Chapter 12

Alternating Multilinear Forms

In order to define the notion of **differential forms** (and to learn how to integrate them), we need concepts from linear algebra. In this chapter, E is a finite dimensional vector space over $\mathbb{R}$ (i.e., $\dim(E) = n \implies E \simeq \mathbb{R}^n$).

12.1 Linear Algebra Notions

A **(linear)** $1-$**form over** E is a linear map $f : E \to \mathbb{R}$; a **(linear)** $p-$**form over** E is a linear map $f : E^p = E \times \cdots \times E \to \mathbb{R}$ which is linear in each of its arguments.

Examples

1. The projection map $f_1 : \mathbb{R}^n \to \mathbb{R}$, defined by $f_1(\mathbf{x}) = f_1(x_1, \ldots, x_n) = x_1$ is a $1-$form over $\mathbb{R}^n$. Generally, the projection $f_i : \mathbb{R}^n \to \mathbb{R}$ defined by $f_i(\mathbf{x}) = x_i$ is a $1-$form over $\mathbb{R}^n$ for all $i = 1, \ldots, n$.

   If $\mathcal{B} = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a basis of E, then for any $\mathbf{x} \in E$ we can write

   $$\mathbf{x} = x_1\mathbf{e}_1 + \cdots + x_1\mathbf{e}_1$$

   and the projection $f_i^B : E \to \mathbb{R}$ defined by $f_i^B(\mathbf{x}) = x_i$ is a $1-$form over E. $\square$

2. The inner product $(\cdot \mid \cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ defined by

   $$(\mathbf{x} \mid \mathbf{y}) = ((x_1, \ldots, x_n) \mid (y_1, \ldots, y_n)) = \sum_{i=1}^{n} x_i y_i$$

   is a **(bilinear)** $2-$form over $\mathbb{R}^n$.

   If $(\mathbf{x} \mid \mathbf{y}) = (\mathbf{y} \mid \mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in E$, the $2-$form is **symmetric**. $\square$

3. The $2-$determinant $\det : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\det(\mathbf{x},\mathbf{y}) = \det \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = x_1y_2 - x_2y_1$$

is a bilinear form over $\mathbb{R}^2$, but it is not symmetric since $\det(\mathbf{x},\mathbf{y}) = -\det(\mathbf{y},\mathbf{x})$ for all $\mathbf{x},\mathbf{y} \in \mathbb{R}^2$. Note as well that $\det(\mathbf{x},\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbb{R}^2$. □

A $p-$form f over E is **alternating** if $f(\mathbf{x}_1,\ldots,\mathbf{x}_p) = 0$ whenever $\mathbf{x}_i = \mathbf{x}_j$ for some $i < j$.

Example: $\det : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is an alternating bilinear form. More generally,

$$\det : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n \text{ times}} \to \mathbb{R}$$

is an alternating linear $n-$form. □

Let $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be an alternating bilinear form on $\mathbb{R}^2$. If $\{\mathbf{e}_1,\mathbf{e}_2\}$ is a basis of $\mathbb{R}^2$, then f is **completely determined** by the value taken by $f(\mathbf{e}_1,\mathbf{e}_2)$. Indeed, let $\mathbf{x},\mathbf{y} \in \mathbb{R}^2$, Then

$$\begin{aligned} f(\mathbf{x},\mathbf{y}) &= f(x_1\mathbf{e}_1 + x_2\mathbf{e}_2, y_1\mathbf{e}_1 + y_2\mathbf{e}_2) = x_1 f(\mathbf{e}_1, y_1\mathbf{e}_1 + y_2\mathbf{e}_2) + x_2 f(\mathbf{e}_2, y_1\mathbf{e}_1 + y_2\mathbf{e}_2) \\ &= x_1y_1 \underbrace{f(\mathbf{e}_1,\mathbf{e}_1)}_{=0} + x_1y_2 f(\mathbf{e}_1,\mathbf{e}_2) + x_2y_1 \underbrace{f(\mathbf{e}_2,\mathbf{e}_1)}_{=-f(\mathbf{e}_1,\mathbf{e}_2)} + x_2y_2 \underbrace{f(\mathbf{e}_2,\mathbf{e}_2)}_{=0} \\ &= (x_1y_2 - x_2y_1) f(\mathbf{e}_1,\mathbf{e}_2) = \det \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} f(\mathbf{e}_1,\mathbf{e}_2). \end{aligned}$$

Let $\{\mathbf{e}_1,\ldots,\mathbf{e}_n\}$ be a basis of $E = \mathbb{R}^n$ and let $\{\mathbf{x}_1,\ldots,\mathbf{x}_n\} \subseteq E = \mathbb{R}^n$. For $1 \leq i \leq n$, Write

$$\mathbf{x}_i = \sum_{j=1}^n s_{i,j}\mathbf{e}_j.$$

If $f : E^n \to \mathbb{R}$ is an alternating (linear) $n-$form, then

$$f(\mathbf{x}_1,\ldots,\mathbf{x}_n) = \det \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix} f(\mathbf{e}_1,\ldots,\mathbf{e}_n) = \det \begin{pmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{pmatrix}^\top f(\mathbf{e}_1,\ldots,\mathbf{e}_n).$$

Let $f_1,\ldots,f_p$ be p linear $1-$forms over E.[1] Define $f : E^p \to \mathbb{R}$ by

$$f(\mathbf{x}_1,\ldots,\mathbf{x}_p) = f_1(\mathbf{x}_1)\cdots f_p(\mathbf{x}_p).$$

Then f is the **tensor product** of the f_i; it is a linear $p-$form over E, which we usually denote by $f = f_1 \otimes \cdots \otimes f_p$.

[1] We can also write this as $f_1,\ldots,f_p \in E^*$, where $E^* = \{f : E \to \mathbb{R} \mid f \text{ linear}\}$ is the **dual space** of E.

If $\mathcal{B} = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a basis of E, then for $1 \leq i \leq n$, we define the **linear functionals** $\mathbf{e}_i^* \in E^*$ by

$$\mathbf{e}_i^*(\mathbf{x}) = \mathbf{e}_i^*(x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n) = x_1\mathbf{e}_i^*(\mathbf{e}_1) + \cdots + x_n\mathbf{e}_i^*(\mathbf{e}_n) = x_i\mathbf{e}_i^*(\mathbf{e}_i) = x_i$$

for all $\mathbf{x} = (x_1, \ldots, x_n) \in E$. In that case, the set

$$\left\{\mathbf{e}_{i_1}^* \otimes \cdots \otimes \mathbf{e}_{i_p}^* \mid i_j \in \{1, \ldots, n\}\right\}$$

forms a basis of the vector space of $p-$forms over E, and $\dim(\{p - \text{forms over } E\}) = n^p$.

