Chapter 13

Differential Forms

In this chapter, we introduce the notion of **differential** $p-$**forms over** $\mathbb{R}^n$, which are derivatives of alternating linear $p-$forms over $\mathbb{R}^n$. This new notion is a generalization of the **differential** of a function and admits a number of applications in mathematical physics (Grand Unified Theories, Yang-Mills theory, superstring theory, etc.)

13.1 Differential $p-$Forms

We start by discussing the situation for $n = 3$. Let $U \subseteq_O \mathbb{R}^3$. A **differential** $1-$**form over** U is a function $U \to (\mathbb{R}^3)^*$; the set of all such differential forms is denoted $\Omega^1(U)$.

If $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the **canonical basis** of $\mathbb{R}^3$, then for any $\mathbf{w} \in \mathbb{R}^3$ we have

$$\mathbf{w} = w_1\mathbf{e}_1 + w_2\mathbf{e}_2 + w_3\mathbf{e}_3.$$

We denote the **dual basis** of $(\mathbb{R}^3)^*$ by $\{\,\mathrm{d}x,\ \mathrm{d}y,\ \mathrm{d}z\}$, which is to say that

$$\mathrm{d}x,\ \mathrm{d}y,\ \mathrm{d}z : \mathbb{R}^3 \to \mathbb{R} \quad \text{and} \quad \mathrm{d}x(\mathbf{w}) = w_1,\ \mathrm{d}y(\mathbf{w}) = w_2,\ \mathrm{d}z(\mathbf{w}) = w_3 \quad \text{for all } \mathbf{w} \in \mathbb{R}^3.$$

Then, if $\alpha \in (\mathbb{R}^3)^*$, there are unique $P, Q, R \in R$ such that

$$\alpha = P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z.$$

In general, if $\omega \in \Omega^1(U)$, $\exists! P, Q, R : U \to \mathbb{R}$ such that

$$\omega(\mathbf{u}) = P(\mathbf{u})\,\mathrm{d}x + Q(\mathbf{u})\,\mathrm{d}y + R(\mathbf{u})\,\mathrm{d}z, \quad \text{for all } \mathbf{u} \in U.$$

Let $f : U \to \mathbb{R}$ be differentiable on U; the **differential of** f is $\mathrm{d}f \in \Omega^1(U)$, where

$$\mathrm{d}f(\mathbf{u}) = \frac{\partial f}{\partial x}(\mathbf{u})\,\mathrm{d}x + \frac{\partial f}{\partial y}(\mathbf{u})\,\mathrm{d}y + \frac{\partial f}{\partial z}(\mathbf{u})\,\mathrm{d}z, \quad \text{for all } \mathbf{u} \in U.$$

Let $\omega \in \Omega^1(U)$. If the constituents $P, Q, R : U \to \mathbb{R}$ are continuous on U (respectively $\mathcal{C}^1$ or $\mathcal{C}^\infty$), then ω is continuous U (respectively $\mathcal{C}^1$ or $\mathcal{C}^\infty$).[1]

[1] These restrictions on P, Q, R make $\Omega^1(U)$ a $\mathcal{C}^0(U,\mathbb{R})-$ **module** (respectively, $\mathcal{C}^1(U,\mathbb{R})$ or $\mathcal{C}^\infty(U,\mathbb{R})$).

Proposition 172
$\Omega^1(U)$ is an infinite-dimensional vector space over $\mathbb{R}$.

Proof: left as an exercise. ■

If $U \subseteq_O \mathbb{R}^n$, $f : U \to \mathbb{R}$ is $\mathcal{C}^0$ (respectively $\mathcal{C}^1$ or $\mathcal{C}^\infty$) and ω is a $\mathcal{C}^0$ (respectively $\mathcal{C}^1$ or $\mathcal{C}^\infty$) differential $1-$form over U, then $f\omega \in \Omega^1(U)$, where

$$f\omega(\mathbf{u}) = f(\mathbf{u})P(\mathbf{u})\,\mathrm{d}x + f(\mathbf{u})Q(\mathbf{u})\,\mathrm{d}y + f(\mathbf{u})R(\mathbf{u})\,\mathrm{d}z, \quad \forall \mathbf{u} \in U.$$

A **differential** $p-$**form** ω **over** U is a map $\omega : U \to \Lambda^p(\mathbb{R}^n)$; the set of all such differential forms is denoted by $\Omega^p(U)$. If $p = 0$, $\Omega^0(U) = \mathbf{C}^k(U, \mathbb{R})$, where $k \in \{0, 1, \infty\}$; Corollary 168 shows that $\Omega^p(U) = \{0\}$ when $p > n$.

Proposition 173
$\Omega^p(U)$ is an infinite-dimensional vector space over $\mathbb{R}$ and a $\mathbf{C}^k(U)-$module (i.e., if $f \in \mathbf{C}^k(U, \mathbb{R})$ and $\omega \in \Omega^p(U)$, then $f\omega \in \Omega^p(U)$ for $k \in \{0, 1, \infty\}$.

Proof: left as an exercise. ■

Let $\omega_1 \in \Omega^{p_1}(U)$ and $\omega_2 \in \Omega^{p_2}(U)$. By definition, $\omega_i(\mathbf{u}) \in \Lambda^{p_i}(U)$ for all $\mathbf{u} \in U$, for $i = 1, 2$; according to Theorem 170, we must have

$$\omega_1(\mathbf{u}) \wedge \omega_2(\mathbf{u}) \in \Lambda^{p_1+p_2}(U),$$

and so the function $\omega_1 \wedge \omega_2 : U \to \Lambda^{p_1+p_2}(U)$ defined by

$$(\omega_1 \wedge \omega_2)(\mathbf{u}) = \omega_1(\mathbf{u}) \wedge \omega_2(\mathbf{u}), \quad \text{for all } \mathbf{u} \in U$$

is a differential $(p_1 + p_2)-$form over U, which is to say that $\omega_1 \wedge \omega_2 \in \Omega^{p_1+p_2}(U)$. This differential form is called the **we dge (or exterior) product of** ω_1 **and** ω_2.[2]

Example: if $n = 3$, we have

- $\Omega^0(U) = \{\omega = f \mid f \in \mathbf{C}^k(U, \mathbb{R})\}$;
- $\Omega^1(U) = \{\omega = f\,\mathrm{d}x + g\,\mathrm{d}y + h\,\mathrm{d}z \mid f, g, h \in \mathbf{C}^k(U, \mathbb{R})\}$;
- $\Omega^2(U) = \{\omega = f\,\mathrm{d}x \wedge \mathrm{d}y + g\,\mathrm{d}x \wedge \mathrm{d}z + h\,\mathrm{d}y \wedge \mathrm{d}z \mid f, g, h \in \mathbf{C}^k(U, \mathbb{R})\}$;
- $\Omega^3(U) = \{\omega = f\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \mid f \in \mathbf{C}^k(U, \mathbb{R})\}$, and
- $\Omega^p(U) = \{0\}$, when $p > 3$. □

[2]It is also sometimes denoted by $\omega_1\omega_2$.

Theorem 174

1. *For* $i = 1, 2$, *let* $\omega_i, \omega_i' \in \Omega^{p_i}(U)$ *and* $f : U \to \mathbb{R}$. *Then:*
   - $(\omega_1 + \omega_1') \wedge \omega_2 = \omega_1 \wedge \omega_2 + \omega_1' \wedge \omega_2$;
   - $\omega_1 \wedge (\omega_2 + \omega_2') = \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_2'$, *and*
   - $(f\omega_1) \wedge \omega_2 = f(\omega_1 \wedge \omega_2) = \omega_1 \wedge (f\omega_2)$.
2. *If* $\omega_1, \ldots, \omega_q \in \Omega^1(U)$, *then*
   - *when* $\omega_i = \omega_j$ *for some* $i \neq j$, *we have* $\omega_1 \wedge \cdots \wedge \omega_q = 0$;
   - *for* $\sigma \in S_q$, $\omega_{\sigma(1)} \wedge \cdots \wedge \omega_{\sigma(q)} = \epsilon(\sigma)\omega_1 \wedge \cdots \wedge \omega_q$.
3. *For* $i = 1, 2, 3$, *let* $\omega_i \in \Omega^{p_i}(U)$. *Then:*
   - $\omega_1 \wedge (\omega_2 \wedge \omega_3) = (\omega_1 \wedge \omega_2) \wedge \omega_3$, *and*
   - $\omega_1 \wedge \omega_2 = (-1)^{p_1 p_2} \omega_2 \wedge \omega_1$.

Proof: left as an exercise. ■

A few examples will help illustrate the main principles.

Examples: let $n = 3$, $f : U \to \mathbb{R}$, and set

$$\omega_1 = \mathrm{d}x_1 = \mathbf{e}_1^*, \quad \omega_2 = \mathrm{d}x_2 = \mathbf{e}_3^*, \quad \omega_3 = \mathrm{d}x_3 = \mathbf{e}_3^* \in \Omega^1(U).$$

- $\mathrm{d}x_1 \wedge \mathrm{d}x_2 = (-1)^{1\cdot 1}\,\mathrm{d}x_2 \wedge \mathrm{d}x_1$;
- $\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 = \mathrm{d}x_3 \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 = -\,\mathrm{d}x_1 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_2$;
- $\mathrm{d}x_1 \wedge \mathrm{d}x_1 = \mathrm{d}x_2 \wedge \mathrm{d}x_2 = \mathrm{d}x_3 \wedge \mathrm{d}x_3 = 0$, and
- $(f\,\mathrm{d}x_1 \wedge \mathrm{d}x_2) \wedge \mathrm{d}x_3 = (-1)^{2\cdot 1}\,\mathrm{d}x_3 \wedge (f\,\mathrm{d}x_1 \wedge \mathrm{d}x_2)$. □

This section's final result will set the stage for the rest of the chapter and the next one.

Theorem 175
Let $\omega \in \Omega^p(U)$. *We can uniquely write*

$$\omega = \sum P_{i_1,\cdots,i_p}\, dx_{i_1} \wedge \cdots \wedge dx_{i_p},$$

where $P_{i_1,\cdots,i_p} : U \to \mathbb{R}$ *for* $i_1 < \cdot < i_p$.

Proof: left as an exercise. ■

13.2 Exterior Derivative

In what follows, we fix $k = \infty$ so that $\Omega^p(U)$ represents the vector space of $\mathcal{C}^\infty$ (smooth) differential $p-$forms over $U \subseteq_O \mathbb{R}^n$.

The exterior derivative (or differential) of $\omega \in \Omega^p(U)$ is defined recursively.

