it is differentiable at all $t \in \mathbb{R}$. Differentiability is also the subject of a limit interchange theorem.

Theorem 153 (Limit Interchange; Differentiable Functions)
Let $(E, \|\cdot\|)$ be a Banach space. If $(f_n) \subseteq \mathcal{C}^1([a,b], E)$ is such that $f_n(x_0) \to f(x_0)$ for some $x_0 \in [a,b]$ and if $\exists g \in \mathcal{C}([a,b], E)$ such that $f_n' \rightrightarrows g$ on $[a,b]$, then $\exists f \in \mathcal{C}^1([a,b], E)$ such that $f_n \rightrightarrows f$ on $[a,b]$ and $f' = g$.

Proof: according to the fundamental theorem of calculus, for all $n \in \mathbb{N}$ we have $f_n(x) - f_n(a) = \int_a^x f_n'(t)\,dt$. Since $f_n' \rightrightarrows g$, then

$$f_n(x) - f_n(a) = \int_a^x f_n'(t)\,dt \rightrightarrows \int_a^x g(t)\,dt \quad \text{on } [a,b],$$

according to Theorem 150. In particular, the sequence $(f_n(x_0) - f(a))_n$ converges, which implies that $(f_n(a))_n$ converges to some $\ell \in E$. It is easy to show that $f_n \rightrightarrows f$, where $f : [a,b] \to E$ is defined by

$$f(x) = \ell + \int_a^x g(t)\,dt.$$

Since all the f_n are continuous and the convergence is uniform, then f is continuous. It is also differentiable, and its derivative is continuous as $f' = g \in \mathcal{C}([a,b], E)$ (again, according to the fundamental theorem of calculus). ■

We can use these theorems to compute various quantities that would be difficult to compute directly.

Examples

1. Compute $\int_0^\infty f(x)\,dx$, where $f(x) = \frac{x^2}{\exp(x)-1}$.

   Solution: consider $(g_n) \subseteq \mathcal{C}(\mathbb{R}^+, \mathbb{R}^+)$ defined by $g_n(x) = \exp(-nx)x^2$ for all $n \in \mathbb{N}^\times$. Then $\sum g_n$ converges pointwise to $f : \mathbb{R}^+ \to \mathbb{R}^+$.

Indeed,

$$\begin{aligned}\sum_{n=1}^{m} g_n(x) = x^2\left(\sum_{n=1}^{m}\exp(-nx)\right) = x^2\left(\sum_{n=1}^{m}(\exp(-x))^n\right)\\ = x^2\left(\frac{\exp(-x)-\exp(-(m+1)x)}{1-\exp(-x)}\right) \leq f(x),\end{aligned}$$

since $\exp(-x) < 1$ for all $x \in \mathbb{R}^+$.

Then,

$$\begin{aligned}\sum_{n\in\mathbb{N}^\times} g_n(x) = \lim_{m\to\infty}\sum_{n=1}^{m} g_n(x) = \lim_{m\to\infty} x^2\left(\frac{\exp(-x)-\exp(-(m+1)x)}{1-\exp(-x)}\right)\\ = \frac{x^2}{\exp(x)-1}.\end{aligned}$$

Furthermore, $\sum g_n$ converges absolutely to f on $[a,b] \subseteq (0,\infty)$.

Indeed, for all $x \in [a,b]$, we have $|g_n(x)| \leq \exp(-na)b^2$. Note that

$$\sum_{n\in\mathbb{N}^\times}\exp(-na)b^2 = b^2\sum_{n\in\mathbb{N}^\times}(\exp(-a))^n = \frac{b^2}{\exp(a)-1}, \quad \text{since } a > 0.$$

Hence $\sum_{n\in\mathbb{N}^\times}\exp(-na)b^2$ converges and so, according to Exercise 1, $\sum g_n$ is absolutely convergent.

Since $\int_0^\infty f(t)\,dt$ converges (use the Comparison Theorem with $\exp(-\sqrt{x})$, for instance), then, according to Theorem 152,

$$\int_0^\infty f(t)\,dt = \int_0^\infty\left(\sum_{n\in\mathbb{N}^\times} g_n(t)\right)dt = \sum_{n\in\mathbb{N}^\times}\left(\int_0^\infty g_n(t)\,dt\right)$$

Repeated integration by parts shows that $\int_0^\infty g_n(t)\,dt = \frac{2}{n^3}$, so that

$$\int_0^\infty \frac{x^2}{\exp(x)-1}\,dx = 2\sum_{n\in\mathbb{N}^\times}\frac{1}{n^3} = 2\zeta(3). \qquad \blacksquare$$

2. Show that uniform convergence is not equivalent to absolute convergence.

   Proof: it will be sufficient to exhibit a series which is uniformly convergent but not absolutely convergent. Consider (u_k) a series of constant functions from an interval I to $\mathbb{R}$ defined by $u_k(x) = \frac{(-1)^k}{k}$ for all $x \in I$.

   Since $\|u_k\|_\infty = \frac{1}{k}$, and since $\sum \frac{1}{k}$ diverges (it is the **harmonic series**, after all), then $\sum u_k$ is not absolutely convergent. However,

   $$\left\| \sum_{k=n}^{m} u_k \right\|_\infty = \left| \sum_{k=n}^{m} \frac{(-1)^k}{k} \right| \leq \frac{1}{n} \to 0 \quad \text{as } n, m \to \infty,$$

   so that $\sum u_k$ is uniformly convergent. ■

11.1.2 Abel's Criterion

In calculus courses and in Chapters 5 and 6, we have seen a number of tests can be used to gauge the convergence of series (whether numerical series or series of functions):

- $p-$test;
- comparison test;
- alternating series test;
- integral test;
- d'Alembert test (also known as the ratio test), or
- Cauchy test (also known as the root test).

In this section, we present a new test.

Proposition 154 (Abel's Criterion)
Let $(\mathbf{a}_n) \subseteq E$, where E is a Banach space over $\mathbb{R}$. If we can write $\mathbf{a}_n = \varepsilon_n \mathbf{b}_n$ with

1. *$\varepsilon_n \searrow 0$ a sequence in $\mathbb{R}$, and*
2. *$\exists \sigma \in \mathbb{R}$ such that $\| \sum_{n \leq N} \mathbf{b}_n \| \leq \sigma$ for all $N \in \mathbb{N}$.*

Then $\sum \mathbf{a}_n$ is pointwise convergent and $\| \sum_{n \geq N} \mathbf{a}_n \| \leq 2\sigma\varepsilon_N$ for all $N \in \mathbb{N}$.

Proof: for any $q > p$, let $S_p^q = \mathbf{b}_{p+1} + \cdots + \mathbf{b}_q$. Since $S_p^q = \sum_{n \leq q} \mathbf{b}_n - \sum_{n \leq p} \mathbf{b}_n$, we have $\left\| S_p^q \right\| \leq 2\sigma$. If we write

$$\mathbf{b}_{p+1} = S_p^{p+1},\ \mathbf{b}_{p+2} = S_p^{p+2} - S_p^{p+1},\ \cdots,\ \mathbf{b}_q = S_p^q - S_p^{q-1},$$

then

$$\begin{aligned}\varepsilon_{p+1}\mathbf{b}_{p+1}+\cdots+\varepsilon_q\mathbf{b}_q &= \varepsilon_{p+1}S_p^{p+1}+\varepsilon_{p+2}\left(S_p^{p+2}-S_p^{p+1}\right)+\cdots+\varepsilon_q\left(S_p^q-S_p^{q-1}\right)\\ &= S_p^{p+1}\left(\varepsilon_{p+1}-\varepsilon_{p+2}\right)+\cdots+S_p^{q-1}\left(\varepsilon_{q-1}-\varepsilon_q\right)+\varepsilon_q S_p^q,\end{aligned}$$

whence

$$\begin{aligned}\left\|\sum_{k=p+1}^{q}\mathbf{a}_k\right\| &= \|\varepsilon_{p+1}\mathbf{b}_{p+1}+\cdots+\varepsilon_q\mathbf{b}_q\|\\ &\leq \left\|S_p^{p+1}\right\|\left|\varepsilon_{p+1}-\varepsilon_{p+2}\right|+\cdots+\left\|S_p^{q-1}\right\|\left|\varepsilon_{q-1}-\varepsilon_q\right|+\left|\varepsilon_q\right|\left\|S_p^q\right\|\\ &\leq 2\sigma\left(\varepsilon_{p+1}-\varepsilon_{p+2}\right)+\cdots+2\sigma\left(\varepsilon_{q-1}-\varepsilon_q\right)+2\sigma\varepsilon_q\\ &= 2\sigma\varepsilon_{p+1}\to 0 \quad \text{as } p,q\to\infty\end{aligned}$$

Hence, $\sum \mathbf{a}_k$ converges by the Cauchy Criterion. ■

We can easily generalize this result to sequences of functions.