12.2 Anti-Symmetric Forms

In introductory linear algebra and group theory courses, we learn that if $A = (a_{i,j}) \subseteq \mathbb{M}_n(\mathbb{R})$, then we can write the determinant of A using the **Laplace expansion**:

$$\det(A) = \sum_{\sigma \in S_n} \epsilon(\sigma) a_{1,\sigma(1)} \cdots a_{n,\sigma(n)},$$

where S_n is the **permutation group** on $\{1, \ldots, n\}$ (whence $|S_n| = n!$) and $\epsilon : S_n \to \{\pm 1\}$ is the **signature** of a permutation σ (more on this in the first footnote of Section 12.3).

Proposition 166
Let f be a linear $p-$form over E. If $g : E^p \to \mathbb{R}$ is defined by

$$g(\mathbf{x}_1, \ldots, \mathbf{x}_p) = \sum_{\sigma \in S_p} \epsilon(\sigma) f(\mathbf{x}_{\sigma(1)}, \ldots, \mathbf{x}_{\sigma(p)}),$$

then g is an alternating $p-$form.

Proof: we only prove the statement for $p = 2$. The proof for $p \geq 3$ is left as an exercise.

Let $p = 2$. Then $S_2 = \{\text{id}, \sigma = \begin{pmatrix} 1 & 2 \end{pmatrix}\}$ and we have $\epsilon(\text{id}) = 1$ and $\epsilon(\sigma) = -1$. Therefore,

$$g(\mathbf{x}_1, \mathbf{x}_2) = f(\mathbf{x}_1, \mathbf{x}_2) - f(\mathbf{x}_2, \mathbf{x}_1).$$

Clearly $g(\mathbf{x}, \mathbf{x}) = 0$, and so g is alternating. ■

The alternating $p-$form g in Proposition 166 is the **anti-symmetric form built from** f.

Let $f_1, \ldots, f_p$ be linear $1-$forms over E. The anti-symmetric form built from the tensor product $f_1 \otimes \cdots \otimes f_p$ is the **wedge product of** $f_1, \ldots, f_p$, denoted by $(f_1 \wedge \cdots \wedge f_p)$.[2]

[2]Formally, we should be using the brackets around the wedge product (and the tensor product) of linear forms, but we will often omit them to simplify the notation.

By definition, then, we have

$$(f_1 \wedge \cdots \wedge f_p)(\mathbf{x}_1, \ldots, \mathbf{x}_p) = \sum_{\sigma \in S_p} \epsilon(\sigma)(f_1 \otimes \cdots \otimes f_p)(\mathbf{x}_{\sigma(1)}, \ldots, \mathbf{x}_{\sigma(p)})$$
$$= \sum_{\sigma \in S_p} \epsilon(\sigma) f_1(\mathbf{x}_{\sigma(1)}) \cdots f_p(\mathbf{x}_{\sigma(p)}) = \det \begin{pmatrix} f_1(\mathbf{x}_1) & \cdots & f_1(\mathbf{x}_p) \\ \vdots & \ddots & \vdots \\ f_p(\mathbf{x}_1) & \cdots & f_p(\mathbf{x}_p) \end{pmatrix}.$$

A few examples will help to illustrate the concept.

Examples: consider the case $p = 2$; let f_1, f_2 be linear $1-$form over $E = \mathbb{R}^2$ and $\mathbf{x}_1, \mathbf{x}_2 \in E$. Then:

1. $f_1 \wedge f_2(\mathbf{x}_1, \mathbf{x}_2) = f_1(\mathbf{x}_1) f_2(\mathbf{x}_2) - f_1(\mathbf{x}_2) f_2(\mathbf{x}_1)$.
2. $f_2 \wedge f_1(\mathbf{x}_1, \mathbf{x}_2) = f_2(\mathbf{x}_1) f_1(\mathbf{x}_2) - f_2(\mathbf{x}_2) f_1(\mathbf{x}_1) = -f_1 \wedge f_2(\mathbf{x}_1, \mathbf{x}_2)$.
3. $f_1 \wedge f_1(\mathbf{x}_1, \mathbf{x}_2) = f_2 \wedge f_2(\mathbf{x}_1, \mathbf{x}_2) = 0$. $\square$

Generally, if $f_i = f_j$ for some $i =\neq j$, then $f_1 \wedge \cdots \wedge f_p = 0$. Furthermore, if $\sigma \in S_p$, then

$$f_{\sigma(1)} \wedge \cdots \wedge f_{\sigma(p)} = \epsilon(\sigma) f_1 \wedge \cdots \wedge f_p.$$

Example: let $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a basis of E (i.e., $n = \dim(E) = 3$) and let $g : E \times E \to \mathbb{R}$ be a bilinear alternating form (i.e., $p = 2$). Then

$$g(\mathbf{x}, \mathbf{y}) = g\left(\sum_{i=1}^{3} x_i \mathbf{e}_i, \sum_{j=1}^{3} y_j \mathbf{e}_j\right) = \sum_{i,j=1}^{3} x_i y_j g(\mathbf{e}_i, \mathbf{e}_j).$$

Since g is alternating, we must have:

$$g(\mathbf{e}_i, \mathbf{e}_j) = -g(\mathbf{e}_j, \mathbf{e}_i), \quad g(\mathbf{e}_i, \mathbf{e}_i) = 0, \quad \text{for all } i, j = 1, \ldots, 3.$$

Thus,

$$\begin{aligned} g(\mathbf{x}, \mathbf{y}) &= x_1 y_2 g(\mathbf{e}_1, \mathbf{e}_2) + x_1 y_3 g(\mathbf{e}_1, \mathbf{e}_3) + x_2 y_3 g(\mathbf{e}_2, \mathbf{e}_3) \\ &\quad - x_2 y_1 g(\mathbf{e}_1, \mathbf{e}_2) - x_3 y_1 g(\mathbf{e}_1, \mathbf{e}_3) - x_3 y_2 g(\mathbf{e}_2, \mathbf{e}_3) \\ &= (x_1 y_2 - x_2 y_1) g(\mathbf{e}_1, \mathbf{e}_2) + (x_1 y_3 - x_3 y_1) g(\mathbf{e}_1, \mathbf{e}_3) + (x_2 y_3 - x_3 y_2) g(\mathbf{e}_2, \mathbf{e}_3). \end{aligned}$$

