Chapter 14

Integrating Differential Forms

The integral of a differential form generalizes the concept of the integral of a function of a single variable (see Chapter 21 for another). In this chapter, we formalize the concepts of the **line**, **surface**, and **flux integral**, and present **Stokes' Theorem**, a deep unifying result of vector analysis.

14.1 Line Integral of a Differential $1-$Form

Let $U \subseteq_O \mathbb{R}^n$. Assume that $\boldsymbol{\gamma}$ is a differentiable path in U and that $\omega \in \Omega^1(U)$. This section's objective is to define $\int_{\boldsymbol{\gamma}} \omega$ **meaningfully**. A **path** in U is a continuous function $\boldsymbol{\gamma} : [a,b] \to U$; $\boldsymbol{\gamma}(a)$ is the **starting point** while $\boldsymbol{\gamma}(b)$ is the path's **finishing point**.

Examples

1. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. The path $\boldsymbol{\gamma} : [0,1] \to \mathbb{R}^n$ defined by $\boldsymbol{\gamma}(t) = t\mathbf{v} + (1-t)\mathbf{u}$ is the (oriented) line segment joining $\mathbf{u}$ and $\mathbf{v}$. □
2. Let $\boldsymbol{\gamma} : [0,2\pi] \to \mathbb{R}^2$ be defined by $\boldsymbol{\gamma}(t) = (\cos t, \sin t)$. Then $\boldsymbol{\gamma}([0,2\pi])$ is the **unit circle** in $\mathbb{R}^2$, starting at $\boldsymbol{\gamma}(0) = (1,0)$ and ending at $\boldsymbol{\gamma}(2\pi) = (1,0)$, travelling counter-clockwise. □

In that last example, $\boldsymbol{\gamma}$ is a **closed**, **simple** curve, which is to say that

$$\boldsymbol{\gamma}(0) = \boldsymbol{\gamma}(2\pi) \quad \text{and} \quad \boldsymbol{\gamma}(t) \neq \boldsymbol{\gamma}(s) \text{ for all } t \neq s \in (0,2\pi).$$

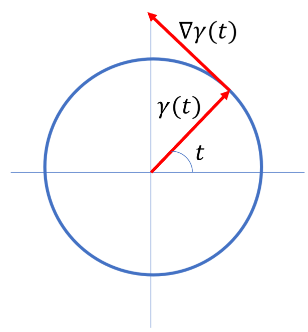
A path $\boldsymbol{\gamma}$ is **continuously differentiable** (denoted $\mathcal{C}^1$) if its derivative $\boldsymbol{\gamma}' : [a,b] \to \text{End}(\mathbb{R}, \mathbb{R}^n)$ varies continuously with t; the derivative is one-sided at the endpoints a and b. In that case,

$$\boldsymbol{\gamma}'(t) : \mathbb{R} \to \mathbb{R}^n, \quad x \mapsto \boldsymbol{\gamma}'(t)x = \nabla\boldsymbol{\gamma}(t)x = (\gamma_1'(t), \ldots, \gamma_n'(t))x.$$

Examples

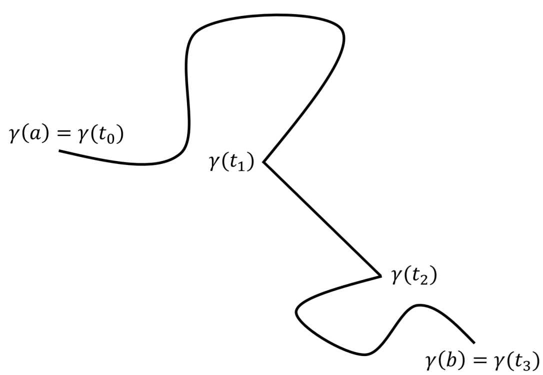
1. In the line segment example above, $\boldsymbol{\gamma}'(t) = \mathbf{v} - \mathbf{u} \in \mathbb{R}^n$. $\square$

2. In the circle example above, $\nabla\boldsymbol{\gamma}(t) = (-\sin t, \cos t)$. Note that $\boldsymbol{\gamma}(t) \perp \nabla\boldsymbol{\gamma}(t)$ for all t. $\square$



If $\boldsymbol{\gamma} : [a, b] \to U \subseteq_O \mathbb{R}^n$ represents the **position** of a particle at time t, then $\boldsymbol{\gamma}'(t)$ represents the **velocity vector** of the particle at time t; $\boldsymbol{\gamma}'(t_0)$ is necessarily tangent to the path $\boldsymbol{\gamma}$ at $t = t_0$.