Proposition 155 (Abel's Criterion (Reprise))
Let $\sum f_n$ be such that $f_n=\varepsilon_n g_n\in\mathcal{F}([a,b],E)$, where E is a Banach space over $\mathbb{R}$. If

1. *$\varepsilon_n(x)\searrow 0$ for all $x\in[a,b]$;*
2. *$\exists\sigma\in\mathbb{R}$ such that $\|\sum_{n\leq N} g_n(x)\|\leq\sigma$ for all $N\in\mathbb{N}$ and all $x\in[a,b]$, and*
3. *$\|\varepsilon_n\|_\infty\to 0$.*

Then $\sum f_n$ is uniformly convergent on $[a,b]$.

Proof: left as an exercise. ■

The three conditions are in fact independent (see Exercise 7). For the next example (and the rest of the chapter), we assume some familiarity with complex numbers (see Chapter 22 if necessary).

Example: consider the series $\sum_{k\in\mathbb{N}} c_k b_k(x)$, where $b_k(x)=e^{ikx}$, $x\in\mathbb{R}$ and $c_k\searrow 0$. Show that the series converges (pointwise) for any $x\in(0,2\pi)$ and that it converges uniformly on $[\delta,2\pi-\delta]$ for any $\delta\in(0,\pi)$.

Proof: since $|e^{ikx}|=1$, then $\sum_{k\in\mathbb{N}} c_k e^{ikx}$ is absolutely convergent whenever $\sum_{k\in\mathbb{N}}|c_k|<\infty$. If $x\neq 2k\pi$, $k\in\mathbb{N}$, then

$$1+e^{ix}+\cdots+e^{inx}=\frac{1-e^{i(n+1)x}}{1-e^{ix}},$$

whence

$$\left|\sum_{k=1}^{n} b_k(x)\right| = |1 + e^{ix} + \cdots + e^{inx}| \leq \frac{2}{|1-e^{ix}|} := \sigma_x.$$

According to Abel's criterion for numerical series, $\sum_{k\in\mathbb{N}} c_k e^{ikx}$ thus converges pointwise for any $x \in (0, 2\pi)$.

Now, let $\pi > \delta > 0$ and $x \in [\delta, 2\pi - \delta]$. Then

$$|1-e^{ix}| = \left|e^{ix/2}(e^{-ix/2} - e^{ix/2})\right| = 2\left|\frac{e^{ix/2} - e^{-ix/2}}{2i}\right| = 2|\sin(x/2)| > \sin\delta,$$

from which we can conclude that

$$\left|\sum_{k=1}^{n} b_k(x)\right| \leq \frac{2}{\sin\delta} := \sigma.$$

Consequently, again according to Abel's criterion applied to series of functions, $\sum_{k\in\mathbb{N}} c_k e^{ikx}$ converges uniformly for any on $[\delta, 2\pi - \delta]$ for any $\pi > \delta > 0$. ■

11.2 Fourier Series

The series $\sum_{k\in\mathbb{N}} c_k e^{ikx}$ in the previous example is continuous on $(0, 2\pi)$ even though it fails to converge uniformly on $(0, 2\pi)$. It is an example of a **Fourier Series**, a monumental idea in the development of modern mathematics. They were first proposed as solutions to the **heat equation**, in which we seek functions $u : U \subseteq_O \mathbb{R}^2 \times (a, b) \to \mathbb{R}$ satisfying the partial differential equation

$$\frac{\partial u}{\partial t} = \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

Since the Fourier series approach gave rise to **already-known solutions** of the heat equation, the process with which they were formed was accepted as valid, even though a number of mathematicians had objections concerning the use of infinity and (possibly divergent) series.

The importance of rigour in mathematics was just starting to be understood by some of the best mathematical minds; while these objections may sound a bit odd nowadays, it is important to remember that the current definitions of the concepts that made some of our predecessors queasy have been distilled of all offending material after years of polishing, which was driven by the very objections that they brought up.

It is no exaggeration to say that analysis would not be what it is today without this particular episode; while it remains in fashion amongst some mathematicians to deride engineers and physicists for "playing with tools beyond their understanding", let us keep in mind that analytical advances mostly arise from the application of mathematics to so-called 'real-world' problems, in the grand tradition of Archimedes and Newton.

In this section, we introduce and discuss the convergence of Fourier Series.

11.2.1 Trigonometric Series and Periodic Functions

A **trigonometric polynomial** is any (finite) linear combination of positive powers of sines and cosines:

$$p(t) = a_0 + \sum_{k=1}^{n} \left(a_k \cos(kt) + b_k \sin(kt)\right), \quad \text{where } a_k, b_k \in \mathbb{C}.$$

Since

$$\cos t = \frac{e^{it} + e^{-it}}{2}, \quad \sin t = \frac{e^{it} - e^{-it}}{2i},$$

we can write

$$p(t) = a_0 + \sum_{k=1}^{n} \left(a_k \cos(kt) + b_k \sin(kt)\right) = \sum_{k=-n}^{n} c_k e^{ikt},$$

with

$$a_0 = c_0, \quad a_k = c_k + c_{-k}, \quad \text{and} \quad b_k = i(c_k - c_{-k}),$$

or

$$c_0 = a_0, \quad c_k = \frac{a_k - ib_k}{2}, \quad \text{and} \quad c_{-k} = \frac{a_k + ib_k}{2},$$

for all $1 \leq k \leq n$.

A **trigonometric series** is a formal expression of the form

$$\sum_{k \in \mathbb{Z}} c_k e^{ikt} = a_0 + \sum_{k \in \mathbb{N}} \left(a_k \cos(kt) + b_k \sin(kt)\right).$$

We say that a series indexed by $\mathbb{Z}$ **converges** if both the series indexed by the positive integers **and** the series indexed by the negative integers converges.

> **Proposition 156**
> *If $\sum_{k\in\mathbb{Z}} c_k e^{ikt}$ converges absolutely for some t, then $\sum_{k\in\mathbb{Z}} |c_k| < \infty$. Furthermore, if $\sum_{k\in\mathbb{Z}} |c_k| < \infty$, then $\exists f \in \mathcal{C}(\mathbb{R},\mathbb{C})$ such that $\sum_{k\in\mathbb{Z}} c_k e^{ikt} \rightrightarrows f$ on $\mathbb{R}$.*
>
> **Proof:** left as an exercise. ■

These ideas will become more clear with a concrete example.

> **Example:** Let $b \in (-1, 1)$. Consider the trigonometric series $\sum_{k\in\mathbb{N}} b^k \sin(kt)$. What is its complex form? Does it converge anywhere? If so, what to?
>
> **Solution:** according to the previous formulas, we formally have
>
> $$c_0 = 0, \quad c_k = \frac{0 - ib^k}{2} = \frac{b^k}{2i} \quad \text{and} \quad c_{-k} = \frac{0 + ib^k}{2} = -\frac{b^k}{2i},$$
>
> for $k \geq 1$.