But note that for $i < j$, we have

$$\mathbf{e}_i^* \wedge \mathbf{e}_j^*(\mathbf{x}, \mathbf{y}) = \mathbf{e}_i^*(\mathbf{x}) \mathbf{e}_j^*(\mathbf{y}) - \mathbf{e}_i^*(\mathbf{y}) \mathbf{e}_j^*(\mathbf{x}) = x_i y_j - x_j y_i.$$

Combining the last two results, we have

$$g(\mathbf{x},\mathbf{y}) = g(\mathbf{e}_1,\mathbf{e}_2)\mathbf{e}_1^* \wedge \mathbf{e}_2^* + g(\mathbf{e}_1,\mathbf{e}_3)\mathbf{e}_1^* \wedge \mathbf{e}_3^* + g(\mathbf{e}_2,\mathbf{e}_3)\mathbf{e}_2^* \wedge \mathbf{e}_3^*.$$

Consequently, g is a linear combination of the wedge products $\{\mathbf{e}_2^* \wedge \mathbf{e}_3^* \mid i < j\}$. Furthermore, $\{\mathbf{e}_1^* \wedge \mathbf{e}_2^*, \mathbf{e}_1^* \wedge \mathbf{e}_3^*, \mathbf{e}_2^* \wedge \mathbf{e}_3^*\}$ are linearly independent.

Indeed, suppose that

$$(d_{1,2}\mathbf{e}_1^* \wedge \mathbf{e}_2^* + d_{1,3}\mathbf{e}_1^* \wedge \mathbf{e}_3^* + d_{2,3}\mathbf{e}_2^* \wedge \mathbf{e}_3^*)(\mathbf{x},\mathbf{y}) = 0 \quad \text{for all } \mathbf{x},\mathbf{y}.$$

In particular, this would hold for $(\mathbf{x},\mathbf{y}) = (\mathbf{e}_1,\mathbf{e}_2)$, and so

$$\begin{aligned}
0 &= d_{1,2}\mathbf{e}_1^* \wedge \mathbf{e}_2^*(\mathbf{e}_1,\mathbf{e}_2) + d_{1,3}\mathbf{e}_1^* \wedge \mathbf{e}_3^*(\mathbf{e}_1,\mathbf{e}_3) + d_{2,3}\mathbf{e}_2^* \wedge \mathbf{e}_3^*(\mathbf{e}_2,\mathbf{e}_3) \\
&= d_{1,2}(\mathbf{e}_1^*(\mathbf{e}_1)\mathbf{e}_2^*(\mathbf{e}_2) - \mathbf{e}_1^*(\mathbf{e}_2)\mathbf{e}_2^*(\mathbf{e}_1)) + d_{1,3}((\mathbf{e}_1^*(\mathbf{e}_1)\mathbf{e}_3^*(\mathbf{e}_2) - \mathbf{e}_1^*(\mathbf{e}_2)\mathbf{e}_3^*(\mathbf{e}_1)) \\
&\qquad + d_{2,3}((\mathbf{e}_2^*(\mathbf{e}_1)\mathbf{e}_3^*(\mathbf{e}_2) - \mathbf{e}_2^*(\mathbf{e}_2)\mathbf{e}_3^*(\mathbf{e}_1)) \\
&= d_{1,2}(1\cdot 1 - 0\cdot 0) + d_{1,3}(1\cdot 0 - 0\cdot 0) + d_{2,3}(0\cdot 0 - 0\cdot 0) = d_{1,2} \implies d_{1,2} = 0.
\end{aligned}$$

Similarly, using $(\mathbf{x},\mathbf{y}) = (\mathbf{e}_1,\mathbf{e}_3)$ and $(\mathbf{x},\mathbf{y}) = (\mathbf{e}_2,\mathbf{e}_3)$ yields $d_{1,3} = d_{2,3} = 0$.

Thus $\{\mathbf{e}_1^* \wedge \mathbf{e}_2^*, \mathbf{e}_1^* \wedge \mathbf{e}_3^*, \mathbf{e}_2^* \wedge \mathbf{e}_3^*\}$ forms a basis for the space of alternating bilinear (2−)forms over E. ■

The **space of alternating** $p-$**forms over** $E \simeq \mathbb{R}^n$ will constantly be appearing in what follows; to lighten the text, we denote it by $\Lambda^p(E)$.

Theorem 167
Let $\{\mathbf{e}_1,\ldots,\mathbf{e}_n\}$ be a basis of $E \simeq \mathbb{R}^n$ and $\{\mathbf{e}_1^*,\ldots,\mathbf{e}_n^*\}$ be the dual basis of E^*. Then

$$\{\mathbf{e}_{i_1}^* \wedge \cdots \wedge \mathbf{e}_{i_p}^* \mid i_1 < \cdots < i_p\}$$

is a basis of $\Lambda^p(E)$.

Proof: left as an exercise. ■

Corollary 168
Let $E \simeq \mathbb{R}^n$. If $1 \leq p \leq n$, then

$$\dim(\Lambda^p(E)) = \binom{n}{p} = \frac{n!}{p!(n-p)!};$$

if $p > n$, then $\dim(\Lambda^p(E)) = 0$.

Proof: left as an exercise. ■

12.3 Wedge Product of Alternating Forms

If $f \in \Lambda^p(E)$ and $g \in \Lambda^q(E)$, is there a **natural way** to build a form $f \wedge g \in \Lambda^{p+q}(E)$? It turns out that it can be done, with a small group theory detour.

Let S_{p+q} be the permutation group on $\{1, \dots, p+q\}$,[3] and set

$$A = \{\sigma \in S_{p+q} \mid \sigma(1) < \cdots < \sigma(p) \quad \text{and} \quad \sigma(p+1) < \cdots < \sigma(p+q)\}.$$

Examples

1. If $p = 1$ and $q = 2$, then $A = \{\sigma \in S_3 \mid \sigma(2) < \sigma(3)\}$. But

$$S_3 = \{\text{id}, \begin{pmatrix}1 & 2\end{pmatrix}, \begin{pmatrix}1 & 3\end{pmatrix}, \begin{pmatrix}2 & 3\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\end{pmatrix}, \begin{pmatrix}1 & 3 & 2\end{pmatrix}\},$$

so that

$$A = \{\text{id}, \begin{pmatrix}1 & 2\end{pmatrix}, \begin{pmatrix}1 & 3 & 2\end{pmatrix}\}.$$