1. If $f \in \Omega^0(U)$ (that is, $f : U \to \mathbb{R}$ is smooth), then its exterior derivative is
$$\mathrm{d}f = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\,\mathrm{d}x_i \in \Omega^1(U).$$

2. If $\omega = \sum_{i=1}^n P_i\,\mathrm{d}x_i \in \Omega^1(U)$, $P_i \in \mathcal{C}^\infty(U,\mathbb{R})$ for $1 \leq i \leq n$, then its exterior derivative is
$$\mathrm{d}\omega = \sum_{i=1}^n \mathrm{d}P_i \wedge \mathrm{d}x_i = \sum_{i=1}^n \left(\sum_{j=1}^n \frac{\partial P_i}{\partial x_j}\,\mathrm{d}x_j\right) \wedge \mathrm{d}x_i = \sum_{i<j}\left(\frac{\partial P_j}{\partial x_i} - \frac{\partial P_i}{\partial x_j}\right)\mathrm{d}x_i \wedge \mathrm{d}x_j \in \Omega^2(U).$$

   ...

p. In general, if
$$\omega = \sum_{i_1<\cdots<i_p} P_{i_1,\cdots,i_p}\,\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p} \in \Omega^p(U),$$
then its exterior derivative is
$$\mathrm{d}\omega = \sum_{i_1<\cdots<i_p} \mathrm{d}P_{i_1,\cdots,i_p} \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p} \in \Omega^{p+1}(U).$$

As we shall see after the next examples, the exterior derivative behaves as a regular derivative with respect to the sum of differential forms and to the product of functions, but there is a twist for a general product of differential forms.

Examples: throughout, let $f, g, h \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ for an appropriate n.

1. In $\mathbb{R}^2$, let $\omega = f\,\mathrm{d}x + g\,\mathrm{d}y \in \Omega^1(\mathbb{R}^2)$. Then
$$\begin{aligned}\mathrm{d}\omega &= \mathrm{d}f \wedge \mathrm{d}x + \mathrm{d}g \wedge \mathrm{d}y = \left(\frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y\right) \wedge \mathrm{d}x + \left(\frac{\partial g}{\partial x}\,\mathrm{d}x + \frac{\partial g}{\partial y}\,\mathrm{d}y\right) \wedge \mathrm{d}y \\ &= \frac{\partial f}{\partial x}\,\mathrm{d}x \wedge \mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y \wedge \mathrm{d}x + \frac{\partial g}{\partial x}\,\mathrm{d}x \wedge \mathrm{d}y + \frac{\partial g}{\partial y}\,\mathrm{d}y \wedge \mathrm{d}y \\ &= \frac{\partial f}{\partial x} \cdot 0 - \frac{\partial f}{\partial y}\,\mathrm{d}x \wedge \mathrm{d}y + \frac{\partial g}{\partial x}\,\mathrm{d}x \wedge \mathrm{d}y + \frac{\partial g}{\partial y} \cdot 0 = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathrm{d}x \wedge \mathrm{d}y \in \Omega^2(\mathbb{R}^2).\end{aligned}$$

2. In $\mathbb{R}^3$, let $\omega = f\,\mathrm{d}x + g\,\mathrm{d}y + h\,\mathrm{d}z \in \Omega^1(\mathbb{R}^3)$. Then

$$\begin{aligned}
\mathrm{d}\omega &= \mathrm{d}f \wedge \mathrm{d}x + \mathrm{d}g \wedge \mathrm{d}y + \mathrm{d}h \wedge \mathrm{d}z \\
&= \left(\frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y + \frac{\partial f}{\partial z}\,\mathrm{d}z\right) \wedge \mathrm{d}x + \left(\frac{\partial g}{\partial x}\,\mathrm{d}x + \frac{\partial g}{\partial y}\,\mathrm{d}y + \frac{\partial g}{\partial z}\,\mathrm{d}z\right) \wedge \mathrm{d}y + \\
&\quad = \left(\frac{\partial h}{\partial x}\,\mathrm{d}x + \frac{\partial h}{\partial y}\,\mathrm{d}y + \frac{\partial h}{\partial z}\,\mathrm{d}z\right) \wedge \mathrm{d}z \\
&= \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\,\mathrm{d}x \wedge \mathrm{d}y - \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\,\mathrm{d}x \wedge \mathrm{d}z + \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\,\mathrm{d}y \wedge \mathrm{d}z \in \Omega^2(\mathbb{R}^3).
\end{aligned}$$

3. In $\mathbb{R}^3$, let $\omega = f\,\mathrm{d}x \wedge \mathrm{d}y + g\,\mathrm{d}x \wedge \mathrm{d}z + h\,\mathrm{d}y \wedge \mathrm{d}z \in \Omega^2(\mathbb{R}^3)$. Then

$$\begin{aligned}
\mathrm{d}\omega &= \mathrm{d}f \wedge \mathrm{d}x \wedge d_y + \mathrm{d}g \wedge \mathrm{d}x \wedge \mathrm{d}z + \mathrm{d}h \wedge \mathrm{d}y \wedge \mathrm{d}z \\
&= \left(\frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y + \frac{\partial f}{\partial z}\,\mathrm{d}z\right) \wedge \mathrm{d}x \wedge \mathrm{d}y + \left(\frac{\partial g}{\partial x}\,\mathrm{d}x + \frac{\partial g}{\partial y}\,\mathrm{d}y + \frac{\partial g}{\partial z}\,\mathrm{d}z\right) \wedge \mathrm{d}x \wedge \mathrm{d}z + \\
&\quad = \left(\frac{\partial h}{\partial x}\,\mathrm{d}x + \frac{\partial h}{\partial y}\,\mathrm{d}y + \frac{\partial h}{\partial z}\,\mathrm{d}z\right) \wedge \mathrm{d}y \wedge \mathrm{d}z \\
&= \frac{\partial f}{\partial z}\,\mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}y + \frac{\partial g}{\partial y}\,\mathrm{d}y \wedge \mathrm{d}x \wedge \mathrm{d}z + \frac{\partial h}{\partial x}\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \\
&= \left(\frac{\partial f}{\partial z} - \frac{\partial g}{\partial y} + \frac{\partial h}{\partial x}\right)\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \in \Omega^3(\mathbb{R}^3). \qquad \square
\end{aligned}$$

Theorem 176
Let $\omega_1, \omega_2 \in \Omega^p(U)$. Then $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$.

Proof: left as an exception. ■

Lemma 177
If $f, g \in \Omega^0(\mathbb{R}^n)$, then $d(fg) = (df)g + f(dg)$.

Proof: the product $fg \in \Omega^0(\mathbb{R}^n)$ is itself a function $\mathbb{R}^n \to \mathbb{R}$. By definition,

$$\begin{aligned}
\mathrm{d}(fg) &= \sum_{i=1}^{n} \frac{\partial (fg)}{\partial x_i}\,\mathrm{d}x_i = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} g + f \frac{\partial g}{\partial x_i}\right)\,\mathrm{d}x_i \\
&= \left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\,\mathrm{d}x_i\right) g + f \left(\sum_{i=1}^{n} \frac{\partial g}{\partial x_i}\,\mathrm{d}x_i\right) = (\mathrm{d}f)g + f(\mathrm{d}g). \qquad ■
\end{aligned}$$

Lemma 177 is a special case (with $p = 0$) of the more general rule for the derivative of the product of differential forms.

Theorem 178
Let $\omega \in \Omega^p(U), \omega' \in \Omega^q(U)$*. Then* $d(\omega \wedge \omega') = d\omega \wedge \omega' + (-1)^p \omega \wedge d\omega'$*.*

Proof: if $\{i_1, \dots, i_\ell\} \subseteq \{1, \dots, n\}$ (in increasing order) and $f \in \mathcal{C}^\infty(U, \mathbb{R})$, then

$$\mathrm{d}(f\,\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_\ell}) = \mathrm{d}f \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_\ell}.$$

Since $\mathrm{d}(\omega_1 + \omega_2) = \mathrm{d}\omega_1 + \mathrm{d}\omega_2$, we only need to verify the conclusion for

$$\begin{aligned} \omega &= f\,\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p}, \quad i_1 < \cdots < i_p \\ \omega' &= g\,\mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_q}, \quad j_1 < \cdots < j_q. \end{aligned}$$

Then

$$\begin{aligned} \mathrm{d}(\omega \wedge \omega') &= \mathrm{d}(f\,\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p} \wedge g\,\mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_q}) \\ \boxed{\text{thm 174.1}} &= \mathrm{d}(fg\,\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p} \wedge \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_q}) \\ &= \mathrm{d}(fg) \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p} \wedge \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_q} \\ \boxed{\text{lemma 177}} &= [(\mathrm{d}f)g + f(\mathrm{d}g)] \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p} \wedge \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_q} \\ &= (\mathrm{d}f)g \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p} \wedge \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_q} \\ &\quad + f(\mathrm{d}g)\,\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p} \wedge \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_q} \\ &= \underbrace{\mathrm{d}f \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p}}_{=\mathrm{d}\omega} \wedge \underbrace{g\,\mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_q}}_{=\omega'} \\ &\quad + (-1)^p \underbrace{f\,\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p}}_{=\omega} \wedge \underbrace{\mathrm{d}g \wedge \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_q}}_{=\mathrm{d}\omega'} \\ &= \mathrm{d}\omega \wedge \omega' + (-1)^p \omega \wedge \mathrm{d}\omega'. \qquad \blacksquare \end{aligned}$$

We illustrate this in the case where $\omega = \sum_{i=1}^n f_i\,\mathrm{d}x_i \in \Omega^1(\mathbb{R}^n)$ and $\omega' = h \in \Omega^0(\mathbb{R}^n)$. Then

$$\omega \wedge \omega' = \sum_{i=1}^n f_i h\,\mathrm{d}x_i \quad \text{and} \quad \mathrm{d}(\omega \wedge \omega') = \mathrm{d}\left(\sum_{i=1}^n f_i h\,\mathrm{d}x_i\right) = \sum_{i=1}^n \mathrm{d}(f_i h\,\mathrm{d}x_i) = \sum_{i=1}^n \mathrm{d}(f_i h) \wedge \mathrm{d}x_i$$

$$\begin{aligned} &= \sum_{i=1}^n [(\mathrm{d}f_i)h + f_i(\mathrm{d}h)] \wedge \mathrm{d}x_i = \sum_{i=1}^n (\mathrm{d}f_i \wedge \mathrm{d}x_i)h + \sum_{i=1}^n f_i\,\mathrm{d}h \wedge \mathrm{d}x_i \\ &= \mathrm{d}\omega \wedge \omega' + \sum_{i=1}^n f_i(-\,\mathrm{d}x_i \wedge \mathrm{d}h) = \mathrm{d}\omega \wedge \omega' - \omega \wedge \mathrm{d}\omega' \\ &= \mathrm{d}\omega \wedge \omega' + (-1)^1 \omega \wedge \mathrm{d}\omega'. \end{aligned}$$

The next result showcases a crucial property of exterior derivatives.