A path $\boldsymbol{\gamma}$ is **piecewise differentiable** if $a = t_0 < t_1 < \cdots < t_n = b$ and $\boldsymbol{\gamma}|_{[t_i, t_{i+1}]}$ is $\mathcal{C}^1$ for all i.



Now we come to the section's important definition. Let $\boldsymbol{\gamma}$ be a $\mathcal{C}^1$ path in $U \subseteq_O \mathbb{R}^n$ and

$$\omega = \sum_{i=1}^{n} P_i(\mathbf{x})\,\mathrm{d}x_i \in \Omega^1(U).$$

The **line integral of** ω **along** $\boldsymbol{\gamma}$ is given by

$$\int_{\boldsymbol{\gamma}} \omega = \int_{\boldsymbol{\gamma}} \sum_{i=1}^{n} P_i(\mathbf{x})\,\mathrm{d}x_i := \int_{[a,b]} \sum_{i=1}^{n} P_i(\boldsymbol{\gamma}(t))\gamma_i(t)\,\mathrm{d}t,$$

where $\boldsymbol{\gamma}(t) = (\gamma_1(t), \ldots, \gamma_n(t))$, $\boldsymbol{\gamma}'(t) = (\gamma_1'(t), \ldots, \gamma_n'(t))$, and $\boldsymbol{\gamma} : [a, b] \to U \subseteq_O \mathbb{R}^n$.

Example: if $\boldsymbol{\gamma} : [-1, 1] \to \mathbb{R}^2$ is $\boldsymbol{\gamma}(t) = (t, t^2)$ and $\omega = -y\,\mathrm{d}x + x\,\mathrm{d}y \in \Omega^1(\mathbb{R}^2)$, then $\boldsymbol{\gamma}'(t) = (1, 2t)$, $P_1(x, y) = -y$, $P_2(x, y) = x$, and

$$\begin{aligned}\int_{\boldsymbol{\gamma}} \omega &= \int_{-1}^{1} \left(P_1(\boldsymbol{\gamma}(t))\gamma_1'(t) + P_2(\boldsymbol{\gamma}(t))\gamma_2'(t)\right)\,\mathrm{d}t \\ &= \int_{-1}^{1} (P_1(t, t^2)(1) + P_2(t, t^2)(2t))\,\mathrm{d}t \\ &= \int_{-1}^{1} (-t^2 + t(2t))\,\mathrm{d}t = \int_{-1}^{1} t^2\,\mathrm{d}t = \left[\frac{t^3}{3}\right]_{-1}^{1} = \frac{2}{3},\end{aligned}$$

using the regular rules of integration. □

But we could also approach the problem from a different (but ultimately equivalent) angle: the pullback of ω by $\boldsymbol{\gamma}$ is

$$\boldsymbol{\gamma}^*(\omega) = \boldsymbol{\gamma}^*(-y\,\mathrm{d}x + x\,\mathrm{d}y) = P_1(\boldsymbol{\gamma}(t))\frac{\partial\gamma_1}{\partial t}\,\mathrm{d}t + P_2(\boldsymbol{\gamma}(t))\frac{\partial\gamma_1}{\partial t}\,\mathrm{d}t = (-\gamma_2(t)\gamma_1'(t) + \gamma_1(t)\gamma_2'(t))\,\mathrm{d}t \in \Omega^1(\mathbb{R}),$$

so that $\int_{\boldsymbol{\gamma}} \omega = \int_{-1}^{1} \boldsymbol{\gamma}^*(\omega)$.

In general, if $\gamma : [a, b] \to U \subseteq_O \mathbb{R}^n$ and $\omega = \sum_{i=1}^{n} P_i\,\mathrm{d}x_i \in \Omega^1(U)$, then

$$\int_{\boldsymbol{\gamma}} \omega = \int_{[a,b]} \boldsymbol{\gamma}^*(\omega) = \int_{[a,b]} \sum_{i=1}^{n} P_i(\boldsymbol{\gamma}(t))\,\mathrm{d}\gamma_i = \int_{[a,b]} \sum_{i=1}^{n} P_i(\boldsymbol{\gamma}(t))\gamma_i(t)\,\mathrm{d}t.$$