2. If $p = 2$ and $q = 2$, then $A = \{\sigma \in S_4 \mid \sigma(1) < \sigma(2) \quad \text{and} \quad \sigma(3) < \sigma(4)\}$; S_4 has $4! = 24$ permutations, and we can show that

$$A = \{\text{id}, \begin{pmatrix}2 & 3\end{pmatrix}, \begin{pmatrix}2 & 4 & 3\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\end{pmatrix}, \begin{pmatrix}1 & 2 & 4 & 3\end{pmatrix}, \begin{pmatrix}1 & 3\end{pmatrix}\begin{pmatrix}2 & 4\end{pmatrix}\}.$$

Permutation calculations can quickly become cumbersome! $\square$

If $f \in \Lambda^p(E)$ and $g \in \Lambda^q(E)$, the **wedge product of f and g** is given by

$$f \wedge g(\mathbf{x}_1, \dots, \mathbf{x}_p, \mathbf{x}_{p+1}, \dots, \mathbf{x}_{p+q}) = \sum_{\sigma \in A} \epsilon(\sigma) f(\mathbf{x}_{\sigma(1)}, \dots, \mathbf{x}_{\sigma(p)}) g(\mathbf{x}_{\sigma(p+1)}, \dots, \mathbf{x}_{\sigma(p+q)}).$$

As $f \wedge g$ depends linearly on each of $\mathbf{x}_1, \dots, \mathbf{x}_{p+q}$, then it is a linear $(p+q)-$form on E. Is it alternating?

Example: if $p = 1$ and $q = 3$, then

$$A = \{\sigma \in S_4 \mid \sigma(2) < \sigma(3) < \sigma(4)\} = \{\text{id}, \begin{pmatrix}1 & 2\end{pmatrix}, \begin{pmatrix}1 & 3 & 2\end{pmatrix}, \begin{pmatrix}1 & 4 & 3 & 2\end{pmatrix}\};$$

the corresponding signatures are $1, -1, 1, -1$. If all we know of f, g is that $f \in \Lambda^1(E)$ and $g \in \Lambda^q(E)$, then we must have:

[3] A permutation $\sigma \in S_n$ is a bijection $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$. We can also write σ in **cycle notation**, as illustrated as follows: suppose that σ acts on $\{1, 2, 3, 4, 5\}$ according to $\sigma(1) = 2$, $\sigma(2) = 5$, $\sigma(5) = 1$, $\sigma(3) = 3$, and $\sigma(4) = 4$. Then we write σ as $\begin{pmatrix}1 & 2 & 5\end{pmatrix}\begin{pmatrix}3\end{pmatrix}\begin{pmatrix}4\end{pmatrix}$, or usually as $\begin{pmatrix}1 & 2 & 5\end{pmatrix}$ since $3, 4$ are left unchanged by σ. The signature $\epsilon(\sigma)$ of a permutation σ is determined as follows. We write σ as a product of disjoint cycles (as above); the signature is -1 if and only if the factorization contains an **odd** number of **even-length** cycles. As $\sigma = \begin{pmatrix}1 & 2 & 5\end{pmatrix}$ contains no even-length cycle, $\epsilon(\sigma) = 1$.

$$f \wedge g(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = f(\mathbf{x}_1)g(\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) - f(\mathbf{x}_2)g(\mathbf{x}_1, \mathbf{x}_3, \mathbf{x}_4) + f(\mathbf{x}_3)g(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_4) - f(\mathbf{x}_4)g(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3).$$

If $\mathbf{x}_1 = \mathbf{x}_2$, $\mathbf{x}_1 = \mathbf{x}_3$, $\mathbf{x}_1 = \mathbf{x}_4$, $\mathbf{x}_2 = \mathbf{x}_3$, $\mathbf{x}_2 = \mathbf{x}_4$, or $\mathbf{x}_3 = \mathbf{x}_4$, the g components of $f \wedge g$ are either 0 because they are alternating and contain a repeated argument, or they cancel one another out (try it!); thus $f \wedge g$ is alternating. □

The wedge product has the right kinds of properties: if $f, f_1, f_2 \in \Lambda^p(E)$, $g, g_1, g_2 \in \Lambda^q(E)$, and $\alpha \in \mathbb{R}$, then

$$\begin{aligned}(f_1 + f_2) \wedge g &= f_1 \wedge g + f_2 \wedge g,\\ f \wedge (g_1 + g_2) &= f \wedge g_1 + f \wedge g_2,\\ (\alpha f) \wedge g &= \alpha(f \wedge g) = f \wedge (\alpha g).\end{aligned}$$

This leads us to the following crucial result.

Lemma 169
Let $f_i \in E^*$, $1 \leq i \leq p+q$. Then $f = f_1 \wedge \cdots \wedge f_p \in \Lambda^p(E)$, $g = g_{p+1} \wedge \cdots \wedge g_{p+q} \in \Lambda^q(E)$ and

$$f \wedge g = f_1 \wedge \cdots \wedge f_{p+q}.$$

Proof: by definition,

$$\begin{aligned}f_1 \wedge \cdots \wedge f_p(\mathbf{x}_1, \ldots, \mathbf{x}_p) &= \sum_{\sigma \in S_p} \epsilon(\sigma) f_1(\mathbf{x}_{\sigma(1)}) \cdots f_p(\mathbf{x}_{\sigma(p)}),\\ f_{p+1} \wedge \cdots \wedge f_{p+q}(\mathbf{x}_{p+1}, \ldots, \mathbf{x}_{p+q}) &= \sum_{\tau \in S_q} \epsilon(\tau) f_{p+1}(\mathbf{x}_{\tau(p+1)}) \cdots f_{p+q}(\mathbf{x}_{\tau(p+q)}).\end{aligned}$$

It is easy to see that

$$S_p \simeq \{\sigma \in S_{p+q} \mid \sigma(j) = j, p+1 \leq j \leq p+q\} \quad \text{and} \quad S_q \simeq \{\tau \in S_{p+q} \mid \tau(j) = j, 1 \leq j \leq p\}.$$