Theorem 179
Let $\omega \in \Omega^p(U)$. Then $d(d\omega) = 0$.

Proof: if $f \in \mathcal{C}^\infty(U, \mathbb{R}) = \Omega^0(U)$, then $\mathrm{d}f \in \Omega^1(U)$ and

$$\mathrm{d}(\mathrm{d}f) = \mathrm{d}\left(\sum_{i=1}^n \frac{\partial f}{\partial x_i}\,\mathrm{d}x_i\right) = \sum_{i=1}^n \mathrm{d}\left(\frac{\partial f}{\partial x_i}\right) \wedge \mathrm{d}x_i = \sum_{i=1}^n \left(\sum_{j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}\,\mathrm{d}x_j\right) \wedge \mathrm{d}x_i.$$

When $i = j$, $\mathrm{d}x_i \wedge \mathrm{d}x_j = 0$; when $i > j$, $\mathrm{d}x_i \wedge \mathrm{d}x_j = -\,\mathrm{d}x_j \wedge \mathrm{d}x_i$, so that

$$\mathrm{d}^2 f = \sum_{i<j} \underbrace{\left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_j \partial x_i}\right)}_{=0 \text{ since } f \in \mathcal{C}^\infty(U,\mathbb{R})} \mathrm{d}x_i \wedge \mathrm{d}x_j = 0.$$

Furthermore,

$$\mathrm{d}(\mathrm{d}x_i) = \mathrm{d}(1 \cdot \mathrm{d}x_i) = d(1) \wedge \mathrm{d}x_i = 0 \wedge \mathrm{d}x_i = 0.$$

Since $\mathrm{d}(\omega + \omega') = \mathrm{d}\omega + \mathrm{d}\omega'$, it is sufficient to show that $\mathrm{d}^2(f\,\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p}) = 0$, where $\{i_1 < \ldots < i_p\} \subseteq \{1, \ldots, n\}$ and f is as above. As

$$\begin{aligned}\mathrm{d}(\mathrm{d}(f\,\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p})) &= \mathrm{d}(\mathrm{d}f \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p}) \\ &= \mathrm{d}(\mathrm{d}f) \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p} + (-1)^{0+1}\mathrm{d}f \wedge \mathrm{d}(\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p}) \\ &= 0 \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p} - \mathrm{d}f \wedge 0 = 0. \qquad \blacksquare\end{aligned}$$

A differential form $\omega \in \Omega^p(U)$ is **closed** if $\mathrm{d}\omega = 0$.

Example: let $n = 1$ and $\omega \in \Omega^1(\mathbb{R}^1)$. Then $\mathrm{d}\omega \in \Omega^2(\mathbb{R}^1)$; since $\Omega^2(\mathbb{R}^1) = \{0\}$, ω is automatically closed. $\square$

13.3 Antiderivative

Let $p > 1$, $U \subseteq_O \mathbb{R}^n$ and $\omega \in \Omega^p(U)$; ω is **exact** if $\exists \eta \in \Omega^{p-1}(U)$ such that $\mathrm{d}\eta = \omega$. The differential form η is an **antiderivative of** ω. If ω is exact, then $\mathrm{d}\omega = \mathrm{d}^2\eta = 0$ and so every exact form is also closed.

If $n = 1$, let $f \in \Omega^0(\mathbb{R})$. Then $\Omega^1(\mathbb{R}) = \{g\,\mathrm{d}x \mid g \in \Omega^0(\mathbb{R})\}$. If $F : \mathbb{R} \to \mathbb{R}$ is such that $F'(x) = f(x)$ for all $x \in \mathbb{R}$, then $F \in \Omega^0(\mathbb{R})$ and

$$dF = \frac{\partial F}{\partial x}\,\mathrm{d}x = f\,\mathrm{d}x.$$

Such an F exists by Theorem 60 since f is continuous on $\mathbb{R}$. Hence, every $\omega \in \Omega^1(\mathbb{R})$ is exact.

Examples

1. Let $\omega = P_1(x,y)\,\mathrm{d}x + P_2(x,y)\,\mathrm{d}y = y\,\mathrm{d}x - x\,\mathrm{d}y \in \Omega^1(\mathbb{R}^2)$. Since

$$\mathrm{d}\omega = \left(\frac{\partial P_2}{\partial x} - \frac{\partial P_1}{\partial y}\right)\mathrm{d}x \wedge \mathrm{d}y = (-1-1)\,\mathrm{d}x \wedge \mathrm{d}y = -2\,\mathrm{d}x \wedge \mathrm{d}y \neq 0;$$

since ω is not closed, it cannot be exact. □

2. Let $\omega = f(x,y)\,\mathrm{d}x + g(x,y)\,\mathrm{d}y = (3x^2+2xy+y^2)\,\mathrm{d}x + (x^2+2xy+3y^2)\,\mathrm{d}y \in \Omega^1(\mathbb{R}^2)$. Since

$$\begin{aligned}\mathrm{d}\omega &= \mathrm{d}f \wedge \mathrm{d}x + \mathrm{d}g \wedge \mathrm{d}y \\ &= \left(\frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y\right) \wedge \mathrm{d}x + \left(\frac{\partial g}{\partial x}\,\mathrm{d}x + \frac{\partial g}{\partial y}\,\mathrm{d}y\right) \wedge \mathrm{d}y \\ &= \frac{\partial f}{\partial y}\,\mathrm{d}y \wedge \mathrm{d}x + \frac{\partial g}{\partial x}\,\mathrm{d}x \wedge \mathrm{d}y = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathrm{d}x \wedge \mathrm{d}y.\end{aligned}$$

But

$$\frac{\partial g}{\partial x} = 2x + 2y = \frac{\partial f}{\partial y}$$

in this specific case, so $\mathrm{d}\omega = 0$, which means that ω is closed.

We can show that this particular closed form is also exact, which is to say that $\exists F \in \Omega^0(\mathbb{R}^2) = \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R})$ such that $\mathrm{d}F = \omega$. If such a F exists,

$$\mathrm{d}F = \frac{\partial F}{\partial x}\,\mathrm{d}x + \frac{\partial F}{\partial y}\,\mathrm{d}y = f(x,y)\,\mathrm{d}x + g(x,y)\,\mathrm{d}y,$$

and we must have

$$\frac{\partial F}{\partial x} = f(x,y) = 3x^2 + 2xy + y^2 \quad \text{and} \quad \frac{\partial F}{\partial y} = g(x,y) = x^2 + 2xy + 3y^2.$$

Integrating the first of these with respect to x yields

$$F(x,y) = x^3 + x^2y + y^2x + \varphi(y).$$

Differentiating with respect to y yields

$$\frac{\partial F}{\partial y} = x^3 + 2xy + \varphi'(y) = x^2 + 2xy + 3y^2,$$

so that $\varphi'(y) = 3y^2$, and so $\varphi(y) = y^3 + C$. Thus the antiderivatives of ω take the form

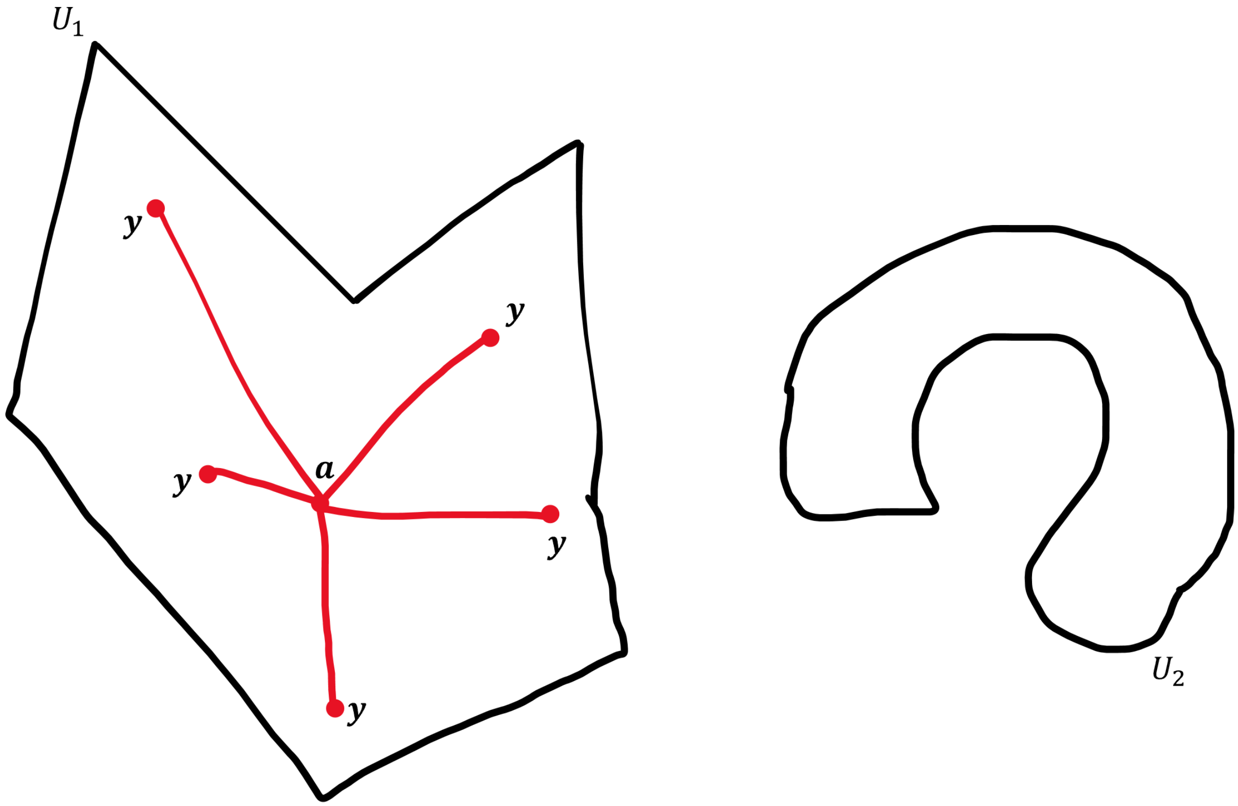
$$F(x,y) = x^3 + x^2y + xy^2 + y^3 + C,$$

where $C \in \mathbb{R}$. □

Exact forms are necessarily closed; the converse is valid when $U \subseteq_O \mathbb{R}^n$ has an additional property. A set $U \subseteq \mathbb{R}$ is **star-shaped** if $\exists \mathbf{a} \in U$ such that $\forall \mathbf{y} \in U$ we have

$$[\mathbf{a}, \mathbf{y}] = \{(1-t)\mathbf{a} + t\mathbf{y} \mid 0 \leq t \leq 1\} = \{\mathbf{a} + t(\mathbf{y} - \mathbf{a}) \mid 0 \leq t \leq 1\} \subseteq U.$$

In $\mathbb{R}^2$, for instance, U_1 (on the left) is star-shaped, whereas U_2 (on the right) is not.