We also have

$$\sum_{k=1}^{n} b^k \sin(kt) = -\frac{1}{2i}\sum_{k=-n}^{-1} b^{-k}e^{ikt} + \frac{1}{2i}\sum_{k=1}^{n} b^k e^{ikt},$$

so that, formally,

$$\sum_{k=1}^{\infty} b^k \sin(kt) = -\frac{1}{2i}\sum_{k=-\infty}^{-1} b^{-k}e^{ikt} + \frac{1}{2i}\sum_{k=1}^{\infty} b^k e^{ikt}.$$

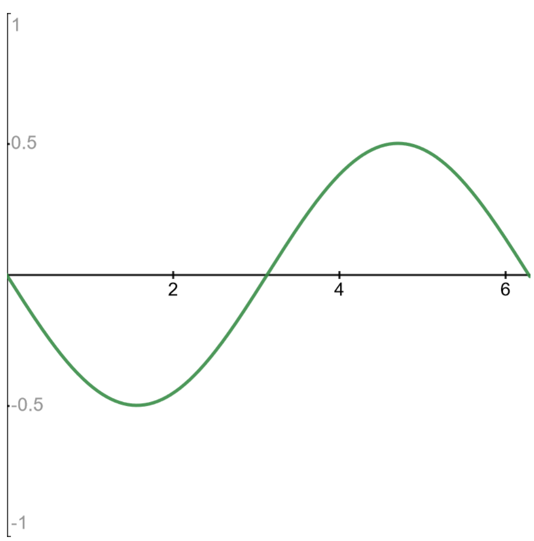
The series converges absolutely (and thus at least pointwise), as

$$\sum_{k\geq 1} \|b^k \sin(kt)\|_\infty = \sum_{k\geq 1} |b|^k = \frac{|b|}{1-|b|} < \infty, \quad \text{since } |b| < 1.$$

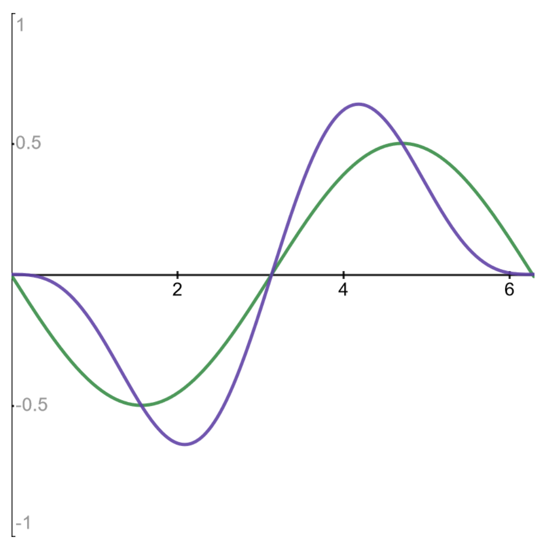
According to Proposition 148, $\exists f \in \mathcal{C}(\mathbb{R},\mathbb{C})$ to which the series converges uniformly on $\mathbb{R}$. We can re-write the convergent series as

$$\begin{aligned}\sum_{k=1}^{\infty} b^k \sin(kt) &= \frac{1}{2i}\left[\sum_{k=1}^{\infty}\left(be^{it}\right)^k - \sum_{k=1}^{\infty}\left(be^{-it}\right)^k\right] = \frac{1}{2i}\left(\frac{be^{it}}{1-be^{it}} - \frac{be^{-it}}{1-be^{-it}}\right)\\ &= \frac{b}{2i}\cdot\frac{e^{it}-e^{-it}}{1-b(e^{it}+e^{-it})+b^2} = b\cdot\underbrace{\frac{e^{it}-e^{-it}}{2i}}_{=\sin t}\cdot\frac{1}{1-2b\underbrace{\frac{e^{it}+e^{-it}}{2}}_{=\cos t}+b^2}.\end{aligned}$$

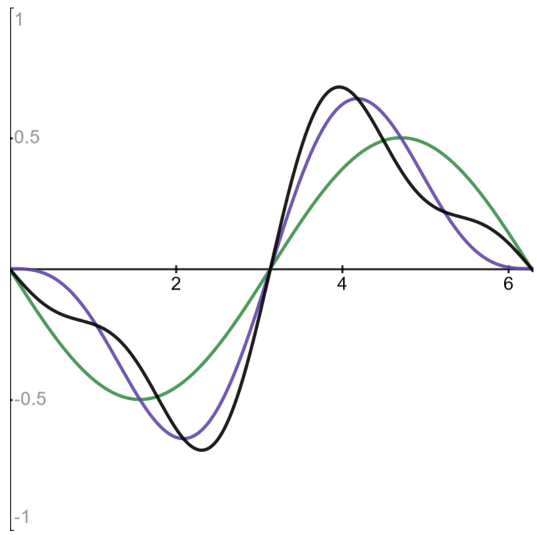
Thus the series converges uniformly to $f : t \mapsto \frac{b\sin t}{1-2b\sin t+b^2}$ on $\mathbb{R}$. ■



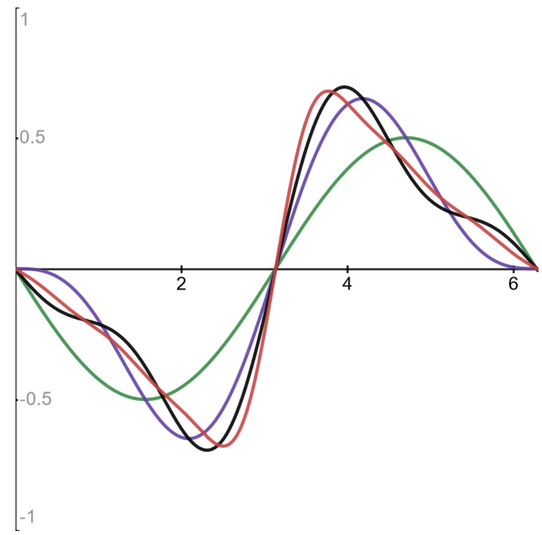
$b = -1/2, k = 1$



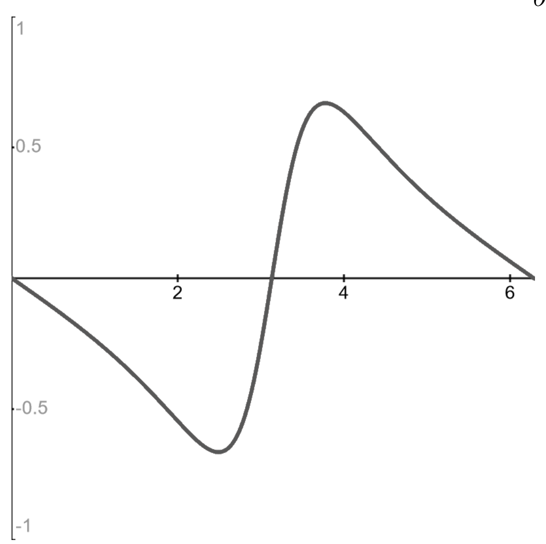
$b = -1/2, k = 1, 2$



$b = -1/2, k = 1, 2, 3$



$b = -1/2, k = 1, \ldots, 6$



$b = -1/2, f(t) = \frac{b \sin t}{1 - 2b \sin t + b^2}$

11.2.2 Again, Abel's Criterion

Proposition 157
Let $\sum_{k \in \mathbb{Z}} c_k e^{ikt}$ be such that $c_k \geq 0$ and $c_k \searrow 0$ both as $k \to \infty$ and as $k \to -\infty$. Then $\sum_{k \in \mathbb{Z}} c_k e^{ikt}$ converges uniformly on $[\delta, 2\pi - \delta]$ for any $\delta \in (0, \pi)$. Consequently, the sum $f(t) = \sum_{k \in \mathbb{Z}} c_k e^{ikt}$ is continuous on $(0, 2\pi)$.

Proof: it suffices to show that

$$\sum_{k\geq 0} c_k e^{ikt} \quad \text{and} \sum_{k\leq -1} c_k e^{ikt}$$

both converge uniformly on $[\delta, 2\pi-\delta]$ for all $0<\delta<\pi$, and to apply Abel's criterion for each of the series.