Example: consider $\omega = -y\,\mathrm{d}x + x\,\mathrm{d}y \in \Omega^1(\mathbb{R}^2)$ and two paths from $(1, 0)$ to $(0, 1)$, $\boldsymbol{\gamma} : [0, \pi/2] \to \mathbb{R}^2$ (a circle arc) and $\boldsymbol{\eta} : [0, 1] \to \mathbb{R}^2$ (a line segment), defined by $\boldsymbol{\gamma}(t) = (\cos t, \sin t)$ and $\boldsymbol{\eta}(t) = (1 - t, t)$. Then

$$\begin{aligned}\int_{\boldsymbol{\gamma}} \omega &= \int_0^{\pi/2} \boldsymbol{\gamma}^*(\omega) = \int_0^{\pi/2} \left[(-\sin t)(\sin t) + (\cos t)(\cos t)\right]\,\mathrm{d}t = \int_0^{\pi/2} 1\,\mathrm{d}t = [t]_0^{\pi/2} = \frac{\pi}{2}, \\ \int_{\boldsymbol{\eta}} \omega &= \int_0^1 \boldsymbol{\eta}^*(\omega) = \int_0^1 \left[(-t)(-1) + (1 - t)(1)\right]\,\mathrm{d}t = \int_0^1 1\,\mathrm{d}t = [t]_0^1 = 1.\end{aligned}$$

Evidently, the value of the line integral depends on the path and the endpoints. □

If $\mathbf{P} : U \to \mathbb{R}^n$ is the vector field corresponding to $\omega \in \Omega^1(U)$, then

$$\sum_{i=1}^{n} P_i(\boldsymbol{\gamma}(t))\gamma_i'(t) = \mathbf{P}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t) = (\mathbf{P}(\boldsymbol{\gamma}(t)) \mid \boldsymbol{\gamma}'(t)),$$

we sometimes write

$$\int_{\boldsymbol{\gamma}} \omega = \int_{[a,b]} \mathbf{P}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\,\mathrm{d}t = \int_{\boldsymbol{\gamma}} \mathbf{P} \cdot \mathrm{d}\mathbf{r},$$

where $\mathbf{r}$ is a parameterization of γ (i.e., $\mathrm{d}\mathbf{r}(t) = \boldsymbol{\gamma}'(t)\mathrm{d}t$).

Let $\varphi : [a', b'] \to [a, b]$ be a $\mathcal{C}^1$ **diffeomorphism**;[1] this entails that $\varphi'(t) \neq 0$ for all $t \in [a', b']$. Since φ' is continuous, there are 2 possibilities:

1. $\varphi'(t) > 0 \Longrightarrow \varphi(a') = a$ and $\varphi(b') = b$, in which case φ **preserves the orientation**;

2. $\varphi'(t) < 0 \Longrightarrow \varphi(a') = b$ and $\varphi(b') = a$, in which case φ **reverses the orientation**.

> **Examples:** $\varphi : [1, 2] \to [1, 4]$ defined by $\varphi(t) = t^2$ preserves the orientation as $\varphi'(t) = 2t > 0$ on $[1, 2]$; but $\varphi : [-2, -1] \to [1, 4]$ defined by $\varphi(t) = t^2$ reverses the orientation as $\varphi'(t) = 2t < 0$ on $[-2, -1]$. $\square$

The distinction comes in at the following level.

> **Proposition 188**
> *Let $\omega = \sum_{i=1}^{n} P_i(\mathbf{x})\, dx_i \in \Omega^1(U)$, $\boldsymbol{\gamma} : [a, b] \to U$, $\boldsymbol{\gamma} \in \mathcal{C}^1$. If $\varphi : [a', b'] \to [a, b]$ is a $\mathcal{C}^1$ diffeomorphism, then*
>
> 1. $\int_{\boldsymbol{\gamma}\circ\varphi} \omega = \int_{\boldsymbol{\gamma}} \omega$ *if φ is orientation-preserving;*
> 2. $\int_{\boldsymbol{\gamma}\circ\varphi} \omega = -\int_{\boldsymbol{\gamma}} \omega$ *if φ is orientation-reversing.*
>
> **Proof:**
>
> 1. By construction, $\boldsymbol{\gamma} \circ \varphi : [a', b'] \to U$ is a $\mathcal{C}^1$ path and $\boldsymbol{\gamma}'(\varphi(t))\varphi'(t)$ exists for all $t \in [a', b']$. If we write $t = \varphi(s)$, then $\mathrm{d}t = \varphi'(s)\,\mathrm{d}s$, $a = \varphi(a')$, and $b = \varphi(b')$, and so
> $$\begin{aligned}\int_{\boldsymbol{\gamma}} \omega &= \int_{t=a}^{t=b} \sum_{i=1}^{n} (P_i \circ \boldsymbol{\gamma}(t))\gamma_i'(t)\,\mathrm{d}t = \int_{s=a'}^{s=b'} \sum_{i=1}^{n} (P_i \circ \boldsymbol{\gamma}(\varphi(s)))\gamma_i'(\varphi(s))\varphi'(s)\,\mathrm{d}s \\ &= \int_{a'}^{b'} \sum_{i=1}^{n} \left[P_i \circ (\boldsymbol{\gamma} \circ \varphi)(s)\right](\boldsymbol{\gamma} \circ \varphi)_i'(s)\,\mathrm{d}s = \int_{\boldsymbol{\gamma}\circ\varphi} \omega.\end{aligned}$$
> 2. The proof is similar, except that the change of variable is $t = \varphi(s)$, then $\mathrm{d}t = \varphi'(s)\,\mathrm{d}s$, $a = \varphi(b')$, and $b = \varphi(a')$, and so
> $$\int_{\boldsymbol{\gamma}} \omega = \int_{s=b'}^{s=a'} \sum_{i=1}^{n} (P_i \circ \boldsymbol{\gamma}(\varphi(s)))\gamma_i'(\varphi(s))\varphi'(s)\,\mathrm{d}s = -\int_{s=a'}^{s=b'} \cdots = -\int_{\boldsymbol{\gamma}\circ\varphi} \omega. \qquad \blacksquare$$