We now present a highly technical lemma that will allow us to prove the desired result.

Theorem 180
Let $U \subseteq_O \mathbb{R}^n$, $I = [0,1]$, and $\varphi : U \times I \to \mathbb{R}$ a continuous function in the Euclidean metric. Then the function $\psi : U \to \mathbb{R}$ defined by

$$\psi(\mathbf{x} = \int_0^1 \varphi(\mathbf{x}, t) dt$$

is continuous.

Furthermore, if $D_{\mathbf{x}}\varphi : U \times I \to \text{End}(\mathbb{R}^n, \mathbb{R}) \simeq (\mathbb{R}^n)^*$ exists and is continuous, then ψ is C^1 and

$$D_{\mathbf{x}}\psi(\mathbf{x}) = \int_0^1 D_{\mathbf{x}}\varphi(\mathbf{x}, t) dt.$$

Proof: we start by proving the continuity of ψ. We want to show that $\forall \varepsilon > 0, \exists \delta_\varepsilon > 0$ such that

$$\|\mathbf{x} - \mathbf{x}'\| < \delta_\varepsilon \implies |\psi(\mathbf{x}) - \psi(\mathbf{x}')| < \varepsilon.$$

For $\mathbf{x}, \mathbf{x}' \in U$, we have

$$|\psi(\mathbf{x}) - \psi(\mathbf{x}')| = \left| \int_0^1 (\varphi(\mathbf{x}, t) - \varphi(\mathbf{x}', t)) dt \right| \leq \int_0^1 |\varphi(\mathbf{x}, t) - \varphi(\mathbf{x}', t)| dt.$$

Let $\varepsilon > 0$ and $(\mathbf{x}, t) \in U \times I$. Since φ is continuous, $\exists \delta_\varepsilon = \delta_\varepsilon(\mathbf{x}, t)$ such that

$$\|\mathbf{x} - \mathbf{x}'\|, |t - t'| < \delta_\varepsilon \implies |\varphi(\mathbf{x}, t) - \varphi(\mathbf{x}', t')| < \varepsilon/12.$$

In particular,

$$\|\mathbf{x} - \mathbf{x}'\| < \delta_\varepsilon \implies |\varphi(\mathbf{x}, t') - \varphi(\mathbf{x}', t')| < \varepsilon/6.$$

For a $\mathbf{x}$ fixed, define $V_t = \{t' \in \mathbb{R} \mid |t - t'| < \delta_\varepsilon(\mathbf{x}, t)\} \cap I$; then $\{V_t\}_{t \in I}$ is an open cover of the subspace $I \subseteq \mathbb{R}$. But I is a compact subspace of $\mathbb{R}$ in the Euclidean topology, and so there is a finite subcover $\{V_{t_1}, \ldots, V_{t_K}\}$ of I with

$$\bigcup_{i=1}^{K} V_{t_i} = I.$$

Let $\delta_\varepsilon(\mathbf{x}) = \min\{\delta(\mathbf{x}, t_i) \mid i = 1, \ldots, K\}$. Thus for any $t' \in I$, we can find a $t_i \in I$ such that $|t_i - t'| < \delta_\varepsilon(\mathbf{x}, t_i)$. If we also have $\|\mathbf{x} - \mathbf{x}'\| < \delta_\varepsilon(\mathbf{x})$, then

$$\begin{aligned} |\varphi(\mathbf{x}, t') - \varphi(\mathbf{x}', t')| &\leq |\varphi(\mathbf{x}, t') - \varphi(\mathbf{x}, t_i)| + |\varphi(\mathbf{x}, t_i) - \varphi(\mathbf{x}', t_i)| + |\varphi(\mathbf{x}', t_i) - \varphi(\mathbf{x}', t')| \\ &< \varepsilon/6 + \varepsilon/6 + \varepsilon/6 = \varepsilon/2. \end{aligned}$$

Set $\delta_\varepsilon = \delta_\varepsilon(\mathbf{x})$. Then for all $\mathbf{x}, \mathbf{x}' \in U$ we have

$$|\psi(\mathbf{x}) - \psi(\mathbf{x}')| \leq \int_0^1 \frac{\varepsilon}{2} dt = \frac{\varepsilon}{2} < \varepsilon.$$

We now tackle the differentiability of ψ. Since $D_{\mathbf{x}}\varphi$ is continuous by assumption, the same argument as above shows that the function

$$\mathbf{x} \in U \mapsto \lambda(\mathbf{x}) = \int_0^1 D_{\mathbf{x}}\varphi(\mathbf{x}, t) dt$$

is continuous. It remains only to show that $\lambda(x) = D_{\mathbf{x}}\psi(\mathbf{x})$, that is, $\forall \varepsilon > 0, \exists \delta_\varepsilon > 0$ such that

$$\|\mathbf{h}\| < \delta_\varepsilon \implies |\psi(\mathbf{x} + \mathbf{h}) - \psi(\mathbf{x}) - \lambda(\mathbf{x})\mathbf{h}| < \varepsilon \cdot \|\mathbf{h}\|.$$

But

$$|\psi(\mathbf{x} + \mathbf{h}) - \psi(\mathbf{x}) - \lambda(\mathbf{x})\mathbf{h}| = \left| \int_0^1 (\varphi(\mathbf{x} + \mathbf{h}, t) - \varphi(\mathbf{x}, t)) dt - \int_0^1 D_{\mathbf{x}}\varphi(\mathbf{x}, t)\mathbf{h} dt \right|$$

$$\leq \int_0^1 |\varphi(\mathbf{x}+\mathbf{h},t)-\varphi(\mathbf{x},t))-D_{\mathbf{x}}\varphi(\mathbf{x},t)\mathbf{h}|dt$$

$$\boxed{\text{Taylor's thm}} = \int_0^1 |D_{\mathbf{x}}\varphi(\mathbf{x}+\boldsymbol{\theta},t)-D_{\mathbf{x}}\varphi(\mathbf{x},t)|dt,$$

for $\boldsymbol{\theta} \in [\mathbf{0},\mathbf{h}]$. But $D_{\mathbf{x}}\varphi$ is continuous so $\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$ such that

$$\|\boldsymbol{\theta}\| \leq \|\mathbf{h}\| < \delta_\varepsilon \Longrightarrow |D_{\mathbf{x}}\varphi(\mathbf{x}+\boldsymbol{\theta},t)-D_{\mathbf{x}}\varphi(\mathbf{x},t)| < \varepsilon.$$

Hence

$$|\psi(\mathbf{x}+\mathbf{h})-\psi(\mathbf{x})-\lambda(\mathbf{x})\mathbf{h}| < \int_0^1 \varepsilon\|\mathbf{h}\|dt = \varepsilon\|\mathbf{h}\|,$$

which completes the proof. ■.

And now, the *pièce de résistance*.

Theorem 181 (Poincaré's Lemma)
Let $U \subseteq \mathbb{R}^n$ be star-shaped and containing **0**. *If $\omega \in \Omega^p(U)$ is closed, then it is exact.*

Proof: we start by proving the result for $n = 1$, $p = 1$. Let $\omega \in \Omega^1(U)$. Then $\omega = f\,\mathrm{d}x$, with $f \in \mathcal{C}^\infty(U,\mathbb{R})$. Since $\Omega^2(U) = \{0\}$, we have $\mathrm{d}\omega = 0 \in \Omega^2(U)$. We show that $\exists F \in \Omega^0(U)$ such that $\mathrm{d}F = \omega$.

Recall that

$$F(x) = \int_0^x f(t)dt = \int_0^1 f(xs)xds = \int_0^1 g(x,s)ds.$$

According to Lemma 180,

$$F'(x) = \int_0^1 \frac{\partial g}{\partial x}(x,s)ds = \int_0^1 (f(xs)+sf'(xs))ds$$
$$= \int_0^1 \frac{d}{ds}[sf(x,s)]ds = 1\cdot f(x,1) - 0\cdot f(x,0) = f(x).$$

Hence $\mathrm{d}F = \frac{\partial F}{\partial x}\,\mathrm{d}x = F'(x)\,\mathrm{d}x = f(x)\,\mathrm{d}x = \omega$.

Now suppose that $n > 1$, $p = 1$. Let $\omega \in \Omega^1(U)$ with $\mathrm{d}\omega = 0$. We want to show $\exists \eta = F \in \Omega^0(U) = \mathcal{C}^\infty(U,\mathbb{R})$ such that $\mathrm{d}\eta = \omega$. By hypothesis,

$$\omega = \sum_{i=1}^n f_i\,\mathrm{d}x_i, \quad \text{with } f_i \in \mathcal{C}^\infty(U,\mathbb{R})$$

and

$$\mathrm{d}\omega = \sum_{i=1}^{n} \mathrm{d}f_i \wedge \mathrm{d}x_i = \sum_{i,j=1}^{n} \frac{\partial f_i}{\partial x_j}\, \mathrm{d}x_i \wedge \mathrm{d}x_j = \sum_{i<j} \left(\frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} \right) \mathrm{d}x_i \wedge \mathrm{d}x_j = 0,$$

and so

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}, \quad \text{for all } 1 \leq i < j \leq n.$$

Let

$$F(\mathbf{x}) = F(x_1, \ldots, x_n) = \sum_{i=1}^{n} f_i(\mathbf{x}s) x_i ds = \sum_{i=1}^{n} \underbrace{f_i(x_1 s, \ldots, x_n s) x_i}_{=g_i(\mathbf{x},s)} ds.$$

We show that $\mathrm{d}F = \omega$:

$$\begin{aligned} \frac{\partial F}{\partial x_1}(\mathbf{x}) &= \sum_{i=1}^{n} \frac{\partial}{\partial x_1} \int_0^1 g_i(\mathbf{x}, s) ds = \sum_{i=1}^{n} \int_0^1 \frac{\partial}{\partial x_1} g_i(\mathbf{x}, s) ds \\ &= \int_0^1 \left[f_1(\mathbf{x}s) + x_1 s \frac{\partial}{\partial x_1} f(\mathbf{x}s) \right] ds + \sum_{j=2}^{n} \int_0^1 x_j s \frac{\partial}{\partial x_1} f_j(\mathbf{x}s) ds \\ &= \int_0^1 \left[f_1(\mathbf{x}s) + \sum_{j=1}^{n} x_j s \frac{\partial}{\partial x_j} f_1(\mathbf{x}s) \right] ds, \end{aligned}$$

by the equality of partial derivatives above. Set $k_1(s) = s f_1(\mathbf{x}s)$. Then

$$k_1'(s) = f_1(\mathbf{x}s) + \sum_{j=1}^{n} x_j s \frac{\partial}{\partial x_j} f_1(\mathbf{x}s),$$

so that

$$\frac{\partial F}{\partial x_1}(\mathbf{x}) = \int_0^1 k'(s) ds = k(1) - k(0) = f_1(\mathbf{x}).$$

In a similar fashion, we can see that

$$\frac{\partial F}{\partial x_i}(\mathbf{x}) = f_i(\mathbf{x}), \quad \text{for all } 1 \leq j \leq n,$$

and so

$$\mathrm{d}F = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}\, \mathrm{d}x_i = \sum_{i=1}^{n} f_i\, \mathrm{d}x_i = \omega.$$

We will not be providing the proof for $p > 1$. ■

Where exactly was the hypothesis that U is star-shaped used?[3]

[3]Hint: look at the definition of $F(x)$ (in the case $n = 1$) and $F(\mathbf{x})$ (in the case $n > 1$).