Let $\delta\in(0,\pi)$. Since

$$\left|\sum_{k=0}^{n} e^{ikt}\right| = \left|1+\cdots+e^{int}\right| = \left|\frac{1-e^{i(n+1)t}}{1-e^{it}}\right| \leq \frac{2}{|1-e^{it}|} \leq \frac{2}{\sin\delta}$$

$$\left|\sum_{k=-n}^{-1} e^{ikt}\right| = \left|e^{-int}+\cdots+e^{-it}\right| = \left|e^{-int}\right|\left|1+\cdots+e^{i(n-1)t}\right|$$

$$= \left|1+\cdots+e^{i(n-1)t}\right| = \left|\frac{1-e^{int}}{1-e^{it}}\right| \leq \frac{2}{|1-e^{it}|} \leq \frac{2}{\sin\delta}$$

for all $t\in[\delta,2\pi-\delta]$, the series converge uniformly on $[\delta,2\pi-\delta]$. ■

Abel's criterion could also be used even in circumstances where c_k is not always positive. For instance, let $\sum_{k\in\mathbb{Z}}(-1)^k c_k e^{ikt}$ where the coefficient c_k are as in the statement of Proposition 157. What does the fact that

$$\left|\sum_{k\in\mathbb{Z}}(-1)^k e^{ikt}\right| = \left|\frac{1+(-1)^{n+1}e^{i(n+1)t}}{1-e^{it}}\right| \leq \frac{2}{|1+e^{it}|}$$

tell you? These results also apply to the real part and the imaginary part of $\sum_{k\in\mathbb{Z}} c_k e^{ikt}$, i.e. to the series

$$a_0+\sum_{k\geq 1} a_k \cos(kt) \quad \text{and} \quad \sum_{k\geq 1} b_k \sin(kt).$$

For instance, $\sum_{k\geq 1}\frac{\sin(kt)}{k}$ converges uniformly on $[\delta,2\pi-\delta]$ for any $\delta>0$. As a result, the sum is continuous on $(0,2\pi)$. However, even though $\sum_{k\geq 1}\frac{\sin(kt)}{k}$ converges for $t=0$ and $t=2\pi$, the function is not continuous on $[0,2\pi]$ (see Exercise 9).

Let $T>0$. A function $f:\mathbb{R}\to\mathbb{C}$ is $T-$**periodic** if $f(t+T)=f(t)$ for all $t\in\mathbb{R}$. The smallest positive T for which this holds is the period of the function. Periodic functions play an important role in Fourier analysis.

Examples

1. The functions cos and sin are $2\pi-$periodic. □
2. The function tan is $\pi-$periodic. □

3. The function defined by e^{ikt} is $\frac{2\pi}{k}-$periodic for any $k \in \mathbb{Z}$. □

4. The function defined by e^{ikwt}, where $w = \frac{2\pi}{T}$ and $k \in \mathbb{Z}$, is $T-$periodic. □

5. Let $f \in \mathcal{C}_c(\mathbb{R}, \mathbb{C})$, with **compact support** K (that is, $f(t) = 0$ when $t \notin K$). Show that $\varphi_f : t \mapsto \sum_{k\in\mathbb{Z}} f(t-k)$ is $1-$periodic.

   Proof: this series converges for all t since there is only a finite set of integers k for which $t - k \in K$ (because K is compact). Then

   $$\varphi(t+1) = \sum_{k\in\mathbb{Z}} f(t+1-k) = \sum_{k\in\mathbb{Z}} f(t-k) = \varphi_f(t),$$

   so φ_f is $1-$periodic. ■

If $f \in \mathcal{C}(\mathbb{R}, \mathbb{C})$ is a $T-$periodic function, then f is bounded on the interval $[0, T]$, with

$$c_0(f) = \frac{1}{T}\int_0^T f(t)\,dt < \infty.$$

The complex number c_0 is the **mean value of** f, also given by

$$c_0(f) = \frac{1}{T}\int_0^T f(t)\,dt.$$

If $w = \frac{2\pi}{T}$ and $k \neq 0$, the function $g : t \mapsto e^{ikwt}$ is $T-$periodic. Then

$$c_0(g) = \frac{1}{T}\int_0^T e^{ikwt}\,dt = \frac{1}{T}\left[\frac{e^{ikwt}}{ikw}\right]_0^T = 0.$$

Hence, if $f(t) = \sum_{k\in\mathbb{Z}} c_k e^{ikwt}$ is uniformly convergent on $[0, T]$ and $T-$periodic, then

$$c_0(f) = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{T}\int_0^T \left(\sum_{k\in\mathbb{Z}} c_k e^{ikwt}\right) dt = \sum_{k\in\mathbb{Z}} \frac{c_k}{T}\int_0^T e^{ikwt}\,dt = c_0$$

The sum and the integral can be interchanged because the series converges uniformly on $[0, T]$. If $f \in \mathcal{C}(\mathbb{R}, \mathbb{C})$ is $T-$periodic, the sequence $(c_k(f))$, where

$$c_k(f) = c_0\left(fe^{-ikwt}\right) = \frac{1}{T}\int_0^T f(t)e^{-ikwt}\,dt, \quad k \in \mathbb{Z},$$

is the sequence of **Fourier coefficients of** f. Clearly, if $w = \frac{2\pi}{T}$ and $f(t) = \sum_{k\in\mathbb{Z}} c_k e^{ikwt}$ is uniformly convergent on $[0, T]$, then $c_k(f) = c_k$.

Proposition 158
The mapping $f \mapsto (c_k(f))_{k\in\mathbb{Z}}$ is a linear map from the vector space of continuous $T-$periodic functions to the space of bounded sequences indexed by $\mathbb{Z}$. More precisely,

$$\sup_{k\in\mathbb{Z}}\{|c_k(f)|\} \leq \|f\|_1 \leq \|f\|_\infty < \infty,$$

where $\|f\|_1 = \frac{1}{T}\int_0^T |f(t)|\, dt$.

Proof: left as an exercise. ■

We can improve on Proposition 158 once we show that

$$\|f\|_2 = \left(\sum_{k\in\mathbb{Z}} |c_k(f)|^2\right)^{1/2}.$$

Proposition 159 Let f be a $2\pi-$periodic function such that $f \in C^n$, $n > 0$. Then

$$c_k(f) = \frac{1}{(ik)^n} c_k\left(f^{(n)}\right), \quad k \neq 0.$$

In particular,

$$|c_k(f)| \leq \frac{\|f^{(n)}\|_\infty}{|k|^n}$$

and so $|c_k(f)| \to 0$ as $|k| \to \infty$.

Proof: this is easily shown by induction on n. If $n = 1$, we have

$$\begin{aligned} c_k(f) &= \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-ikt}\, dt = \frac{1}{2\pi}\left[\left.\frac{f(t)e^{-ikt}}{-ik}\right|_0^{2\pi} + \frac{1}{ik}\int_0^{2\pi} f'(t)e^{-ikt}\, dt\right] \\ &= \frac{1}{ik} c_k(f'). \end{aligned}$$

A sequence of integrations by parts yields the result for general n. ■

As a corollary, if $f \in C^2$ is $2\pi-$periodic, then $\sum_{k\in\mathbb{Z}} c_k(f)e^{ikt}$ converges absolutely (and so uniformly) on $\mathbb{R}$.

All that precedes leads us to the crucial definition: the **Fourier series** of a $2\pi-$periodic function f is the series $\sum_{k\in\mathbb{Z}} c_k(f)e^{ikt}$; in that case, we write $f(t) \sim \sum_{k\in\mathbb{Z}} c_k(f)e^{ikt}$. Note that it is possible for f **not to equal** its Fourier series.

11.2.3 Convergence of Fourier Series

The next results discuss the **convergence** of Fourier series.

Theorem 160
Let f be $2\pi-$periodic. If $f \in C^2$, then the Fourier series $\sum_{k\in\mathbb{Z}} c_k(f)e^{ikt}$ converges absolutely (and so uniformly) to f on $\mathbb{R}$.

Proof: according to the corollary to Proposition 159, the Fourier series $g(t) = \sum_{k\in\mathbb{Z}} c_k(f)e^{ikt}$ converges absolutely on $\mathbb{R}$, and thus g is continuous and $2\pi-$periodic. We want to show that $g = f$.

Let $h = f - g$. Then h is continuous and $2\pi-$periodic. We also have

$$c_k(h) = c_k(f) - c_k(g) = 0,$$

so that $c_k(f) = c_k(g)$ for all $k \in \mathbb{Z}$.

It remains only to show that when h is continuous, $2\pi-$periodic, and $c_k(h) = 0$ for all $k \in \mathbb{Z}$, then $h \equiv 0$. According to a corollary of the Stone-Weierstrass theorem (see Chapter 23), $\exists (p_n)_{n\in\mathbb{N}}$ such that $p_n(t) = \sum_{k\in\mathbb{Z}} a_k(n)e^{ikt}$ and $p_n \rightrightarrows \overline{h}$. Note that for a fixed k, we must have $a_k(n) \to 0$ when $n \to \infty$.