The line integral has two properties that are the counterparts of Theorems 55.1 and 56.

[1] That is, both φ and φ^{-1} are $\mathcal{C}^1$.

Proposition 189
Let $U \subseteq_O \mathbb{R}^n$, $\omega, \omega_1, \omega_2 \in \Omega^1(U)$, and $\boldsymbol{\gamma}, \boldsymbol{\eta}$ be $\mathcal{C}^1$ paths in U such that the finishing point of $\boldsymbol{\gamma}$ is the starting point of $\boldsymbol{\eta}$. The concatenation $\boldsymbol{\gamma} + \boldsymbol{\eta}$ is piecewise $\mathcal{C}^1$. Then:

1. *the line integral is linear in the sum (concatenation) of paths:*
$$\int_{\boldsymbol{\gamma}+\boldsymbol{\eta}} \omega = \int_{\boldsymbol{\gamma}} \omega + \int_{\boldsymbol{\eta}} \omega;$$
2. *the line integral is linear in the sum of differential forms:*
$$\int_{\boldsymbol{\gamma}} (\omega_1 + \omega_2) = \int_{\boldsymbol{\gamma}} \omega_1 + \int_{\boldsymbol{\gamma}} \omega_2.$$

Proof: left as an exercise. ■

Proposition 189, together with the next property, justifies the naming of the line integral: if it looks like an integral and it behaves like an integral...

Theorem 190 (Fundamental Theorem of Line Integrals)
Let $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}$ be a piecewise $\mathcal{C}^1$ path and $\omega = df \in \Omega^1(U)$ for some vector field $f \in \mathcal{C}^\infty(U, \mathbb{R})$. Then
$$\int_{\boldsymbol{\gamma}} \omega = \int_{\boldsymbol{\gamma}} df = f(\boldsymbol{\gamma}(b)) - f(\boldsymbol{\gamma}(a)).$$

Proof: according to Proposition 189.1, it is sufficient to show the result for $\mathcal{C}^1$ paths $\boldsymbol{\gamma}$; according to Proposition 184, we know that $\mathrm{d}(\boldsymbol{\gamma}^*(f)) = \boldsymbol{\gamma}^*(\mathrm{d}f)$. Then
$$\begin{aligned}\int_{\boldsymbol{\gamma}} \omega &= \int_{\boldsymbol{\gamma}} \mathrm{d}f = \int_{[a,b]} \gamma^*(f)(\mathrm{d}f) = \int_a^b \mathrm{d}(\boldsymbol{\gamma}^*(f)) = \int_a^b \mathrm{d}(f \circ \boldsymbol{\gamma}) = \int_a^b (f \circ \boldsymbol{\gamma})'(t)\,\mathrm{d}t \\ &= [f \circ \boldsymbol{\gamma}(t)]_a^b = f(\boldsymbol{\gamma}(b)) - f(\boldsymbol{\gamma}(a)),\end{aligned}$$
which completes the proof. ■

In the example on page 335, we have $\int_{\boldsymbol{\gamma}} -y\,\mathrm{d}x + x\,\mathrm{d}x \neq \int_{\boldsymbol{\eta}} -y\,\mathrm{d}x + x\,\mathrm{d}x$, even though $\boldsymbol{\gamma}$ and $\boldsymbol{\eta}$ have the same starting points and finishing points, and so Theorem 190 does not apply. What is the problem?