In Lemma 171, we will see that every $\tilde{\sigma} \in S_{p+q}$ can be written uniquely as $\tilde{\sigma} = \sigma\sigma'\sigma''$, with $\sigma \in A$, $\sigma' \in S_p$, and $\sigma'' \in S_q$. Then

$$\begin{aligned}f_1 \wedge \cdots \wedge f_{p+q}(\mathbf{x}_1, \ldots, \mathbf{x}_{p+q}) &= \sum_{\tilde{\sigma} \in S_{p+q}} \epsilon(\tilde{\sigma}) f_1(\mathbf{x}_{\tilde{\sigma}(1)}) \cdots f_{p+q}(\mathbf{x}_{\tilde{\sigma}(p+q)})\\ &= \sum_{\sigma, \sigma', \sigma''} \epsilon(\sigma\sigma'\sigma'') f_1(\mathbf{x}_{\sigma\sigma'\sigma''(1)}) \cdots f_{p+q}(\mathbf{x}_{\sigma\sigma'\sigma''(p+q)})\\ &= \sum_{\sigma, \sigma', \sigma''} \epsilon(\sigma)\epsilon(\sigma')\epsilon(\sigma'') f_1(\mathbf{x}_{\sigma\sigma'(1)}) \cdots f_p(\mathbf{x}_{\sigma\sigma'(p)}) f_{p+1}(\mathbf{x}_{\sigma\sigma''(p+1)}) \cdots f_{p+q}(\mathbf{x}_{\sigma\sigma''(p+q)})\end{aligned}$$

so that

$$f_1 \wedge \cdots \wedge f_{p+q}(\mathbf{x}_1, \ldots, \mathbf{x}_{p+q})$$
$$= \sum_{\sigma \in A} \epsilon(\sigma) \Big(\sum_{\sigma' \in S_p} \epsilon(\sigma'') f_1(\mathbf{x}_{\sigma\sigma'(1)}) \cdots f_p(\mathbf{x}_{\sigma\sigma'(p)}) \Big) \Big(\sum_{\sigma'' \in S_q} \epsilon(\sigma'') f_{p+1}(\mathbf{x}_{\sigma\sigma''(p+1)}) \cdots f_{p+q}(\mathbf{x}_{\sigma\sigma''(p+q)}) \Big)$$
$$= \sum_{\sigma \in A} f(\mathbf{x}_{\sigma(1)}, \ldots, \mathbf{x}_{\sigma(p)}) g(\mathbf{x}_{\sigma(p+1)}, \ldots, \mathbf{x}_{\sigma(p+q)}) = f \wedge g(\mathbf{x}_{\sigma(p)}, \ldots, \mathbf{x}_{\sigma(p+q)}).$$

That this $(p+q)-$form is alternating is left as an exercise. ■

This leads us to the main result of this section.

Theorem 170
Let $f \in \Lambda^p(E)$ and $g \in \Lambda^q(E)$. Then $f \wedge g \in \Lambda^{p+q}(E)$.

Proof: according to Theorem 167, f is a linear combination of wedge products of $p-$forms over E of the form $\mathbf{e}^*_{i_1} \wedge \cdots \wedge \mathbf{e}^*_{i_p}$; similarly, g is a linear combination of wedge products of $q-$forms over E of the form $\mathbf{e}^*_{j_1} \wedge \cdots \wedge \mathbf{e}^*_{j_q}$.

According to Lemma 169, expressions of the form

$$(\mathbf{e}^*_{i_1} \wedge \cdots \wedge \mathbf{e}^*_{i_p}) \wedge (\mathbf{e}^*_{j_1} \wedge \cdots \wedge \mathbf{e}^*_{j_q}) \tag{12.1}$$

are alternating $(p+q)-$forms.

Thus $f \wedge g$ is a linear combination of alternating $(p+q)-$forms as in (12.1); since $\Lambda^{p+q}(E)$ is a vector space over E (see Corollary 168), $f \wedge g$ is alternating. ■

The wedge product of alternating forms is thus well-defined, and it has a set of useful properties. Let $f \in \Lambda^p(E), g \in \Lambda^q(E), h \in \Lambda^r(E)$. Then:

1. $f \wedge (g \wedge h) = (f \wedge g) \wedge h$ (the wedge product is **associative**);
2. $f \wedge g = (-1)^{pq} g \wedge f$ (it is **not commutative**), and
3. if $u : E \to F$ is a linear transformation, $f \in \Lambda^p(E)$, and $g \in \Lambda^q(F)$, then $u(f) \in \Lambda^p(E)$, where
$$u(f)(\mathbf{x}_1, \ldots, \mathbf{x}_p) = f(u(\mathbf{x}_1), \ldots, u(\mathbf{x}_p));$$
$u(g) \in \Lambda^q(E)$, where
$$u(g)(\mathbf{x}_1, \ldots, \mathbf{x}_p) = g(u(\mathbf{x}_1), \ldots, u(\mathbf{x}_q)),$$
and $u(f \wedge g) = u(f) \wedge u(g) \in \Lambda^{p+q}(E)$ (the proof is left as an exercise).

We finish this section with the promised lemma.

> **Lemma 171**
> *If $\tilde{\sigma} \in S_{p+q}$, there is a unique triplet $\sigma \in A$, $\sigma' \in S_p$, and $\sigma'' \in S_q$ such that $\tilde{\sigma} = \sigma\sigma'\sigma''$.*
>
> **Proof:** let $A' = \{\tilde{\sigma}(1), \ldots, \tilde{\sigma}(p)\} \subseteq \{1, \ldots, p+q\}$. List the integers in A' in increasing order, and define σ' by $\sigma'(j) =$rank of $\tilde{\sigma}(j)$ in A', for $1 \leq j \leq p$.
>
> Similarly, define σ'' by $\sigma''(j) =$rank of $\tilde{\sigma}(i)$ in $A'' =$ ordered$\{\tilde{\sigma}(p+1), \ldots, \tilde{\sigma}(p+q)\}$, for $p+1 \leq i \leq p+q$.
>
> If we write $A' = \{i_1 < \cdots < i_p\}$ and $A'' = \{i_{p+1} < \cdots < i_{p+q}\}$, we can then define σ by $\sigma(j) = i_j$, $1 \leq j \leq p+q$. Then $\tilde{\sigma} = \sigma\sigma'\sigma''$. ■

12.4 Solved Problems

1. Let E be a finite-dimensional vector space over $\mathbb{R}$, with dim$(E) = 3$. If $\mathbf{x}, \mathbf{y}, \mathbf{z} \in E$ are linearly dependent, show that $f(\mathbf{x}, \mathbf{y}, \mathbf{z}) = 0$ for any alternating linear $3-$form f.