Then

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} |h(t)|^2\,dt &= \frac{1}{2\pi}\int_0^{2\pi} h(t)\overline{h(t)}\,dt \overset{\text{thm 150}}{=} \lim_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi} h(t)p_n(t)\,dt \\ &\overset{\text{thm 152}}{=} \sum_{k\in\mathbb{Z}}\left(\lim_{n\to\infty} a_k(n)\frac{1}{2\pi}\int_0^{2\pi} h(t)e^{ikt}\,dt\right) = \sum_{k\in\mathbb{Z}}\left(\lim_{n\to\infty} a_k(n)c_{-k}(h)\right) = 0.\end{aligned}$$

Since $|h(t)|^2$ is continuous, $|h(t)|^2 = 0$ for all $t \in [0, 2\pi]$, so that $h(t) = 0$ for all $t \in [0, 2\pi]$. ■

The next result provides a sufficient condition for a function to be equal to its Fourier series.

Theorem 161
Let f be a continuous $2\pi-$periodic function such that

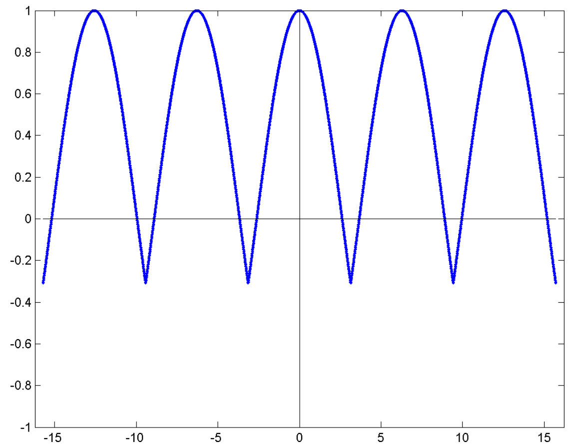
$$\sum_{k\in\mathbb{Z}} |c_k(f)| = M < \infty.$$

Then the Fourier series of f converges absolutely to f on $\mathbb{R}$ and is equal to f on $\mathbb{R}$.

Proof: left as an exercise. ■

Let us take a look at an example.

Example: fix $a \in \mathbb{R}$ and let $f_a(t) = \cos(at)$, $|t| \leq \pi$. Extend f_a to $\mathbb{R}$ by periodicity. What is the Fourier series of f_a? Is it equal to f_a on $\mathbb{R}$? **Solution:** if $a \notin \mathbb{Z}$, f_a is not differentiable (see example below).



If $a \in \mathbb{Z}$ then $\cos(at)$ is already a trigonometric polynomial so the Fourier series of f_a is simply $\cos(at)$. So assume that $a \notin \mathbb{Z}$.

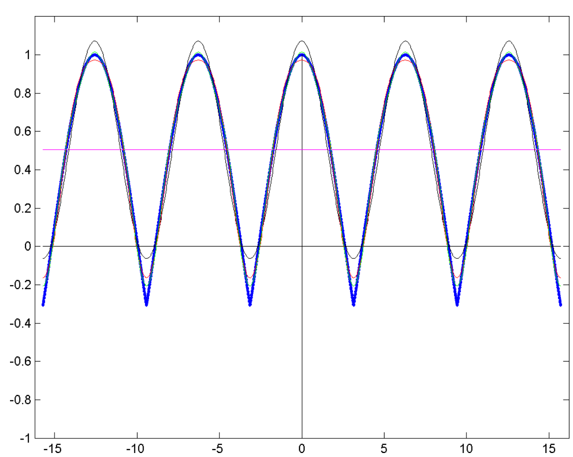
Let $k \in \mathbb{Z}$. Then

$$c_k(f_a) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(at)e^{-ikt}\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{iat} - e^{-iat}}{2}e^{-ikt}\,dt = \frac{a(-1)^k \sin(\pi a)}{\pi(a^2 - k^2)}$$

Using the comparison test with $|c_k(f)| \sim \frac{1}{k^2}$, we see that $\sum_{k\in\mathbb{Z}} |c_k(f)| < \infty$. According to Theorem 161,

$$f_a(t) = \sum_{k\in\mathbb{Z}} \frac{a(-1)^k \sin(\pi a)}{\pi(a^2 - k^2)} e^{ikt}$$

converges absolutely on $\mathbb{R}$. ■



11.2.4 Dirichlet's Convergence Theorem

Let $f:\mathbb{R}\to\mathbb{C}$ be a $2\pi-$periodic Riemann-integrable function. For $k\in\mathbb{Z}$, set

$$e_k(t)=e^{ikt}=\left(e^{it}\right)^k=(e_1(t))^k.$$

Let $N\in\mathbb{N}$. Define

$$S_N(f)(t):=\sum_{k=-N}^{N}c_k(f)e_k(t);$$

$S_N(f)$ is the **partial sum of degree** N for the Fourier series of f.[3] We can write these partial sums as **convolutions**: indeed, we have

$$\begin{aligned}
S_N(f)(t):=\sum_{k=-N}^{N}c_k(f)e_k(t)&=\sum_{k=-N}^{N}e_k(t)\int f(y)e_k(-y)\,dy\\
&=\int f(y)\left\{\sum_{k=-N}^{N}e_k(t)e_k(-y)\right\}dy\\
&=\int f(y)\left\{\sum_{k=-N}^{N}e_k(t-y)\right\}dy\\
&=\int f(y)K_N(t-y)\,dy:=(\hat{D}_N*f)(t),
\end{aligned}$$

where the **Dirichlet kernel** of order N is, formally,

$$\begin{aligned}
K_N(t)=\sum_{k=-N}^{N}e_k(t)=\sum_{k=-N}^{N}e^{ikt}&=\frac{e^{-iNt}-e^{i(N+1)t}}{1-e^{it}}\\
=\frac{1}{e^{iNt}}\left(\frac{1-e^{i(2N+1)t}}{1-e^{it}}\right)&=\frac{\sin((N+1/2)t)}{\sin(t/2)},\quad\text{when } t\notin 2\pi\mathbb{Z}.
\end{aligned}$$

Proposition 162
The Dirichlet kernel is even, $2\pi-$periodic, $c_0(K_N)=1$, $\int_0^\pi K_N(t)\,dt=\pi$, and

$$K_N(0)=\lim_{t\to}K_N(t)=2N+1.$$

Proof: left as an exercise. ∎

The next result is substantially more difficult to prove.

[3] In what follows, we will write $\int:=\frac{1}{2\pi}\int_0^{2\pi}=\frac{1}{2\pi}\int_a^{a+2\pi}$ for any $a\in\mathbb{R}$.

Lemma 163 (Riemann-Lebesgue Lemma)
Let $f : [a,b] \to \mathbb{C}$ *be integrable over* $[a,b]$*. Then* $\lim_{n\to\infty} \int_a^b f(t)e^{int}\,dt = 0$.

Proof: left as a (difficult) exercise. ■

We can now state and prove this section's main result.