Corollary 191
If $\omega = dg \in \Omega^1(U)$ and $\boldsymbol{\gamma}$ is a $\mathcal{C}^1$ path in U, then $\int_{\boldsymbol{\gamma}} \omega = \int_{\boldsymbol{\gamma}} dg$ depends only on the endpoints of $\boldsymbol{\gamma}$. **Proof:** immediately follows from Theorem 190. ■

An open subset $U \subseteq_O \mathbb{R}^n$ is **path-connected** if for all $\mathbf{u}, \mathbf{v} \in U$, there is a path $\boldsymbol{\gamma} : [a,b] \to U$ such that $\boldsymbol{\gamma}(a) = \mathbf{u}$ and $\boldsymbol{\gamma}(b) = \mathbf{v}$; open balls and open annulii/torii are path-connected in $\mathbb{R}^2/\mathbb{R}^3$, but a set made up of disjoint open balls isn't.

A **loop** $\boldsymbol{\gamma}$ is a path $\boldsymbol{\gamma} : [a,b] \to U$ for which $\boldsymbol{\gamma}(a) = \boldsymbol{\gamma}(b)$; the path $\boldsymbol{\gamma} : [0, 2\pi] \to \mathbb{R}^2 \simeq \mathbb{C}$ defined by $\boldsymbol{\gamma}(t) = (\cos t, \sin t) \simeq e^{it}$ is a loop.

Theorem 192
Let $U \subseteq_O \mathbb{R}^n$ be path-connected. For a continuous differential form $\omega \in \Omega^1(U)$, the following are equivalent:

1. *ω is exact in U;*
2. *$\int_{\boldsymbol{\gamma}} \omega = 0$ for any loop $\boldsymbol{\gamma} : [a,b] \to U$;*
3. *if $\boldsymbol{\gamma}$ is any path in U, $\int_{\boldsymbol{\gamma}} \omega$ only depends on the endpoints of $\boldsymbol{\gamma}$.*

Proof: $\boxed{1. \implies 2.}$ follows from Theorem 190 since $\boldsymbol{\gamma}(a) = \boldsymbol{\gamma}(b)$ for any loop $\boldsymbol{\gamma} : [a,b] \to U$.

For $\boxed{2. \implies 3.}$, let $\boldsymbol{\gamma}, \boldsymbol{\eta}$ be two paths in U with the same endopoints. Then $\boldsymbol{\gamma} - \boldsymbol{\eta}$ is a loop in U, and

$$0 = \int_{\boldsymbol{\gamma}-\boldsymbol{\eta}} \omega = \int_{\boldsymbol{\gamma}} \omega + \int_{-\boldsymbol{\eta}} \omega = \int_{\boldsymbol{\gamma}} \omega - \int_{\boldsymbol{\eta}} \omega \Longrightarrow \int_{\boldsymbol{\gamma}} \omega = \int_{\boldsymbol{\eta}} \omega.$$

For $\boxed{3. \implies 1.}$, let $\mathbf{x}_0 \in U$ be fixed. For any $\mathbf{x} \in U$, let $\boldsymbol{\gamma}_{\mathbf{x}}$ be a path in U from $\mathbf{x}_0$ to $\mathbf{x}$. Define $f : U \to \mathbb{R}$ by $f(\mathbf{x}) = \int_{\boldsymbol{\gamma}_{\mathbf{x}}} \omega$. By assumption, if $\tilde{\boldsymbol{\gamma}}_{\mathbf{x}}$ is any other path in U from $\mathbf{x}_0$ to $\mathbf{x}$, then $\boldsymbol{\gamma}_{\mathbf{x}} - \tilde{\boldsymbol{\gamma}}_x$ is a loop in U and

$$0 = \int_{\boldsymbol{\gamma}_{\mathbf{x}} - \tilde{\boldsymbol{\gamma}}_{\mathbf{x}}} \omega = \int_{\boldsymbol{\gamma}_{\mathbf{x}}} \omega - \int_{\tilde{\boldsymbol{\gamma}}_{\mathbf{x}}} \omega \Longrightarrow f(\mathbf{x}) = \int_{\boldsymbol{\gamma}_{\mathbf{x}}} \omega = \int_{\tilde{\boldsymbol{\gamma}}_{\mathbf{x}}} \omega,$$

no matter which path $\boldsymbol{\gamma}_{\mathbf{x}}$ we use. Hence, f is well-defined.