   Proof: let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the canonical basis of E. Since $\mathbf{x}, \mathbf{y}, \mathbf{z} \in E$ are linearly dependent, (at least) one of them may be expressed as a linear combination of the other two. Without loss of generality, say $\mathbf{x} = a\mathbf{y} + b\mathbf{z}$, with $a, b \in \mathbb{R}$. Then

   $$f(\mathbf{x}, \mathbf{y}, \mathbf{z}) = f(a\mathbf{y} + b\mathbf{z}, \mathbf{y}, \mathbf{z}) = af(\mathbf{y}, \mathbf{y}, \mathbf{z}) + bf(\mathbf{z}, \mathbf{y}, \mathbf{z}) = a \cdot 0 + b \cdot 0 = 0,$$

   since f is alternating. ■

2. Let E be a finite-dimensional vector space over $\mathbb{R}$, with dim$(E) = 3$. If $\mathbf{x}, \mathbf{y}, \mathbf{z} \in E$ are linearly independent, show that $f(\mathbf{x}, \mathbf{y}, \mathbf{z}) \neq 0$ for any alternating linear $3-$form $f \neq 0$.

   Proof: let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the canonical basis of E. Since $f \neq 0$, $f(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \neq 0$. Write

   $$\begin{aligned} \mathbf{x} &= x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 \\ \mathbf{y} &= y_1\mathbf{e}_1 + y_2\mathbf{e}_2 + y_3\mathbf{e}_3 \\ \mathbf{z} &= z_1\mathbf{e}_1 + z_2\mathbf{e}_2 + z_3\mathbf{e}_3 \end{aligned}$$

   Since $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ are linearly independent,

   $$\det \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} \neq 0.$$

   Then

   $$f(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \sum_{\substack{i \neq j \\ i \neq k \\ j \neq k}} x_i y_j z_k f(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = \det \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} \cdot f(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \neq 0. \quad ■$$

3. Show that the inner product $(\cdot \mid |\cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a bilinear form.

    Proof: the inner product $(\cdot \mid \cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined by

    $$(\mathbf{x} \mid \mathbf{y}) = \sum_{i=1}^{n} x_i y_i.$$

    In order to show it is bilinear, we need to show that for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$, $a, b \in \mathbb{R}$, we have

    - $(a\mathbf{x} + b\mathbf{y} \mid \mathbf{z}) = a(\mathbf{x}|\mathbf{z}) + b(\mathbf{y} \mid \mathbf{z})$
    - $(\mathbf{x} \mid a\mathbf{y} + b\mathbf{z}) = a(\mathbf{x} \mid \mathbf{y}) + b(\mathbf{x} \mid \mathbf{z})$

    But

    $$(a\mathbf{x} + b\mathbf{y} \mid \mathbf{z}) = \sum_{i=1}^{n} (ax_i + by_i) z_i = a \sum_{i=1}^{n} x_i z_i + b \sum_{i=1}^{n} y_i z_i = a(\mathbf{x} \mid \mathbf{z}) + b(\mathbf{y} \mid \mathbf{z})$$

    and

    $$(\mathbf{x} \mid a\mathbf{y} + b\mathbf{z}) = \sum_{i=1}^{n} x_i (ay_i + bz_i) = a \sum_{i=1}^{n} x_i y_i + b \sum_{i=1}^{n} x_i z_i = a(\mathbf{x} \mid \mathbf{y}) + b(\mathbf{x} \mid \mathbf{z})$$

    so that the inner product is indeed bilinear. It is not alternating, however, since we would need $(\mathbf{x} \mid \mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbb{R}^n$ but $(\mathbf{e}_1 \mid \mathbf{e}_1) = 1$. ■

4. Show that det : $\mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ is a bilinear form.

    Proof: that this form is both multilinear and alternating is immediate due to the properties of the determinant that you have seen/will see in your linear algebra courses:

    - Firstly, $\det(\mathbf{x}_1, \ldots, a_1\mathbf{y}_1 + a_2\mathbf{y}_2, \ldots, \mathbf{x}_n) = \sum_{j=1}^{2} a_j \det(\mathbf{x}_1, \ldots, \mathbf{y}_j, \ldots, \mathbf{x}_n)$
    - Secondly, $\det(\mathbf{x}_1, \ldots, \mathbf{x}_n) = 0$ if $x_i = x_j$ for some $i \neq j$.

    ■

5. Show that $\{\mathbf{e}^*_{i_1} \otimes \mathbf{e}^*_{i_2} \otimes \cdots \mathbf{e}^*_{i_p} : 1 \leq i_j \leq n\}$ forms a basis of the vector space of linear $p-$forms over E. What is the dimension of that vector space?

    Proof: recall that $\mathbf{e}^*_i : E \to \mathbb{R}$ is the linear functional such that $\mathbf{e}^*_i(\mathbf{e}_j) = \delta_{i,j}$.

    Let us first assume that the set in question is indeed a basis of the space of all linear (but not necessarily alternating) $p-$forms. There are n possible choices for each $1-$form $\mathbf{e}^*_{i_j}$ appearing in the tensor product. Since there are p such forms, there is a total of n^p tensor products. Hence, dim$(\{\text{space of } p-\text{linear forms over } E\}) = n^p$.

We now show that the set is such a basis. First, note that for any choice of indices i_j, $1 \leq j \leq p$, $\mathbf{e}^*_{i_1} \otimes \cdots \mathbf{e}^*_{i_p}$ is a $p-$linear form over E; indeed,

$$\begin{aligned}
&\mathbf{e}^*_{i_1} \otimes \cdots \otimes \mathbf{e}^*_{i_j} \otimes \cdots \otimes \mathbf{e}^*_{i_p}(\mathbf{x}_1, \ldots, a\mathbf{y}_1 + b\mathbf{y}_2, \ldots, \mathbf{x}_p) \\
&\quad = \mathbf{e}^*_{i_1}(\mathbf{x}_1) \cdots \mathbf{e}^*_{i_j}(a\mathbf{y}_1 + b\mathbf{y}_2) \cdots \mathbf{e}^*_{i_p}(\mathbf{x}_p) \\
&\quad = a\mathbf{e}^*_{i_1}(\mathbf{x}_1) \cdots \mathbf{e}^*_{i_j}(\mathbf{y}_1) \cdots \mathbf{e}^*_{i_p}(\mathbf{x}_p) + b\mathbf{e}^*_{i_1}(\mathbf{x}_1) \cdots \mathbf{e}^*_{i_j}(\mathbf{y}_2) \cdots \mathbf{e}^*_{i_p}(\mathbf{x}_p) \\
&\quad = a\mathbf{e}^*_{i_1} \otimes \cdots \otimes \mathbf{e}^*_{i_j} \otimes \cdots \otimes \mathbf{e}^*_{i_p}(\mathbf{x}_1, \ldots, \mathbf{y}_1, \ldots, \mathbf{x}_p) + b\mathbf{e}^*_{i_1} \otimes \cdots \otimes \mathbf{e}^*_{i_j} \otimes \cdots \otimes \mathbf{e}^*_{i_p}(\mathbf{x}_1, \ldots, \mathbf{y}_2, \ldots, \mathbf{x}_p)
\end{aligned}$$

since $\mathbf{e}^*_{i_j}$ is linear. Hence,

$$\text{Span}\{\mathbf{e}^*_{i_1} \otimes \mathbf{e}^*_{i_2} \otimes \cdots \mathbf{e}^*_{i_p} : 1 \leq i_j \leq n\} \subseteq \{\text{space of } p\text{−linear forms over } E\}$$