Theorem 164 (Dirichlet's Convergence Theorem)
Let $f : \mathbb{R} \to \mathbb{C}$ *be piecewise (with a finite number of discontinuities) and* $2\pi-$*periodic. If the following one-sided limits exist* $\forall x \in \mathbb{R}$*:*

$$f(x^{\pm}) = \lim_{h\searrow 0} f(x \pm h), \quad f'(x^{\pm}) = \lim_{h\searrow 0} \frac{f(x\pm h) - f(x)}{h},$$

then

$$S_N(f)(x) = \sum_{k=-N}^{N} c_k(f)e_k(x) \to \frac{f(x^+) + f(x^-)}{2}, \quad \textit{as } N \to \infty.$$

Proof: without loss of generality, we can assume that $x = 0$ by translating the variable x to the origin as needed. Consider the sequence of partial sums

$$s_N := S_N(f)(0) = \sum_{k=-N}^{N} c_k(f)e_k(0) = \sum_{k=-N}^{N} c_k(f).$$

For $N \in \mathbb{N}$, we have

$$s_N = \sum_{|k|\leq N} \int f(t)e^{-ikt}\,dt = \int f(t)K_N(t)\,dt.$$

Since $K_N(t)$ is even, then

$$\int_{-\pi}^{0} f(t)K_N(t)\,dt = \int_0^{\pi} f(-t)K_N(t)\,dt,$$

whence (remember the notation convention for integrals)

$$s_N = \frac{1}{2\pi}\int_0^{\pi} \{f(t) + f(-t)\}K_N(t)\,dt.$$

Write

$$u_N = s_N - \tfrac{f(0^+)+f(0^-)}{2}.$$

Then

$$\begin{aligned} u_N &= \frac{1}{2\pi}\int_0^\pi \{f(t)+f(-t)\}K_N(t)\,dt - \tfrac{f(0^+)+f(0^-)}{2}\cdot\frac{1}{\pi}\int_0^\pi K_N(t)\,dt \\ &= \frac{1}{2\pi}\int_0^\pi \{f(t)+f(-t)-f(0^+)-f(0^-)\}K_N(t)\,dt \\ &= \frac{1}{2\pi}\int_0^\pi g(t)\sin((N+1/2)t)\,dt, \end{aligned}$$

where

$$g(t)=\begin{cases}\dfrac{f(t)-f(0^+)+f(-t)-f(0^-)}{\sin(t/2)}, & \text{if } t\in(0,\pi] \\ 0, & \text{otherwise}\end{cases}$$

By construction, g is clearly piecewise continuous on $(0,\pi]$. It is necessarily bounded in a neighbourhood of $t=0$ according to de l'Hôpital's Rule:

$$\lim_{t\searrow 0} g(t)=\lim_{t\searrow 0}\frac{2(f'(t)-f'(-t))}{\cos(t/2)}=2(f'(0^+)+f'(0^-))<\infty.$$

The function g is thus nicely-behaved: it is bounded and piecewise continuous (with at most a finite number of discontinuities) over $[0,\pi]$ and so is integrable on every continuous piece of $[0,\pi]$, using an easy generalization of Theorem 54 (see Chapter 4).

According to the Riemann-Lebesgue lemma 155,

$$\lim_{n\to\infty}\int_0^\pi g(t)e^{int}\,dt=0.$$

The relation still holds with the change of variable $n=N+1/2$.

Since $2\pi u_N$ is the imaginary part of $\int_0^\pi g(t)e^{i(N+1/2)t}\,dt$, then $2\pi u_N \to 0$ and $s_N\to\frac{f(0^+)+f(0^-)}{2}$ when $N\to\infty$. ■

In other words, if a periodic function f is "**nice enough**" (piecewise C^1), then it is **equal to its Fourier series wherever f is continuous**. At discontinuities of f, the Fourier series converges to the **mean of the one-sided limits**.[4]

Example: let $f:[0,2\pi]\to\mathbb{R}$ be defined by $f(t)=t^2$. Extend f to $\mathbb{R}$ by periodicity. What is the Fourier series of f. Is it equal to f on $\mathbb{R}$?

[4]Be careful: some piecewise C^0 periodic functions have **divergent** Fourier series.

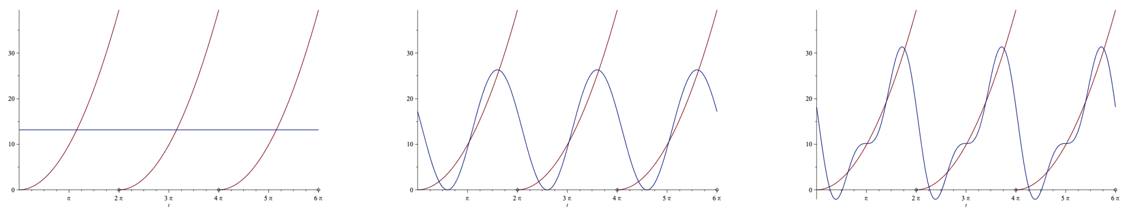
Solution: the Fourier coefficients of f are

$$c_k(f_a) = \frac{1}{2\pi}\int_0^{2\pi} t^2 e^{-ikt}\,dt = \begin{cases} [c]\frac{2}{n^2}(i\pi k+1), & k \neq 0 \\ \frac{4\pi^2}{3}, & k = 0 \end{cases}$$

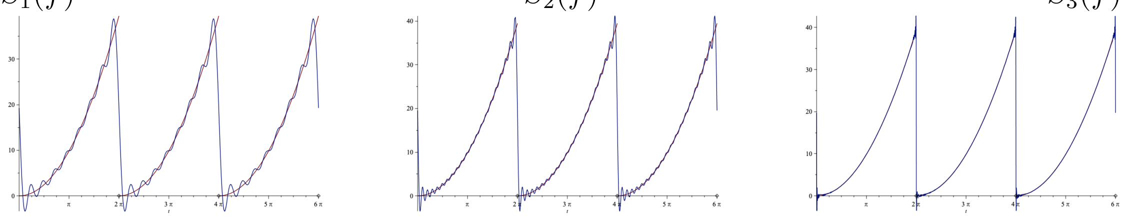
According to Dirichlet's convergence theorem,

$$\sum_{k\in\mathbb{Z}} c_k(f)e^{ikt} = \frac{4\pi^2}{3} + \sum_{k\in\mathbb{Z}^\times} \frac{2}{k^2}(i\pi k+1)e^{ikt}$$

converges (at least pointwise) to t^2 for $t \notin 2\pi\mathbb{Z}$, and to $\frac{f(2\pi)+f(0)}{2} = 2\pi^2$ for $t \in 2\pi\mathbb{Z}$, since f is piecewise C^1.



$S_1(f)$ $S_2(f)$ $S_3(f)$



$S_8(f)$ $S_{20}(f)$ $S_{200}(f)$

The convergence turns out to be uniform on $[2\pi\ell+\delta, 2\pi(\ell+1)-\delta]$, for all $\delta \in (0,\pi)$, $\ell \in \mathbb{Z}$ (more on this in the next section), but only pointwise over $\mathbb{R}$ as a whole, in keeping with Theorem 164. ■

Notice the overshooting of the partial sums as $t \to 2\pi\ell$, $\ell \in \mathbb{Z}$, which does not seem to dampen when $N \to \infty$. This "universal" behaviour at discontinuities is termed **Gibbs' Phenomenon** (contrast the behaviour of the Fourier series of t^2 with that of $\cos(at)$ discussed earlier).

The explanation of the problem is linked with the lim sup and lim inf of the partial sums $S_n(f)(x_N)$ at points x_N that approach a discontinuity at x_0, but we will not discuss this any further.

11.2.5 Quadratic Mean Convergence

The set of $2\pi-$periodic piecewise continuous functions from $\mathbb{R}$ to $\mathbb{C}$ is an **inner product space** together with

$$(f \mid g) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g(t)}\,dt,$$

with **associated norm** $\|f\|_2 = \sqrt{(f \mid f)}$.

Note that for $\mu, \nu \in \mathbb{Z}$, we have

$$(e_\mu \mid e_\nu) = \frac{1}{2\pi}\int_0^{2\pi} e^{i\mu t}e^{-i\nu t}\,dt = \frac{1}{2\pi}\int_0^{2\pi} e^{i(\mu-\nu)t}\,dt = \delta_{\mu,\nu} = \begin{cases} 0, & \mu \neq \nu \\ 1, & \mu = \nu \end{cases}$$

For a given $N \in \mathbb{N}$ and a function f in the inner product space of the previous page, consider the partial sum

$$S_N(f) = \sum_{|k|\leq N} c_k(f)e_k(t).$$

For any $|k| \leq N$, we must have

$$c_k(f) = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-ikt}\,dt = (f \mid e_k).$$

But

$$(S_N(f) \mid e_k) = \sum_{|\ell|\leq N} c_\ell(f)(e_\ell \mid e_k) = \sum_{|\ell|\leq N} c_\ell(f)\delta_{\ell,k} = c_k(f).$$

Thus, $(f - S_N(f) \mid e_k) = 0$ for all $|k| \leq N$ and we can write

$$f = S_N(f) + (f - S_N(f)),$$

with $S_N(f) \in \mathcal{P}_N = \text{Span}\{e_k \mid -N \leq k \leq N\}$ and $f - S_N(f) \in \mathcal{P}_N^\perp$.