It remains to see that $\mathrm{d}f = \omega$. Since

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \, \mathrm{d}x_i \quad \text{and} \quad \omega = \sum_{i=1}^{n} P_i \, \mathrm{d}x_i,$$

we need to show that $\frac{\partial f}{\partial x_i} = P_i$, $1 \leq i \leq n$. We know that

$$\frac{\partial f}{\partial x_i} = \lim_{t \to 0} \frac{f(\mathbf{x} + t\mathbf{e}_i) - f(\mathbf{x})}{t},$$

for $1 \leq i \leq n$ if the limit exists. Since U is open, $\mathbf{x} + t\mathbf{e}_i \in U$ for all i if t is small enough.

For each i, we have

$$\frac{1}{t}\left(f(\mathbf{x}+t\mathbf{e}_i)-f(\mathbf{x})\right)=\frac{1}{t}\left[\int_{\boldsymbol{\gamma}_{\mathbf{x}+t\mathbf{e}_i}}\omega-\int_{\boldsymbol{\gamma}_{\mathbf{x}}}\omega\right]=\frac{1}{t}\int_{\boldsymbol{\gamma}_i^t}\omega,$$

where $\boldsymbol{\gamma}_i^t$ is the straight line path from $\mathbf{x}$ to $\mathbf{x}+t\mathbf{e}_i$ (which is possible, again, if t is small enough), that is $\boldsymbol{\gamma}_i^t:[0,1]\to U$ defined by

$$\boldsymbol{\gamma}_i^t(s)=s(\mathbf{x}+t\mathbf{e}_i)+(1-s)\mathbf{x}=\mathbf{x}+st\mathbf{e}_i;$$

then $(\boldsymbol{\gamma}_i^t)'(s)=t\mathbf{e}_i$. In particular, for $1\leq j\leq n$ we have

$$\mathrm{d}(\boldsymbol{\gamma}_i^t)_j=\sum_{j=1}^n\frac{\partial(\boldsymbol{\gamma}_i^t)_j}{\partial s}\,\mathrm{d}s=\begin{cases}0 & \text{if } i\neq j\\ t\,\mathrm{d}s & \text{if } i=j\end{cases}$$

so that the pullback of ω by $\boldsymbol{\gamma}_i^t$ is

$$(\boldsymbol{\gamma}_i^t)^*(\omega)=\sum_{j=1}^n\left(P_j\circ\boldsymbol{\gamma}_i^t\right)\,\mathrm{d}(\boldsymbol{\gamma}_i^t)_j$$

and so

$$\begin{aligned}\frac{1}{t}\int_{\boldsymbol{\gamma}_i^t}\omega&=\frac{1}{t}\int_0^1(\boldsymbol{\gamma}_i^t)^*(\omega)=\frac{1}{t}\int_0^1\sum_{j=1}^n P_j\circ\boldsymbol{\gamma}_i^t(s)\,\mathrm{d}(\boldsymbol{\gamma}_i^t)_j=\frac{1}{t}\int_0^1P_i(\boldsymbol{\gamma}_i^t(s))t\,\mathrm{d}s\\&=\int_0^1P_i(\mathbf{x}+st\mathbf{e}_i)\,\mathrm{d}s=\int_0^1\left(P_i(\mathbf{x})+P_i(\mathbf{x}+st\mathbf{e}_i)-P_i(\mathbf{x})\right)\mathrm{d}s\\&=P_i(\mathbf{x})+\int_0^1\left(P_i(\mathbf{x}+st\mathbf{e}_i)-P_i(\mathbf{x})\right)\mathrm{d}s.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{\partial f}{\partial x_i}&=\lim_{t\to0}\left[P_i(\mathbf{x}+\int_0^1\left(P_i(\mathbf{x}+st\mathbf{e}_i)-P_i(\mathbf{x})\right)\mathrm{d}s\right]\\&=P_i(\mathbf{x})+\int_0^1\underbrace{\lim_{t\to0}\Big(P_i(\mathbf{x}+st\mathbf{e}_i)-P_i(\mathbf{x})\Big)}_{=0\text{ since }\omega\text{ is }\mathcal{C}^0}\,\mathrm{d}s=P_i(\mathbf{x}),\end{aligned}$$

which completes the proof. ■

We extract a specific implication from this result, for future ease of access.

Corollary 193
With the same hypotheses as in Theorem 192, if $\int_{\gamma}\omega=0$ for any loop γ in U, then ω is exact.

Finally, we show how to build an antiderivative for $\omega \in \Omega^1(U)$.