Now, let f be a $p-$linear form, and suppose $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be the canonical basis of E. For $1 \leq j \leq p$, write

$$\mathbf{x}_j = \sum_{i=1}^{n} x_{j,i}\mathbf{e}_i.$$

Then

$$\begin{aligned}
f(\mathbf{x}_1, \ldots, \mathbf{x}_p) &= \sum_{j_1,\ldots,j_p=1}^{n} x_{j_1,1} \cdots x_{j_p,1} f(\mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_p}) \\
&= \sum_{j_1,\ldots,j_p=1}^{n} \mathbf{e}^*_{j_1}(\mathbf{x}_1) \cdots \mathbf{e}^*_{j_p}(\mathbf{x}_p) f(\mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_p}) \\
&= \sum_{j_1,\ldots,j_p=1}^{n} f(\mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_p}) \mathbf{e}^*_{j_1} \otimes \cdots \otimes \mathbf{e}^*_{j_p}(\mathbf{x}_1, \ldots, \mathbf{x}_p)
\end{aligned}$$

and so $f \in \text{Span}\{\mathbf{e}^*_{i_1} \otimes \mathbf{e}^*_{i_2} \otimes \cdots \mathbf{e}^*_{i_p} : 1 \leq i_j \leq n\}$. Consequently,

$$\text{Span}\{\mathbf{e}^*_{i_1} \otimes \mathbf{e}^*_{i_2} \otimes \cdots \mathbf{e}^*_{i_p} : 1 \leq i_j \leq n\} = \{\text{space of } p\text{−linear forms over } E\}$$

It remains only to show that the tensor products are linearly independent. To do so, suppose that

$$\sum_{j_1,\ldots,j_p=1}^{n} a_{j_1,\ldots,j_p} \mathbf{e}^*_{j_1} \otimes \cdots \otimes \mathbf{e}^*_{j_p} = 0$$

Then

$$\sum_{j_1,\ldots,j_p=1}^{n} a_{j_1,\ldots,j_p} \mathbf{e}^*_{j_1} \otimes \cdots \otimes \mathbf{e}^*_{j_p}(\mathbf{x}_1, \ldots, \mathbf{x}_p) = 0$$

for all $(\mathbf{x}_1, \ldots, \mathbf{x}_p) \in E^p$. Fix $j^*_1, \ldots, j^*_p$. Then $(\mathbf{e}_{j^*_1}, \ldots, \mathbf{e}_{j^*_p}) \in E^p$ and so

$$\sum_{j_1,\ldots,j_p=1}^{n} a_{j_1,\ldots,j_p} \mathbf{e}^*_{j_1} \otimes \cdots \otimes \mathbf{e}^*_{j_p}(\mathbf{e}_{j^*_1}, \ldots, \mathbf{e}_{j^*_p}) = 0$$

But

$$\sum_{j_1,\ldots,j_p=1}^{n} a_{j_1,\ldots,j_p}\mathbf{e}_{j_1}^* \otimes \cdots \otimes \mathbf{e}_{j_p}^*(\mathbf{e}_{j_1^*},\ldots,\mathbf{e}_{j_p^*}) = a_{j_1^*,\ldots,j_p^*}$$

so that $a_{j_1^*,\ldots,j_p^*} = 0$. But $j_1^*,\ldots,j_p^*$ were arbitrary, so that we indeed have $a_{j_1,\ldots,j_p} = 0$ for all $1 \leq j_1,\ldots,j_p \leq n$, and the tensor products are linearly independent. ■

6. Let $f_1, f_2, \ldots, f_p$ be linear $1-$forms over E and $\sigma \in S_p$. Show that

$$f_{\sigma(1)} \wedge \cdots \wedge f_{\sigma(p)} = \varepsilon(\sigma) f_1 \wedge \cdots \wedge f_p.$$

Proof: by definition, we have

$$\begin{aligned} f_{\sigma(1)} \wedge \cdots \wedge f_{\sigma(p)}(\mathbf{x}_1,\ldots,\mathbf{x}_p) &= \det \begin{pmatrix} f_{\sigma(1)}(\mathbf{x}_1) & \cdots & f_{\sigma(1)}(\mathbf{x}_p) \\ \vdots & & \vdots \\ f_{\sigma(p)}(\mathbf{x}_1) & \cdots & f_{\sigma(p)}(\mathbf{x}_p) \end{pmatrix} \\ &= \epsilon(\sigma) \det \begin{pmatrix} f_1(\mathbf{x}_1) & \cdots & f_1(\mathbf{x}_p) \\ \vdots & & \vdots \\ f_p(\mathbf{x}_1) & \cdots & f_p(\mathbf{x}_p) \end{pmatrix} \\ &= \epsilon(\sigma) f_1 \wedge \cdots \wedge f_p(\mathbf{x}_1,\ldots,\mathbf{x}_p) \end{aligned}$$

■

7. Let $f_1, f_2, \ldots, f_p$ be linear $1-$forms over E such that $f_i = f_j$ for some $i \neq j$. Show that $f_1 \wedge \cdots \wedge f_p = 0$.