In a nutshell, we have shown the following result.

> **Proposition 182**
> *Let $U \subseteq_O \mathbb{R}$ and $\omega = \sum_{i=1}^n f_i\, dx_i \in \Omega^1(U)$. Consider the following conditions:*
>
> 1. *ω is exact in U;*
> 2. *ω is closed in U;*
> 3. *$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$ for all i, j.*
>
> Then $1. \implies 2. \iff 3.$ Furthermore, if U is star-shaped, then the three conditions are equivalent.

13.4 Pullback of a Differential Form

Let $U \subseteq_O \mathbb{R}^m$, $V \subseteq_O \mathbb{R}^n$, $\mathbf{g} \in \mathcal{C}^\infty(U, V)$.[4] The **pullback** function $\mathbf{g}^* : \Omega^k(V) \to \Omega^k(U)$ satisfies

$$\mathbf{g}^*(\bigwedge_i \omega_i) = \bigwedge_i \mathbf{g}^*(\omega_i).$$

We define it as follows.

Case $k = 0$: if $f \in \mathcal{C}^\infty(V, \mathbb{R}) = \Omega^0(V)$, the pullback is

$$\mathbf{g}^*(f) = f \circ \mathbf{g} : U \to \mathbb{R} \in \mathcal{C}^\infty(U, V) = \Omega^0(U).$$

Case $k = 1$: if a smooth $\mathbf{g} : U \subseteq_o \mathbb{R}^m \to V \subseteq_O \mathbb{R}^n$ maps

$$\mathbf{u} = (u_1, \ldots, u_m) \in U \mapsto \mathbf{v} = \mathbf{g}(\mathbf{u}) = (g_1(\mathbf{u}), \ldots, g_n(\mathbf{u})) \in V,$$

and $\omega \in \Omega^1(V)$, then $\omega = \sum_{i=1}^n f_i\, \mathrm{d}x_i$ and the pullback is

$$\mathbf{g}^*(\omega) = \sum_{i=1}^n \mathbf{g}^*(f_i)\mathbf{g}^*(\mathrm{d}x_i) = \sum_{i=1}^n (f_i \circ \mathbf{g})\, \mathrm{d}g_i = \sum_{i=1}^n (f_i \circ \mathbf{g}) \left(\sum_{j=1}^m \frac{\partial g_i}{\partial u_j}\, \mathrm{d}u_j \right).$$

Let us take a look at some examples.

> **Examples**
>
> 1. Let $\mathbf{g} : U = \mathbb{R} \to V = \mathbb{R}$ and consider $\omega = f\, \mathrm{d}x \in \Omega^1(V)$. Then the pullback $\mathbf{g}^*(\omega) \in \Omega^1(U)$ is given by
>
> $$\mathbf{g}^*(\omega)(u) = (f \circ g)\mathbf{g}^*(\mathrm{d}x)(u) = f(\mathbf{g}(u)) \cdot \mathbf{g}'(u)\, \mathrm{d}u. \qquad \square$$

[4] We will encounter such functions when we discuss vector fields in Section 13.5.

2. Let $\mathbf{g} : U = \mathbb{R} \to V = \mathbb{R}^2$ be defined by

$$\mathbf{g}(t) = (\cos t, \sin t)$$

and $\omega = -y\,\mathrm{d}x + x\,\mathrm{d}y \in \Omega^1(V)$. Then

$$\mathbf{g}^*(\mathrm{d}x)(t) = (\mathrm{d}g_1)(t) = -\sin t\,\mathrm{d}t, \quad \mathbf{g}^*(\mathrm{d}y)(t) = (\mathrm{d}g_2)(t) = \cos t\,\mathrm{d}t,$$

and the pullback $\mathbf{g}^*(\omega) \in \Omega^1(U)$ is given by

$$\begin{aligned}\mathbf{g}^*(\omega)(t) &= f_1(\mathbf{g}(t))(\mathrm{d}g_1)(t) + f_2(\mathbf{g}(t))(\mathrm{d}g_2)(t) \\ &= (-\sin t)(-\sin t\,\mathrm{d}t) + (\cos t)(\cos t\,\mathrm{d}t) = (\sin^2 t + \cos^2 t)\,\mathrm{d}t = \mathrm{d}t. \qquad \square\end{aligned}$$

3. Let $\mathbf{g} : U = \mathbb{R}^2 \to V = \mathbb{R}^2$ be defined by

$$\mathbf{g}(\mathbf{u}) = (g_1(u_1,u_2), g_2(u_1,u_2)) = (u_1 \cos u_2, u_1 \sin u_2)$$

and $\omega = f_1(x_1,x_2)\,\mathrm{d}x_1 + f_2(x_1,x_2)\,\mathrm{d}x_2 = x_1\,\mathrm{d}x_1 + x_2\,\mathrm{d}x_2 \in \Omega^1(V)$. Then

$$\begin{aligned}\mathbf{g}^*(\mathrm{d}x_1)(u_1,u_2) &= (\mathrm{d}g_1)(u_1,u_2) = \frac{\partial g_1(u_1,u_2)}{\partial u_1}\,\mathrm{d}u_1 + \frac{\partial g_1(u_1,u_2)}{\partial u_2}\,\mathrm{d}u_2 \\ &= \cos u_2\,\mathrm{d}u_1 - u_1 \sin u_2\,\mathrm{d}u_2 \\ \mathbf{g}^*(\mathrm{d}x_2)(u_1,u_2) &= (\mathrm{d}g_2)(u_1,u_2) = \frac{\partial g_2(u_1,u_2)}{\partial u_1}\,\mathrm{d}u_1 + \frac{\partial g_2(u_1,u_2)}{\partial u_2}\,\mathrm{d}u_2 \\ &= \sin u_2\,\mathrm{d}u_1 + u_1 \cos u_2\,\mathrm{d}u_2,\end{aligned}$$

and the pullback $\mathbf{g}^*(\omega) \in \Omega^1(U)$ is given by

$$\begin{aligned}\mathbf{g}^*(\omega)(u_1,u_2) &= f_1(\mathbf{g}(u_1,u_2))(\mathrm{d}g_1)(u_1,u_2) + f_2(\mathbf{g}(u_1,u_2))(\mathrm{d}g_2)(u_1,u_2) \\ &= u_1 \cos u_2(\cos u_2\,\mathrm{d}u_1 - u_1 \sin u_2\,\mathrm{d}u_2) + u_1 \sin u_2(\sin u_2\,\mathrm{d}u_1 + u_1 \cos u_2\,\mathrm{d}u_2) \\ &= u_1(\cos^2 u_2 + \sin^2 u_2), \mathrm{d}u = u_1\,\mathrm{d}u_1. \qquad \square\end{aligned}$$

Case $k > 1$: if $\mathbf{g} : U \subseteq_O \mathbb{R}^m \to V \subseteq_O \mathbb{R}^n$ is smooth and $\omega = \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k} \in \Omega^k(V)$, we define the pullback

$$\mathbf{g}^*(\omega) = \mathbf{g}^*(\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}) = \mathrm{d}g_{i_1} \wedge \cdots \wedge \mathrm{d}g_{i_k} \in \Omega^k(U).$$

If

$$\omega = \sum_{i_1<\cdots<i_k} P_{i_1,\cdots,i_k} \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k} \in \Omega^k(V),$$

then the pullback is

$$\mathbf{g}^*(\omega) = \sum_{i_1<\cdots<i_k} \mathbf{g}^*(P_{i_1,\cdots,i_k})\mathbf{g}^*(\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}) = \sum_{i_1<\cdots<i_k} (P_{i_1,\cdots,i_k} \circ \mathbf{g})\mathrm{d}g_{i_1} \wedge \cdots \wedge \mathrm{d}g_{i_k} \in \Omega^k(U).$$

Example: let $\mathbf{g} : U = \mathbb{R}^2 \to V = \mathbb{R}^2$ be defined by

$$\mathbf{g}(\mathbf{u}) = (g_1(u_1, u_2), g_2(u_1, u_2)) = (u_1 \cos u_2, u_1 \sin u_2)$$

and $\omega = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \in \Omega^2(V)$. Then

$$(\mathrm{d}g_1)(u_1, u_2) = \cos u_2 \,\mathrm{d}u_1 - u_1 \sin u_2 \,\mathrm{d}u_2, \quad (\mathrm{d}g_2)(u_1, u_2) = \sin u_2 \,\mathrm{d}u_1 + u_1 \cos u_2 \,\mathrm{d}u_2,$$

and the pullback $\mathbf{g}^*(\omega) \in \Omega^2(U)$ is given by

$$\begin{aligned}\mathbf{g}^*(\omega)(u_1, u_2) &= \mathbf{g}^*(\mathrm{d}x_1 \wedge \mathrm{d}x_2)(u_1, u_2) = (\mathrm{d}g_1)(u_1, u_2) \wedge (\mathrm{d}g_2)(u_1, u_2) \\ &= (\cos u_2 \,\mathrm{d}u_1 - u_1 \sin u_2 \,\mathrm{d}u_2) \wedge (\sin u_2 \,\mathrm{d}u_1 + u_1 \cos u_2 \,\mathrm{d}u_2) \\ &= u_1 \cos^2 u_2 \,\mathrm{d}u_1 \wedge \mathrm{d}u_2 - u_1 \sin^2 u_2 \,\mathrm{d}u_2 \wedge \mathrm{d}u_1 \\ &= u_1(\cos^2 u_2 + \sin^2 u_2)\,\mathrm{d}u_1 \wedge \mathrm{d}u_2 = u_1 \,\mathrm{d}u_1 \wedge \mathrm{d}u_2. \qquad \square\end{aligned}$$

While none of the computations are particularly difficult to perform (although they can be tedious), there is a simpler way to express pullbacks, as the following discussion illustrates.