Example: consider the differential form

$$\omega = P_1(x,y)\,\mathrm{d}x + P_2(x,y)\,\mathrm{d}y = (e^x + 2xy)\,\mathrm{d}x + (x^2 + \cos y)\,\mathrm{d}y \in \Omega^1(\mathbb{R}^2).$$

Since

$$\mathrm{d}\omega = \left(\frac{\partial P_2}{\partial x} - \frac{\partial P_1}{\partial y}\right)\,\mathrm{d}x \wedge \mathrm{d}y = (2x - 2x)\,\mathrm{d}x \wedge \mathrm{d}y = 0,$$

then ω is closed. According to Poincaré's lemma, since $\mathbb{R}^2$ is star-shaped (and thus path-connected), then ω is exact, so it has an antiderivative $f : \mathbb{R}^2 \to \mathbb{R}$. We will compute f in two ways, exploiting Theorem 192.

1. Let $\mathbf{z}_0 = (x_0, y_0) \in U$ be fixed and consider the path $\boldsymbol{\gamma} : [0,1] \to \mathbb{R}^2$ given by $\boldsymbol{\gamma}(t) = t\mathbf{z}_0$ ($\boldsymbol{\gamma}$ is the line segment joining the origin to $\mathbf{z}_0$). Then $\boldsymbol{\gamma}'(t) = \mathbf{z}_0$. Set

$$\begin{aligned} f(\mathbf{z}_0) &= \int_{\boldsymbol{\gamma}} \omega = \int_0^1 \boldsymbol{\gamma}^*(\omega) = \int_0^1 P_1(\boldsymbol{\gamma}(t))\gamma_1'(t)\,\mathrm{d}t + P_2(\boldsymbol{\gamma}(t))\gamma_2'(t)\,\mathrm{d}t \\ &= \int_0^1 P_1(tx_0, ty_0)x_0\,\mathrm{d}t + P_2(tx_0, ty_0)y_0\,\mathrm{d}t \\ &= \int_0^1 \Big(e^{tx_0} + 2(tx_0)(ty_0)x_0\Big)\,\mathrm{d}t + \int_0^1 \Big((tx_0)^2 + \cos(ty_0)y_0\Big)\,\mathrm{d}t \\ &= \left[e^{tx_0} + \frac{2}{3}t^3x_0^2y_0 + \frac{1}{3}t^3x_0^2y_0 + \sin(ty_0)\right]_0^1 = e^{x_0} + x_0^2y_0 + \sin y_0 - 1. \end{aligned}$$

2. If instead we join the origin to $\mathbf{z}_0 = (x_0, y_0)$ by first travelling horizontally to $(x_0, 0)$ along $\boldsymbol{\gamma}_1$, then travelling vertically to (x_0, y_0) along $\boldsymbol{\gamma}_2$, we have

$$\boldsymbol{\gamma}_1 : [0, x_0] \to \mathbb{R}^2, t \mapsto (t, 0), \quad \boldsymbol{\gamma}_2 : [0, y_0] \to \mathbb{R}^2, t \mapsto (x_0, t),$$

and $\boldsymbol{\gamma}_1'(t) = (1,0)$, $\boldsymbol{\gamma}_2'(t) = (0,1)$, so that

$$\begin{aligned} f(\mathbf{z}_0) &= \int_{\boldsymbol{\gamma}_1} \omega + \int_{\boldsymbol{\gamma}_2} \omega = \int_{\boldsymbol{\gamma}_1} \omega + \int_{\boldsymbol{\gamma}_2} \omega \\ &= \int_0^{x_0} e^t\,\mathrm{d}t + \int_0^{y_0} (x_0^2 + \cos t)\,\mathrm{d}t = e^{x_0} - 1 + [x_0^2 t + \sin t]_0^{y_0} \\ &= e^{x_0} - 1 + x_0^2 y_0 + \sin y_0. \end{aligned}$$

No surprise there: they're the same! $\square$

Interpretation of the Line Integral Suppose a point particle proceeds along the path $\boldsymbol{\gamma}$ and is subjected to the effects of a vector field $\mathbf{F}$. Then the **work** done by the particle on its journey is given by $\int_{\boldsymbol{\gamma}} \Phi_1 \circ \mathbf{F} = \int_{\boldsymbol{\gamma}} \omega_{\mathbf{F}}$.