Proof: by definition, we have

$$f_1 \wedge \cdots \wedge f_p(\mathbf{x}_1,\ldots,\mathbf{x}_p) = \det \begin{pmatrix} f_1(\mathbf{x}_1) & \cdots & f_1(\mathbf{x}_p) \\ \vdots & & \vdots \\ f_p(\mathbf{x}_1) & \cdots & f_p(\mathbf{x}_p) \end{pmatrix}$$

If $f_i = f_j$ for $i \neq j$, two of the rows in the above matrix are identical; as a result, the determinant is 0. ■

8. Provide a proof of Corollary 168.

Proof: you should be able to make an informal argument for this one. In essence, the proof runs as follows:

a) $\Lambda^p(E)$ is a subspace of the space of linear $p-$forms over E.

b) $\mathbf{e}_{i_1}^* \wedge \cdots \wedge \mathbf{e}_{i_p}^* \in \Lambda^p(E)$ for all $1 \leq i_1,\ldots,i_p \leq n$, so that

$$\text{Span}\{\mathbf{e}_{i_1}^* \wedge \cdots \wedge \mathbf{e}_{i_p}^* : 1 \leq i_1,\ldots,i_p \leq n\} \subseteq \Lambda^p(E).$$

c) Any $f \in \Lambda^p(E)$ can be written as

$$f = \sum_{i_1<\cdots<i_p} f(\mathbf{e}_{i_1},\ldots,\mathbf{e}_{i_p})\mathbf{e}_{i_1}^* \wedge \cdots \wedge \mathbf{e}_{i_p}^*$$

so that

$$\Lambda^p(E) \subseteq \text{Span}\{\mathbf{e}_{i_1}^* \wedge \cdots \wedge \mathbf{e}_{i_p}^* : 1 \leq i_1, \ldots, i_p \leq n\}.$$

Consequently,

$$\Lambda^p(E) = \text{Span}\{\mathbf{e}_{i_1}^* \wedge \cdots \wedge \mathbf{e}_{i_p}^* : 1 \leq i_1, \ldots, i_p \leq n\}.$$

d) For each fixed choice of $i_1, \ldots, i_p$ there are two possibilities:

i. if the indices are all distinct, let

$$A_{i_1,\ldots,i_p} = \big\{\sigma \in S_p : \sigma(\{i_1, \ldots, i_p\}) \subseteq \{i_1, \ldots, i_p\}\big\}.$$

Then $\mathbf{e}_{i_1}^* \wedge \cdots \wedge \mathbf{e}_{i_p}^* = \epsilon(\sigma)\mathbf{e}_{\sigma(i_1)}^* \wedge \cdots \wedge \mathbf{e}_{\sigma(i_p)}^*$ for each $\sigma \in A_{i_1,\ldots,i_p}$. Consequently, all wedge products containing $\mathbf{e}_{i_1}^*, \ldots, \mathbf{e}_{i_p}^*$ are linearly dependent. Remove all of them except the canonical one, i.e. the one for which $i_1 < \ldots < i_p$ (this can be done since all indices are distinct);

ii. if some of the indices repeat, then $\mathbf{e}_{i_1}^* \wedge \cdots \wedge \mathbf{e}_{i_p}^* = 0$ (see exercise 8). Consequently, all such wedge products are linearly dependent. Remove all of them.

e) The remaining wedge products $\{\mathbf{e}_{i_1}^* \wedge \cdots \wedge \mathbf{e}_{i_p}^* : i_1 < \cdots < i_p\}$ span $\Lambda^p(E)$. One can show that they are linearly independent just as was done at the end of exercise 6. Thus

$$\{\mathbf{e}_{i_1}^* \wedge \cdots \wedge \mathbf{e}_{i_p}^* : i_1 < \cdots < i_p\}$$

is a basis of $\Lambda^p(E)$.

f) If $n \leq p$, there are $\binom{n}{p}$ ways of selecting p distinct indices from a set of n indices, and so $\dim(\Lambda^p(E)) = \binom{n}{p}$.

g) In the event where $p > n$, there is no way of selecting p distinct indices from a set of n indices, and so $\Lambda^p(E) = \{0\}$. ■

9. Let $f = f_1 \wedge f_2$ and $g = g_1 \wedge g_2$ be alternating $p-$forms over E. Work out the details and express $f \wedge g$ in terms of f and g, and show that $f \wedge g$ is alternating.

Proof: we have

$$\begin{aligned} A &= \{\sigma \in S_4 | \sigma(1) < \sigma(2) \ \underline{\text{and}}\ \sigma(3) < \sigma(4)\} \\ &= \{\text{id}, (2\,3), (2\,4\,3), (1\,2\,3), (1\,2\,4\,3), (1\,3)(2\,4)\} \end{aligned}$$

Consequently,

$$\begin{aligned} f \wedge g(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) &= \sum_{\sigma \in A} \epsilon(\sigma) f(\mathbf{x}_{\sigma(1)}, \mathbf{x}_{\sigma(2)}) g(\mathbf{x}_{\sigma(3)}, \mathbf{x}_{\sigma(4)}) \\ &= f(\mathbf{x}_1, \mathbf{x}_2) g(\mathbf{x}_3, \mathbf{x}_4) - f(\mathbf{x}_1, \mathbf{x}_3) g(\mathbf{x}_2, \mathbf{x}_4) + f(\mathbf{x}_1, \mathbf{x}_2) g(\mathbf{x}_3, \mathbf{x}_4) \\ &\quad + f(\mathbf{x}_2, \mathbf{x}_3) g(\mathbf{x}_1, \mathbf{x}_4) - f(\mathbf{x}_2, \mathbf{x}_4) g(\mathbf{x}_1, \mathbf{x}_3) + f(\mathbf{x}_3, \mathbf{x}_4) g(\mathbf{x}_1, \mathbf{x}_2) \end{aligned}$$

We can easily verify that $f \wedge g$ is alternating, using the fact that both f and g are alternating. ■

12.5 Exercises

1. Prepare a 2-page summary of this chapter, with important definitions and results.
2. Prove Proposition 166 for $p \geq 3$.
3. Let $f_1, \ldots, f_p \in E^*$. Show that if $f_i = f_j$ for some $i \neq j$, then $f_1 \wedge \cdots \wedge f_p = 0$.
4. Prove Theorem 167.
5. Show that if $p = q = 2$, the set $A \subseteq S_4$ contains only 6 permutations.
6. Let f, f_1, f_2 be alternating $p-$forms over E, g, g_1, g_2 be alternating $q-$forms over E, and $\alpha \in \mathbb{R}$. Show that
   a) $(f_1 + f_2) \wedge g = f_1 \wedge g + f_2 \wedge g$
   b) $f \wedge (g_1 + g_2) = f \wedge g_1 + f \wedge g_2$
   c) $(\alpha f) \wedge g = \alpha(f \wedge g) = f \wedge (\alpha g)$
7. Complete the proof of Lemma 169.
8. Let $f \in \Lambda^p(E)$, $g \in \Lambda^q(E)$, and $h \in \Lambda^r(E)$. Show that $f \wedge (g \wedge h) = (f \wedge g) \wedge h \in \Lambda^{p+q+r}(E)$ and that $f \wedge g = (-1)^{pq} g \wedge f$.
9. Prove property 3 on p. 304.
10. Let k be odd and $\omega \in \Lambda^k(\mathbb{R}^n)$. Show that $\omega \wedge \omega = 0$. Is the condition on k necessary, sufficient, or both?