If $\mathbf{g} : U = \mathbb{R}^2 \to V = \mathbb{R}^2$ is smooth, then the pullback of $\mathrm{d}x_1 \wedge \mathrm{d}x_2 \in \Omega^2(V)$ by $\mathbf{g}$ is

$$\begin{aligned}\mathbf{g}^*(\mathrm{d}x_1 \wedge \mathrm{d}x_2) = \mathrm{d}g_1 \wedge \mathrm{d}g_2 &= \left(\frac{\partial g_1}{\partial u_1}\,\mathrm{d}u_1 + \frac{\partial g_1}{\partial u_2}\,\mathrm{d}u_2\right) \wedge \left(\frac{\partial g_2}{\partial u_1}\,\mathrm{d}u_1 + \frac{\partial g_2}{\partial u_2}\,\mathrm{d}u_2\right) \\ &= \left(\frac{\partial g_1}{\partial u_1}\frac{\partial g_2}{\partial u_2} - \frac{\partial g_1}{\partial u_2}\frac{\partial g_2}{\partial u_1}\right)\mathrm{d}u_1 \wedge \mathrm{d}u_2 = \det(D\mathbf{g})\,\mathrm{d}u_1 \wedge \mathrm{d}u_2 \in \Omega^2(U),\end{aligned}$$

where $D\mathbf{g}$ is the **Jacobian matrix of g** (see Section 21.7).

Generally, if $\mathbf{g} : U \subseteq_O \mathbb{R}^m \to V \subseteq_O \mathbb{R}^m$ is smooth, then the pullback of $\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k} \in \Omega^k(V)$ by $\mathbf{g}$ is

$$\begin{aligned}\mathbf{g}^*(\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}) = \mathrm{d}g_{i_1} \wedge \cdots \wedge \mathrm{d}g_{i_k} &= \left(\sum_{j=1}^m \frac{\partial g_{i_1}}{\partial u_j}\,\mathrm{d}u_j\right) \wedge \cdots \wedge \left(\sum_{j=1}^m \frac{\partial g_{i_k}}{\partial u_j}\,\mathrm{d}u_j\right) \\ &= \sum_{j_1 < \cdots < j_k} \det \begin{pmatrix} \frac{\partial g_{i_1}}{\partial u_{j_1}} & \cdots & \frac{\partial g_{i_1}}{\partial u_{j_k}} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_{i_k}}{\partial u_{j_1}} & \cdots & \frac{\partial g_{i_k}}{\partial u_{j_k}} \end{pmatrix} \mathrm{d}u_{j_1} \wedge \cdots \wedge \mathrm{d}u_{j_k} \in \Omega^k(U).\end{aligned}$$

If $U, V \subseteq_O \mathbb{R}^n$, $g : U \to V$ smooth, $f \in \mathcal{C}^\infty(V, \mathbb{R})$, and $\omega = f\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \in \Omega^n(V)$, then the pullback of ω by $\mathbf{g}$ is

$$\mathbf{g}^*(\omega) = (f \circ \mathbf{g})\,\mathrm{d}g_1 \wedge \cdots \wedge \mathrm{d}g_n = \mathbf{g}^*(f)\det(D\mathbf{g})\,\mathrm{d}u_1 \wedge \cdots \wedge \mathrm{d}u_n \in \Omega^n(U).$$

The pullback commutes with the exterior derivative for $0-$differential forms.

Lemma 183
With the usual assumptions of this section, if $f \in \Omega^0(V)$, then $d(\mathbf{g}^*(f)) = \mathbf{g}^*(df)$.

Proof: we use the definition and see that

$$\begin{aligned} \mathrm{d}(\mathbf{g}^*(f)) = \mathrm{d}(f \circ \mathbf{g}) &= \sum_{j=1}^{m} \frac{\partial (f \circ \mathbf{g})}{\partial u_j}\, \mathrm{d}u_j = \sum_{j=1}^{m} \left(\sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \circ \mathbf{g} \right) \frac{\partial g_i}{\partial u_j} \right) \mathrm{d}u_j \\ &= \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \circ \mathbf{g} \right) \left(\sum_{j=1}^{m} \frac{\partial g_i}{\partial u_j}\, \mathrm{d}u_j \right) = \mathbf{g}^* \left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\, \mathrm{d}x_i \right) = \mathbf{g}^*(\mathrm{d}f), \end{aligned}$$

which completes the proof. ■

But this result does not apply solely to $\Omega^0(V)$.

Proposition 184
Let $\mathbf{g} : U \subseteq_O \mathbb{R}^m \to V \subseteq_O \mathbb{R}^m$ be smooth. If $\omega \in \Omega^0(V)$, then $d(\mathbf{g}^*(\omega)) = \mathbf{g}^*(d\omega)$.

Proof: the case $k = 0$ was proven in Lemma 183. For $k > 0$, since $\mathrm{d}(\omega_1 + \omega_2) = \mathrm{d}\omega_1 + \mathrm{d}\omega_2$ and

$$\omega = \sum_{i_1 < \cdots < i_k} f_{i_1,\ldots,i_k}\, \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_l}, \quad f_{i_1,\ldots,i_k} \in \Omega^0(V),$$

it is sufficient to show that

$$\mathbf{g}^*(\mathrm{d}(f\, \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k})) = \mathrm{d}\left(\mathbf{g}^*(f\, \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k})\right).$$

But the left side of this equation reduces to

$$\begin{aligned} \mathbf{g}^*(\mathrm{d}(f\, \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k})) &= \mathbf{g}^*(\mathrm{d}f) \wedge \mathbf{g}^*(\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}) \\ \boxed{\text{lemma 183}} &= \mathrm{d}(\mathbf{g}^*(f)) \wedge \mathbf{g}^*(\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}) \\ &= \mathrm{d}(\mathbf{g}^*(f)) \wedge (\mathrm{d}g_{i_1} \wedge \cdots \wedge \mathrm{d}g_{i_k}). \end{aligned}$$

Thanks to repeated use of Theorem 177, the right side, on the other hand, reduces to

$$\begin{aligned} \mathrm{d}(f \circ \mathbf{g}\, \mathrm{d}g_{i_1} \wedge \cdots \wedge \mathrm{d}g_{i_k}) &= \mathrm{d}(f \circ \mathbf{g}) \wedge \mathrm{d}g_{i_1} \wedge \cdots \wedge \mathrm{d}g_{i_k} + (-1)^0 (f \circ \mathbf{g})\, \underbrace{\mathrm{d}(\mathrm{d}g_{i_1} \wedge \cdots \wedge \mathrm{d}g_{i_k})}_{=0} \\ &= \mathrm{d}(f \circ \mathbf{g}) \wedge \mathrm{d}g_{i_1} \wedge \cdots \wedge \mathrm{d}g_{i_k}. \end{aligned}$$ ■

The machinery we have developed up to now may seem hopelessly formal and mechanical; its practical value comes through once we identify differential forms with vector fields.

13.5 Vector Fields

Let $U \subseteq_O \mathbb{R}^n$. A **vector field** is a function $\mathbf{F} : U \to \mathbb{R}^n$; it is of class $\mathbf{C}^k$ if $\mathbf{F} \in \mathcal{C}^k(U, \mathbb{R}^n)$. A function $f : U \to \mathbb{R}$ is called a **scalar field.**

> **Example:** let $f : U \to \mathbb{R}$ be continuously differentiable and consider $\nabla f : U \to \mathbb{R}^n$ defined by
> $$\nabla f(\mathbf{u}) = \left(\frac{\partial f(\mathbf{u})}{\partial x_1}, \ldots, \frac{\partial f(\mathbf{u})}{\partial x_n}\right).$$
> Then f is a scalar field and ∇f is a vector field.

We can associate to any vector field $\mathbf{F} : U \to \mathbb{R}^n$, defined by $\mathbf{F}(\mathbf{x}) = (F_1(\mathbf{x}), \ldots, F_n(\mathbf{x}))$ a unique differential form $\omega_{\mathbf{F}} \in \Omega^1(U)$ defined by

$$\omega_{\mathbf{F}} = F_1 \, \mathrm{d}x_1 + \cdots + F_n \, \mathrm{d}x_n.$$

In particular, if $f : U \to \mathbb{R}$ is smooth, the differential form associated to ∇f is

$$\omega_{\nabla f} = \frac{\partial f}{\partial x_1} \, \mathrm{d}x_1 + \cdots + \frac{\partial f}{\partial x_n} \, \mathrm{d}x_n = \mathrm{d}f \in \Omega^1(U).$$

> **Theorem 185**
> *Let $U \subseteq_O \mathbb{R}^n$ and $\mathbf{F} : U \to \mathbb{R}^n$ be smooth. Consider the following conditions:*
>
> 1. *$\mathbf{F} = \nabla f$ for some $f : U \to \mathbb{R}$ smooth;*
> 2. *$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$ for all i, j.*
>
> *Then $1. \implies 2.$ If U is star-shaped then, the conditions are equivalent.*
>
> **Proof:** if $\mathbf{F} = \nabla f$, then $\omega_{\mathbf{F}} = \omega_{\nabla f} = \mathrm{d}f \in \Omega^1(U)$ is exact and so condition 2. holds according to Proposition 182.
>
> If U is star-shaped and $\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$ for all i, j, then $\omega_{\mathbf{F}} = F_1 \, \mathrm{d}x_1 + \cdots + \mathrm{d}x_n$ is exact (again, by Theorem 182), so that
> $$\omega_{\mathbf{F}} = \mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \, \mathrm{d}x_i$$
> for some $f : U \to \mathbb{R} \in \Omega^0(U)$. By unicity of $\omega_{\mathbf{F}}$, we must have $F_i = \frac{\partial f}{\partial x_i}$ for all i, which is to say that $\mathbf{F} = \nabla f$. ■

When $\mathbf{F} = \nabla f$, we say that $\mathbf{F}$ is a **conservative vector field** (or a **gradient field**) and that f is a **scalar potential** for $\mathbf{F}$.