Note furthermore that since $(S_N \mid f - S_N(f)) = 0$, then

$$\begin{aligned} \|f\|_2^2 &= (f \mid f) = (S_N(f) + (f - S_N(f)) \mid S_N(f) + (f - S_N(f))) \\ &= (S_N(f) \mid S_N(f)) + 2\text{Re}\underbrace{(S_N(f) \mid f - S_N(f))}_{=0} + (f - S_N(f) \mid f - S_N(f)) \\ &= \|S_N(f)\|_2^2 + \|f - S_N(f)\|_2^2. \end{aligned}$$

For any other function $g \in \mathcal{P}_N$, we see that

$$\begin{aligned} \|f - g\|^2 &= \|\underbrace{f - S_N(f)}_{\in \mathcal{P}_N^\perp} + \underbrace{S_N(f) - g}_{\in \mathcal{P}_N}\|_2^2 \\ &= \|f - S_N(f)\|_2^2 + \|S_N(f) - g\|_2^2 \geq \|f - S_N(f)\|_2^2. \end{aligned}$$

Since g was arbitrary,

$$\inf_{g\in\mathcal{P}_N} \|f-g\|_2^2 = \|f-S_N(f)\|_2^2 = \|f\|_2^2 - \|S_N(f)\|_2^2. \tag{11.1}$$

The partial sum $S_N(f)$ is thus the nearest trigonometric polynomial to f in $\mathcal{P}_N$, in the sense of the **quadratic mean**.

Theorem 165 (Parseval's Identity)
Let f be a $2\pi-$periodic piecewise continuous function from $\mathbb{R}$ to $\mathbb{C}$. Then

$$\frac{1}{2\pi}\int_0^{2\pi} |f(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |c_k(f)|^2.$$

Proof: as $|f|^2$ is Riemann-integrable on $[0,2\pi]$, the convergence of the series will be assured once the equality is established. By construction,

$$\begin{aligned}\|S_N(f)\|_2^2 &= \left(\sum_{|k|\leq N} c_k(f)e^{ikt} \,\middle|\, \sum_{|\ell|\leq N} c_\ell(f)e^{i\ell t}\right) = \sum_{k,\ell=-N}^{N} c_k(f)\overline{c_\ell(f)}(e_k \mid e_\ell) \\ &= \sum_{k,\ell=-N}^{N} c_k(f)\overline{c_\ell(f)}\delta_{k,\ell} = \sum_{k=-N}^{N} |c_k(f)|^2.\end{aligned}$$

The sequence of infimums given in (11.1) by

$$(x_N) = \left(\inf_{g\in\mathcal{P}_N} \{\|f-g\|_2^2\}\right)$$

is bounded below by 0.

Let $N\in\mathbb{N}$. Clearly, $\|S_N(f)\|_2^2 \leq \|S_{N+1}(f)\|_2^2$, and so

$$x_N = \|f-S_N(f)\|_2^2 = \|f\|_2^2 - \|S_N(f)\|_2^2 \geq \|f\|_2^2 - \|S_{N+1}(f)\|_2^2 = x_{N+1}.$$

Thus (x_N) is a decreasing and bounded sequence; as such, it converges to $0 \leq x_* = \inf\{x_N \mid N\in\mathbb{N}\}$ by the bounded monotone convergence theorem.

In particular, this means that

$$x_* = \lim_{N\to\infty} x_N = \|f\|_2^2 - \lim_{N\to\infty} \|S_N(f)\|_2^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(t)|^2\,dt \; - \sum_{k=-\infty}^{\infty} |c_k(f)|^2,$$

which guarantees the convergence of the series, as $|f|^2$ is Riemann-integrable over $[0,2\pi]$ (being continuous).

Write $\mathcal{P} = \bigcup_{N\in\mathbb{N}} \mathcal{P}_N$. Since $\mathcal{P}_N \subseteq \mathcal{P}$ for all $N \in \mathbb{N}$, we have

$$\inf_{g\in\mathcal{P}} \|f-g\|_2^2 \leq \inf_{g\in\mathcal{P}_N} \|f-g\|_2^2 = x_N, \quad \text{for all } N \in \mathbb{N},$$

which implies that

$$0 \leq \inf_{g\in\mathcal{P}} \|f-g\|_2^2 \leq x_*.$$

Conversely, $x_* \leq \|f-g\|_2^2$ for all $g \in \mathcal{P}_N$, $N \in \mathbb{N}$. Thus $x_* \leq \|f-g\|_2^2$ for all $g \in \mathcal{P}$, so that

$$x_* \leq \inf_{g\in\mathcal{P}} \|f-g\|_2^2.$$

Combining these, we obtain

$$\inf_{g\in\mathcal{P}} \|f-g\|_2^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(t)|^2\,dt - \sum_{k=-\infty}^{\infty} |c_k(f)|^2.$$

Let $\varepsilon > 0$. As f is a $2\pi-$periodic piecewise continuous function, we can find a $2\pi-$periodic continuous function f_c such that

$$\|f-f_c\|_2 < K\varepsilon, \quad \text{for some } K > 0.$$

If f is constant, simply set $f_c = f$; we do the same if f is continuous.

Otherwise, assume that f admits m discontinuities at

$$x_1 < \ldots < x_m \in (\delta, 2\pi+\delta), \quad \text{for some } \delta > 0,$$

and denote the closed ε^2-neighbourhood around x_α by

$$B_{\alpha,\varepsilon^2} = [y_{\alpha,\varepsilon^2}, y_{\alpha,\varepsilon^2} + 2\varepsilon^2],$$

for $\alpha = 1,\ldots,m$, and their union by B_{ε^2} (restrict ε as needed to ensure that the $B_{\alpha,\varepsilon^2} = [y_{\alpha,\varepsilon^2}, y_{\alpha,\varepsilon^2} + 2\varepsilon^2]$ do not overlap).

Outside of B_{ε^2} but in $[\delta, 2\pi+\delta]$, define $f_c \equiv f$. In each of the $B_{\alpha,\varepsilon^2} \cap [\delta, 2\pi+\delta]$, let f_c be the linear function joining the points

$$(y_{\alpha,\varepsilon^2}, f(y_{\alpha,\varepsilon^2})) \quad \text{and} \quad (y_{\alpha,\varepsilon^2} + 2\varepsilon^2, f(y_{\alpha,\varepsilon^2} + 2\varepsilon^2)).$$

The function $f_c : [\delta, 2\pi+\delta] \to \mathbb{C}$ is "clearly" continuous, and can be extended to a $2\pi-$periodic continuous function over $\mathbb{R}$.

In particular, $|f-f_c|^2$ is real-valued and continuous over $[\delta, 2\pi+\delta]$. Consequently, the latter reaches its maximum $M > 0$ somewhere on $[\delta, 2\pi+\delta]$, by the max/min theorem.

Thus, for any $\delta > 0$,

$$\|f - f_c\|_2^2 = \frac{1}{2\pi}\int_{\delta}^{2\pi+\delta} |f(t) - f_c(t)|^2\,dt = \frac{1}{2\pi}\sum_{\alpha=1}^{m}\int_{B_{\alpha,\varepsilon^2}} |f(t) - f_c(t)|^2\,dt$$
$$\leq \frac{1}{2\pi}\sum_{\alpha=1}^{m}\int_{B_{\alpha,\varepsilon^2}} M\,dt = \frac{1}{2\pi}\sum_{\alpha=1}^{m} 2\varepsilon^2 \cdot M = \underbrace{\frac{mM}{\pi}}_{>0}\varepsilon^2 := K^2\varepsilon^2$$

According to the Stone-Weierstrass theorem (see Chapter 23), the set of $2\pi-$periodic trigonometric polynomials $\mathcal{P}$ is dense in the set of $2\pi-$periodic continuous functions w.r.t. to $\|\cdot\|_2$, and so $\exists g \in \mathcal{P}$ with $\|f_c - g\|_2 < \varepsilon$.