Until the end of the chapter, we work with vector fields $\mathbf{F} : U \subseteq_O \mathbb{R}^3 \to \mathbb{R}^3$. Recall that, seen as a vector field over $\mathbb{R}$,

$$\dim\left(\Lambda^p(\mathbb{R}^n)\right) = \binom{n}{p},$$

according to Corollary 168; in that case, we have

$$\dim(\Lambda^1(\mathbb{R}^3)) = \dim(\Lambda^2(\mathbb{R}^3)), \quad \dim(\Lambda^0(\mathbb{R}^3)) = \dim(\Lambda^3(\mathbb{R}^3)) = 1.$$

Consider the vector space isomorphism $\Phi_1 : \mathbb{R}^3 \to \Lambda^1(\mathbb{R}^3)$ defined by

$$\Phi_1(\mathbf{a}) = \Phi_1(a_1, a_2, a_3) = a_1\,\mathrm{d}x_1 + a_2\,\mathrm{d}x_2 + a_3\,\mathrm{d}x_3.$$

If we "multiply" two vectors in $\mathbb{R}^3$, we should get the same "result" as if we "multiply" two $1-$forms over $\mathbb{R}^3$; the problem is that we while the wedge product can play the role of a multiplication, the wedge product of two $1-$forms over $\mathbb{R}^3$ is a $2-$form over $\mathbb{R}^3$.

Over other spaces this would be a deal-breaker, but over $\mathbb{R}^3$ the problem evaporates once we introduce a second vector space isomorphism $\Phi_2 : \mathbb{R}^3 \to \Lambda^2(\mathbb{R}^3)$, defined by

$$\Phi_2(\mathbf{a}) = \Phi_2(a_1, a_2, a_3) = a_1\,\mathrm{d}x_2 \wedge \mathrm{d}x_3 + a_2\,\mathrm{d}x_3 \wedge \mathrm{d}x_1 + a_3\,\mathrm{d}x_1 \wedge \mathrm{d}x_2,$$

and define the **cross-product** over $\mathbb{R}^3$ by

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= (a_1, a_2, a_3) \times (b_1, b_2, b_3) \\
&\simeq \Phi_1(a_1, a_2, a_3) \wedge \Phi_1(b_1, b_2, b_3) \\
&= (a_1\,\mathrm{d}x_1 + a_2\,\mathrm{d}x_2 + a_3\,\mathrm{d}x_3) \wedge (b_1\,\mathrm{d}x_1 + b_2\,\mathrm{d}x_2 + b_3\,\mathrm{d}x_3) \\
&= (a_2b_3 - a_3b_2)\,\mathrm{d}x_2 \wedge \mathrm{d}x_3 + (a_3b_1 - a_1b_3)\,\mathrm{d}x_1 \wedge \mathrm{d}x_2 + (a_1b_2 - a_2b_1)\,\mathrm{d}x_1 \wedge \mathrm{d}x_2 \\
&\simeq \Phi_2^{-1}((a_2b_3 - a_3b_2)\,\mathrm{d}x_2 \wedge \mathrm{d}x_3 + (a_3b_1 - a_1b_3)\,\mathrm{d}x_1 \wedge \mathrm{d}x_2 + (a_1b_2 - a_2b_1)\,\mathrm{d}x_1 \wedge \mathrm{d}x_2) \\
&= (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1),
\end{aligned}$$

which should go some way towards elucidating the mystery of where the apparently random definition of the cross-product come from when it is first introduced in linear algebra courses.

In applications, it is typical to use $x = x_1$, $y = x_2$, and $z = x_3$. In that case, we could also write the vector field $\mathbf{F} : U \to \mathbb{R}^3$ as

$$\mathbf{F}(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z));$$

the composition

$$\Phi_1 \circ \mathbf{F} = \omega_{\mathbf{F}} = P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z \in \Omega^1(U)$$

is the corresponding differential $1-$form over U.

Then, we have:

$$\begin{aligned}
\mathrm{d}\omega_{\mathbf{F}} &= \mathrm{d}P \wedge \mathrm{d}x + \mathrm{d}Q \wedge \mathrm{d}y + \mathrm{d}R \wedge \mathrm{d}z \\
&= \left(\frac{\partial P}{\partial x}\,\mathrm{d}x + \frac{\partial P}{\partial y}\,\mathrm{d}y + \frac{\partial P}{\partial z}\,\mathrm{d}z \right) \wedge \mathrm{d}x + \left(\frac{\partial Q}{\partial x}\,\mathrm{d}x + \frac{\partial Q}{\partial y}\,\mathrm{d}y + \frac{\partial Q}{\partial z}\,\mathrm{d}z \right) \wedge \mathrm{d}y \\
&\quad + \left(\frac{\partial R}{\partial x}\,\mathrm{d}x + \frac{\partial R}{\partial y}\,\mathrm{d}y + \frac{\partial R}{\partial z}\,\mathrm{d}z \right) \wedge \mathrm{d}z \\
&= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathrm{d}y \wedge \mathrm{d}z + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathrm{d}z \wedge \mathrm{d}x + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x \wedge \mathrm{d}y \in \Omega^2(U).
\end{aligned}$$

The vector field $\Phi_2^{-1}(\mathrm{d}\omega_{\mathbf{F}}) = \Phi_2^{-1}(\Phi_1(\mathbf{F}))$ associated with $\mathrm{d}\omega_{\mathbf{F}}$ is the **curl of F** and is denoted by $\mathrm{curl}(\mathbf{F}) = \nabla \times \mathbf{F} : U \to \mathbb{R}^3$ and

$$\mathrm{curl}(\mathbf{F}) = \nabla \times \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right).$$

Theorem 186
Let $U =\subseteq_O \mathbb{R}^3$ and $\mathbf{F} : U \to \mathbb{R}^3$ be smooth. Consider the following conditions:

1. $\mathbf{F} = \nabla f$ *for some smooth* $f : U \to \mathbb{R}$;
2. $\nabla \times \mathbf{F} = \mathbf{0}$.

Then $1. \implies 2.$ *If U is star-shaped then, the conditions are equivalent.*

Proof: direct application of Theorem 185. ■

If instead we consider the composition

$$\Phi_2 \circ \mathbf{F} = \varphi_{\mathbf{F}} = P\,\mathrm{d}y \wedge \mathrm{d}x + Q\,\mathrm{d}z \wedge \mathrm{d}x + R\,\mathrm{d}x \wedge \mathrm{d}z \in \Omega^2(U),$$

then we have

$$\begin{aligned}
\mathrm{d}\varphi_{\mathbf{F}} &= \mathrm{d}P \wedge \mathrm{d}y \wedge \mathrm{d}z + \mathrm{d}Q \wedge \mathrm{d}z \wedge \mathrm{d}x + \mathrm{d}R \wedge \mathrm{d}x \wedge \mathrm{d}y \\
&= \left(\frac{\partial P}{\partial x}\,\mathrm{d}x + \frac{\partial P}{\partial y}\,\mathrm{d}y + \frac{\partial P}{\partial z}\,\mathrm{d}z \right) \wedge \mathrm{d}y \wedge \mathrm{d}z + \left(\frac{\partial Q}{\partial x}\,\mathrm{d}x + \frac{\partial Q}{\partial y}\,\mathrm{d}y + \frac{\partial Q}{\partial z}\,\mathrm{d}z \right) \wedge \mathrm{d}z \wedge \mathrm{d}x \\
&\quad + \left(\frac{\partial R}{\partial x}\,\mathrm{d}x + \frac{\partial R}{\partial y}\,\mathrm{d}y + \frac{\partial R}{\partial z}\,\mathrm{d}z \right) \wedge \mathrm{d}x \wedge \mathrm{d}y \\
&= \frac{\partial P}{\partial x}\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z + \frac{\partial Q}{\partial y}\,\mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}x + \frac{\partial R}{\partial z}\,\mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}y \\
&= \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \in \Omega^3(U).
\end{aligned}$$

The scalar field associated with $\mathrm{d}\varphi_{\mathbf{F}}$ is the **divergence of F** and is denoted by $\mathrm{div}(\mathbf{F}) = \nabla \cdot \mathbf{F} : U \to \mathbb{R}$ and

$$\mathrm{div}(\mathbf{F}) = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

As a consequence of Poincaré's lemma, we obtain the following result.

Theorem 187
Let $U =\subseteq_O \mathbb{R}^3$ and $\mathbf{F} : U \to \mathbb{R}^3$ be smooth. If there is a vector field $\mathbf{G} : U \to \mathbb{R}^3$ such that curl$(\mathbf{G}) = \nabla \times \mathbf{G} = \mathbf{F}$, then div$(\mathbf{F}) = \nabla \cdot \mathbf{F} = 0$. If U is star-shaped and div$(\mathbf{F}) = \nabla \cdot \mathbf{F} = 0$, then there is a $\mathbf{G} : U \to \mathbb{R}^3$ such that curl$(\mathbf{G}) = \nabla \times \mathbf{G} = \mathbf{F}$.

Proof: let $\omega_\mathbf{G} \in \Omega^1(U)$ and $\varphi_\mathbf{F} \in \Omega^2(U)$ be the associated differential forms. If $\text{curl}(\mathbf{G}) = \mathbf{F}$, then $\text{d}\omega_\mathbf{G} = \varphi_\mathbf{F}$, so that $\text{d}\varphi_\mathbf{F} = \text{d}(\text{d}\omega_\mathbf{G}) = 0$, and thus $\text{div}(\mathbf{F}) = 0$.

If U is star-shaped and $\text{div}(\mathbf{F}) = 0$, then $\text{d}\varphi_\mathbf{F} = 0$, and so $\varphi_\mathbf{F}$ is closed. According to Poincaré's lemma, $\varphi_\mathbf{F}$ is exact, which is to say that $\exists \omega \in \Omega^1(U)$ such that $\text{d}\omega = \varphi_\mathbf{F}$. If $\mathbf{G}$ is the vector field corresponding to ω, then we have $\text{curl}(\mathbf{G}) = \mathbf{F}$. ■