Putting this together, we see that

$$\|f - g\|_2 \leq \|f - f_c\|_2 + \|f_c - g\|_2 < K\varepsilon + \varepsilon = (K+1)\varepsilon.$$

Thus

$$\inf_{g\in\mathcal{P}} \|f - g\|_2 < (K+1)\varepsilon \quad \text{for all } \varepsilon \implies \inf_{g\in\mathcal{P}} \|f - g\|_2 = 0. \qquad \blacksquare$$

Parseval's identity remains valid for locally Riemann-integrable functions ($\int_K |f|\,dt < \infty$ for all $K \subseteq_K [0, 2\pi]$), instead of piecewise continuous, with multiple consequences: the series

$$\sum_{k\in\mathbb{Z}} |c_k(f)|^2$$

converges, which shows that $|c_k(f)|^2 \to 0$, and thus $c_k(f) \to 0$ as $k \to \pm\infty$ (by the Riemann-Lebesgue lemma). It can also be used to show that any $2\pi-$periodic continuous function $f : \mathbb{R} \to \mathbb{C}$ whose Fourier series converges uniformly on $\mathbb{R}$ must be equal to said series (compare with Dirichlet's convergence theorem).

11.3 Exercises

1. Let (g_n) be a sequence of functions. Show that $\sum g_n$ converges absolutely if and only if $\exists(a_n) \subseteq \mathbb{R}^+$ such that $\sum a_n$ converges and $\|g_n\|_\infty \leq a_n$ for all n. Use that result to show that the series of functions $\sum g_n$, where $g_n : [0,1] \to \mathbb{R}$ is defined by $g_n(x) = \frac{x^n}{n^2}$, is absolutely convergent on $[0,1]$.
2. For each of the theorems of Section 11.1.1 (save for Theorem 152), find an example showing that the result does not hold if uniform convergence is replaced by pointwise convergence.
3. Prove Theorems 152, 153, and 161, as well as Propositions 156 and 158.

4. Find some examples showing that the result of Theorem 152 does not hold in general if absolute convergence is replaced by a weaker type of convergence.

5. Let $g_n : \mathbb{R} \to \mathbb{R}$ be defined by $g_n(x) = \frac{x^n}{n!}$ for each $n \in \mathbb{N}$. Show that each of the following series of functions converges absolutely on $\mathbb{R}$.

   a) $S = \sum (-1)^{n+1} g_{2n+1}$
   b) $C = \sum (-1)^n g_{2n}$
   c) $E = \sum g_n$

6. Let S, C, E be as in the previous question. Using the appropriate theorems, show that for any $x \in \mathbb{R}$ show that $S'(x) = C(x)$, $C'(x) = -S(x)$, and $E'(x) = E(x)$.

7. Find examples showing that the three conditions in the statement of Proposition 155 are independent from one another.

8. Improve the bound in Proposition 158 by showing that

$$\|f\|_2 = \left(\sum_{k\in\mathbb{Z}} |c_k(f)|^2 \right)^{1/2}.$$

9. Show that the function $f : [0, 2\pi] \to \mathbb{R}$ defined by $f(t) = \sum_{k\geq 1} \frac{\sin(kt)}{k}$ is not continuous on $[0, 2\pi]$.

10. Using the Fourier series of the cosine, show that $\pi \cot(a\pi) = \sum_{k\in\mathbb{Z}} \frac{a}{a^2-k^2}$ for all $a \notin \mathbb{Z}$ (also known as **Euler's Formula**).

11. Prove the properties of the Dirichlet kernel (Proposition 11.2.4).

12. Show that $(f \mid g)$ (see page 289) defines an inner product on the set of $2\pi-$periodic piecewise continuous functions from $\mathbb{R}$ to $\mathbb{C}$.

13. Prove the Riemann-Lebesgue lemma without using Parseval's identity.

14. Show that any $2\pi-$periodic continuous function $f : \mathbb{R} \to \mathbb{C}$ whose Fourier coefficients are all 0 must be the zero function.

15. Let $(a_n) \subseteq \mathbb{C}$ be such that $a_n \to \ell$ and let $(\varepsilon_n) \subseteq \mathbb{R}^+$ be a divergent sequence. Define a sequence $(b_n) \subseteq \mathbb{C}$ by

$$b_n = \frac{\sum_{i=1}^n a_i \varepsilon_i}{\sum_{i=1}^n \varepsilon_i}.$$

Show that $b_n \to \ell$.

16. a) Let (f_n) be the sequence of functions defined by

$$f_n : \mathbb{R}_0^+ \to \mathbb{R}, \quad f_n(x) = \begin{cases} \left(1 - \frac{x}{n}\right)^n & x \in [0, n] \\ 0 & x > n \end{cases}$$

Show that $f_n \rightrightarrows f$ on $\mathbb{R}_0^+$, where $f : \mathbb{R}_0^+ \to \mathbb{R}$ is defined by $f(x) = e^{-x}$.

b) Let $U \subseteq_K \mathbb{C}$ and let (f_n) be the sequence of functions defined by

$$f : \mathbb{C} \to \mathbb{C}, \quad f(z) = \left(1 + \frac{z}{n}\right)^n.$$

Show that $f_n \rightrightarrows f$ on K, where $f : \mathbb{C} \to \mathbb{C}$ is defined by $f(z) = e^z$.

17. For any $n \in \mathbb{N}^\times$, let $u_n : \mathbb{R}_0^+ \to \mathbb{R}$ be defined by $u(x) = \frac{x}{n^2+x^2}$.

   a) Show that $\sum u_n \to f$ for some $f \in \mathcal{C}(\mathbb{R}_0^+, \mathbb{R})$, but that $\sum u_n \not\rightrightarrows f$ on $\mathbb{R}_0^+$.
   b) Show that $\sum (-1)^n u_n \rightrightarrows g$ on $\mathbb{R}_0^+$ for some $g \in \mathcal{C}(\mathbb{R}_0^+, \mathbb{R})$, but that $\sum (-1)^n u_n$ is not absolutely convergent on $\mathbb{R}_0^+$.

18. What can you say about a function $f : \mathbb{R} \to \mathbb{R}$ which is the uniform limit of a sequence of polynomials (P_n)?

19. Consider the sequence of functions $(f_n) \subseteq \mathcal{C}([0, \pi/2], \mathbb{R})$ defined by $f_n(x) = \cos^n x \sin x$ for all $n \in \mathbb{N}$.

   a) Let $\mathcal{O} : [0, \pi/2] \to \mathbb{R}$ be the zero function. Show that $f_n \rightrightarrows \mathcal{O}$ on $[0, \pi/2]$.
   b) Consider the sequence of functions (g_n) defined by $g_n = (n+1)f_n$. Let $\delta > 0$. Show that $g_n \rightrightarrows \mathcal{O}$ on $[\delta, \pi/2]$ but that

$$\int_0^{\pi/2} g_n(t)\, dt \not\to 0.$$

20. Theses results are due to Dini.

   a) Let $(f_n) \in \mathcal{C}([a, b], \mathbb{R})$ be an increasing sequence of functions (i.e. for all $x \in [a, b]$ and for all $n \in \mathbb{N}$, we have $f_n(x) \leq f_{n+1}(x)$). If $f_n \to f$ on $[a, b]$ where $f \in \mathcal{C}([a, b], \mathbb{R})$, show that $f_n \rightrightarrows f$ on $[a, b]$.
   b) Let $(f_n) \in \mathcal{C}([a, b], \mathbb{R})$ be a sequence of increasing functions (i.e. for all $x \geq y \in [a, b]$ and for all $n \in \mathbb{N}$, we have $f_n(x) \geq f_n(y)$). If $f_n \to f$ on $[a, b]$ where $f \in \mathcal{C}([a, b], \mathbb{R})$, show that $f_n \rightrightarrows f$ on $[a, b]$.

21. Determine whether $\sum \mathbf{x}_n$ converges in $(\mathbb{R}^2, \|\cdot\|_2)$, where

$$\mathbf{x}_n = \left(\frac{(\sin n)^n}{n^2}, \frac{1}{n^2}\right).$$

   If so, does $\sum \mathbf{x}_n$ converge absolutely?

22. Compute the values of the following convergent series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}, \quad \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}, \quad \sum_{n=1}^{\infty} \frac{1}{n^4},$$

   using the $2\pi-$periodic function defined by $f(x) = 1 - x^2/\pi^2$ over the interval $[-\pi, \pi]$.

23. Prepare a 2-page summary of this chapter, with important definitions and results.