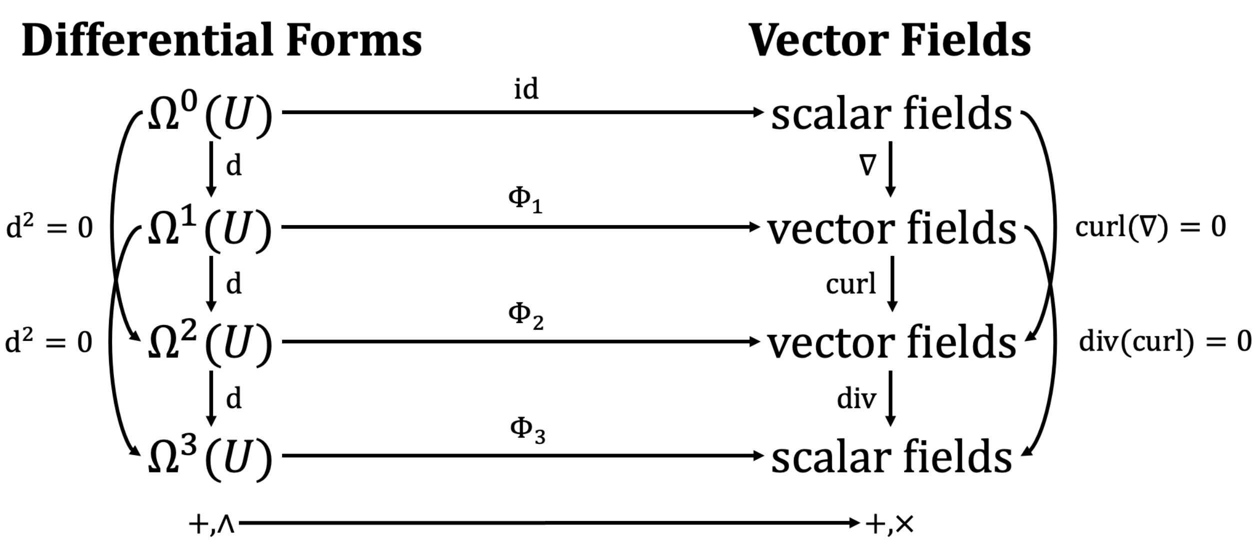
When $\mathbf{F} = \text{curl}(\mathbf{G})$ for some $\mathbf{G} : U \to GR^3$, the vector field $\mathbf{G}$ is a **vector potential** for $\mathbf{F}$. Such a vector potential is not unique; indeed if $f : U \to \mathbb{R}$ is smooth, then $\text{curl}(\mathbf{G} + \nabla f) = \text{curl}(\mathbf{G})$, as we can see below: if

$$\mathbf{G} \leftrightsquigarrow \omega_\mathbf{G} \in \Omega^1(U), \quad \text{curl}(\mathbf{G}) \leftrightsquigarrow \text{d}\omega_\mathbf{G} \in \Omega^2(U), \quad \nabla f \leftrightsquigarrow \text{d}f \in \Omega^1(U),$$

then

$$\text{curl}(\mathbf{G} + \nabla f) \leftrightsquigarrow \text{d}(\omega_\mathbf{G} + \text{d}f) = \text{d}\omega_\mathbf{G} \leftrightsquigarrow \text{curl}(\mathbf{G}).$$

In short, differential forms provide a tool to work with vector fields, which are the objects of interests in applications; the correspondence is diagrammed below.



13.6 Solved Problems

1. Are the following $1-$forms exact?

a) $\omega = 2xy\,\mathrm{d}x + x^2\,\mathrm{d}y$ b) $\omega = (x^2+yz)\,\mathrm{d}x + (xz+\cos y)\,\mathrm{d}y + (z+xy)\,\mathrm{d}z$
c) $\omega = y\,\mathrm{d}x + z\,\mathrm{d}y + x\,\mathrm{d}z$ d) $\omega = \frac{x}{x^2+y^2}\,\mathrm{d}x + \frac{y}{x^2+y^2}\,\mathrm{d}y$

Solution:

a) We have $\omega = 2xy\,\mathrm{d}x + x^2\,\mathrm{d}y \in \Omega^1(\mathbb{R}^2)$, where $\mathbb{R}^2$ is star-shaped. Since

$$\mathrm{d}\omega = 2\big[(\mathrm{d}x)y + x(\mathrm{d}y)\big] \wedge \mathrm{d}x + (2x\,\mathrm{d}x)\wedge \mathrm{d}y = 2x\big[\,\mathrm{d}y\wedge\mathrm{d}x + \mathrm{d}x\wedge\mathrm{d}y\big] = 0,$$

ω is closed. According to Poincaré's lemma, ω is also exact. In fact, $\eta = x^2y$ is an anti-derivative of ω (i.e. $\mathrm{d}\eta=\omega$).

b) We have $\omega = (x^2+yz)\,\mathrm{d}x + (xz+\cos y)\,\mathrm{d}y + (z+xy)\,\mathrm{d}z \in \Omega^1(\mathbb{R}^3)$, where $\mathbb{R}^3$ is star-shaped. Since

$$\mathrm{d}\omega = z\,\mathrm{d}y\wedge\mathrm{d}x + y\,\mathrm{d}z\wedge\mathrm{d}x + x\,\mathrm{d}z\wedge\mathrm{d}y + z\,\mathrm{d}x\wedge\mathrm{d}y + x\,\mathrm{d}y\wedge\mathrm{d}z + y\,\mathrm{d}x\wedge\mathrm{d}z = 0,$$

ω is closed. According to Poincaré's lemma, ω is also exact. In fact,

$$\eta = \frac{x^3}{3} + xyz + \sin y + \frac{z^2}{2}$$

is an anti-derivative of ω (i.e. $\mathrm{d}\eta=\omega$).

c) Since $\mathrm{d}\omega = \mathrm{d}y\wedge\mathrm{d}x + \mathrm{d}z\wedge\mathrm{d}y + \mathrm{d}x\wedge\mathrm{d}z \neq 0$, ω is not closed. Consequently, ω is not exact (remember, this has nothing to do with Poincaré's lemma).

d) We have $\omega = \frac{x}{x^2+y^2}\,\mathrm{d}x + \frac{y}{x^2+y^2}\,\mathrm{d}y \in \Omega^1(\mathbb{R}^2-\{(0,0)\})$. Note that $U=\mathbb{R}^2-\{(0,0)\}$ is NOT star-shaped, and so we cannot use Poincaré's lemma to determine whether ω is exact or not. If ω is not closed, then it will necessarily not be exact, by contraposition. However,

$$\mathrm{d}\omega = \frac{-2xy}{(x^2+y^2)^2}\,\mathrm{d}y\wedge\mathrm{d}x - \frac{2xy}{(x^2+y^2)^2}\,\mathrm{d}x\wedge\mathrm{d}y = 0,$$

and so ω is closed and we cannot use this approach. We are left with no other option than to try to find an anti-derivative. The brute force method yields $\eta = \ln(\sqrt{x^2+y^2})$ as an anti-derivative of ω (i.e. $\mathrm{d}\eta=\omega$). □

2. Are the following $2-$forms exact?

a) $\omega = \mathrm{d}x\wedge\mathrm{d}y$

b) $\omega = z\,\mathrm{d}x\wedge\mathrm{d}y + y\,\mathrm{d}x\wedge\mathrm{d}z + z\,\mathrm{d}y\wedge\mathrm{d}z$

Solution:

a) We have $\omega = \mathrm{d}x\wedge\mathrm{d}y \in \Omega^2(\mathbb{R}^2)$, where $\mathbb{R}^2$ is star-shaped. Since

$$\mathrm{d}\omega = d(\mathrm{d}x\wedge\mathrm{d}y) = d^2x\wedge\mathrm{d}y -\ \mathrm{d}x\wedge\mathrm{d}y^2 = 0-0=0,$$

ω is closed. According to Poincaré's lemma, ω is also exact.

b) We have

$$\omega = z\,dx \wedge dy + y\,dx \wedge dz + z\,dy \wedge dz \in \Omega^2(\mathbb{R}^3),$$

where $\mathbb{R}^3$ is star-shaped. Since

$$\begin{aligned} d\omega &= dz \wedge dx \wedge dy + dy \wedge dx \wedge dz + dz \wedge dy \wedge dz \\ &= dz \wedge dx \wedge dy - dz \wedge dx \wedge dy + 0 = 0, \end{aligned}$$

ω is closed. According to Poincaré's lemma, ω is also exact. In fact, $\eta = xz\,dy + xy\,dz$ is an anti-derivative of ω (i.e. $d\eta = \omega$). □

13.7 Exercises

1. Prepare a 2-page summary of this chapter, with important definitions and results.
2. Prove results 172, 173, 174, 175, 176, 180 (try, at least), and 184.
3. If $f \in \Omega^0(U)$ and $\omega \in \Omega^p(U)$, show that $f \wedge \omega = f\omega$.
4. Show that if ω and φ are two closed differential forms, then so is $\omega \wedge \varphi$. Show that if ω is also exact, then $\omega \wedge \varphi$ is exact.
5. What is the pullback $\mathbf{g}^*(\omega) \in \Omega^1(\mathbb{R})$ if $\mathbf{g} : \mathbb{R} \to \mathbb{R}^2$ is defined by $\mathbf{g}(v) = (3\cos 2v, 3\sin 2v)$ and $\omega = -y\,dx + x\,dy \in \Omega^1(\mathbb{R}^2)$? Simplify your answer as much as possible.
6. What is the pullback $\mathbf{g}^*(\omega) \in \Omega^1(\mathbb{R}^2)$ if $g : \mathbb{R}^2 \to \mathbb{R}^3$ is defined by $\mathbf{g}(u, v) = (\cos u, \sin u, v)$ and $\omega = z\,dx + x\,dy + y\,dz \in \Omega^1(\mathbb{R}^3)$? Simplify your answer as much as possible.
7. What is the pullback $\mathbf{g}^*(\omega) \in \Omega^2(\mathbb{R}^2)$ if $g : \mathbb{R}^2 \to \mathbb{R}^3$ is defined by $\mathbf{g}(u, v) = (\cos u, \sin u, v)$ and $\omega = z\,dx \wedge dy + y\,dz \wedge dx \in \Omega^2(\mathbb{R}^3)$? Simplify your answer as much as possible.
8. For each of the three previous exercises, compute $\mathbf{g}^*(d\omega)$ and $d(\mathbf{g}^*\omega)$.
9. Let $\mathbf{g} : (0, \infty) \times (0, \pi) \times (0, 2\pi) \to \mathbb{R}^3$ the map defining the spherical coordinates in $\mathbb{R}^3$. Compute $g^*(dx \wedge dy \wedge dz)$.
10. Let $\mathbf{F}, \mathbf{G} : \mathbb{R}^3 \to \mathbb{R}^3$, $f : \mathbb{R}^3 \to \mathbb{R}$ be smooth mappings and $\cdot$ and $\times$ represent the inner product and cross product in $\mathbb{R}^3$, respectively. Show that
    a) $\text{div}(\mathbf{F} + \mathbf{G}) = \text{div}(\mathbf{F}) + \text{div}(\mathbf{G})$
    b) $\text{div}(f\mathbf{F}) = f\text{div}(\mathbf{F}) + \mathbf{F} \cdot \nabla f$
    c) $\text{div}(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \text{curl}\,\mathbf{F} - \mathbf{F} \cdot \text{curl}\,\mathbf{G}$
    d) $\text{curl}(f\mathbf{F}) = f\,\text{curl}(\mathbf{F}) + (\nabla f) \times \mathbf{F}$
    e) $\text{div}(f\nabla f) = |\nabla f|^2$
11. Let $U \subseteq_O \mathbb{R}^n$ and $p \geq 0$. Show that $\Omega^p(U)$ is a vector space over $\mathbb{R}$.
12. Let $U \subseteq_O \mathbb{R}^n$, $p \geq 0$ and $\omega_1, \omega_2 \in \Omega^p(U)$. Show that $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$.