14.2 Integral of a Differential $p-$Form

Let $U \subseteq_O \mathbb{R}^n$. Given a differential form $\omega \in \Omega^1(U)$ and a $\mathcal{C}^1$ function $\boldsymbol{\gamma} : V = [a,b] \subseteq \mathbb{R}^1 \to U$, we have seen how we could define a quantity, the line integral $\int_{\boldsymbol{\gamma}} \omega$, that behaves in many ways like the Riemann integral.

If we remember that dim$(\Lambda^1(\mathbb{R}^1)) = 1$, we can define an vector space isomorphism

$$\tilde{\Phi}_1 : \mathbb{R}^1 \to \Lambda$$

by $\tilde{\Phi}_1(a) = a\,\mathrm{d}t$ and thus re-write the line integral formulation as

$$\int_{\boldsymbol{\gamma}} \omega = \int_V \boldsymbol{\gamma}^*(\omega) := \int_V \tilde{\Phi}_1^{-1}(\boldsymbol{\gamma}^*(\omega))\,\mathrm{d}m = \int_{[a,b]} \tilde{\Phi}_1^{-1}(\boldsymbol{\gamma}^*(\omega))\,\mathrm{d}m,$$

where m is the Borel-Lebesgue measure on $\mathbb{R}$ (see Chapter 21).[2]

We can generalize this definition to **differential** $p-$**forms**. Let $V \subseteq \mathbb{R}^p$ and consider a $\mathcal{C}^1$ function $\boldsymbol{\sigma} : V \to U$ and a differential form $\varphi \in \Omega^p(U) \subseteq \Omega^p(\mathbb{R}^n)$. The pullback of φ by $\boldsymbol{\sigma}$ is itself a differential form $\boldsymbol{\sigma}^*(\varphi) \in \Omega^1(V) \subseteq \Omega^p(\mathbb{R}^p)$. Since dim $\Lambda^p(\mathbb{R}^p) = 1$, we there is a vector space isomorphism

$$\tilde{\Phi}_p : \mathbb{R}^1 \to \Lambda^p(\mathbb{R}^p)$$

given by $\tilde{\Phi}_p(a) = a\,\mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_p$. Suppose that $\boldsymbol{\sigma}$ is **orientable** (more on this later), then we define the "surface" integral of φ on V by

$$\int_{\boldsymbol{\sigma}} \varphi = \int_V \boldsymbol{\sigma}^*(\varphi) := \int_V \tilde{\Phi}_p^{-1}(\boldsymbol{\gamma}^*(\varphi))\,\mathrm{d}m.$$

Example: consider $\boldsymbol{\sigma} : [0,1]^2 \to \mathbb{R}^3$, which is defined by $\boldsymbol{\sigma}(s,t) = (s,t,s^2+t^2)$, and $\varphi = \mathrm{d}x \wedge \mathrm{d}z - \mathrm{d}x \wedge \mathrm{d}y \in \Omega^2(\mathbb{R}^3)$. Then

$$\begin{aligned}\boldsymbol{\sigma}^*(\varphi) &= \boldsymbol{\sigma}(\mathrm{d}x \wedge \mathrm{d}z) - \boldsymbol{\sigma}^*(\mathrm{d}x \wedge \mathrm{d}y) = \mathrm{d}\sigma_1 \wedge \mathrm{d}\sigma_3 - \mathrm{d}\sigma_1 \wedge \mathrm{d}\sigma_2 \\ &= \left(\frac{\partial \sigma_1}{\partial s}\,\mathrm{d}s + \frac{\partial \sigma_1}{\partial t}\,\mathrm{d}t\right) \wedge \left(\frac{\partial \sigma_3}{\partial s}\,\mathrm{d}s + \frac{\partial \sigma_3}{\partial t}\,\mathrm{d}t\right) - \left(\frac{\partial \sigma_1}{\partial s}\,\mathrm{d}s + \frac{\partial \sigma_1}{\partial t}\,\mathrm{d}t\right) \wedge \left(\frac{\partial \sigma_2}{\partial s}\,\mathrm{d}s + \frac{\partial \sigma_2}{\partial t}\,\mathrm{d}t\right) \\ &= (1 \cdot\, \mathrm{d}s + 0 \cdot\, \mathrm{d}t) \wedge (2s\,\mathrm{d}s + 2t\,\mathrm{d}t) - (1 \cdot\, \mathrm{d}s + 0 \cdot\, \mathrm{d}t) \wedge (0 \cdot \mathrm{d}s + 1 \cdot \mathrm{d}t) = (2t-1)\,\mathrm{d}s \wedge \mathrm{d}t.\end{aligned}$$

Hence $\tilde{\Phi}_2^{-1}(\boldsymbol{\sigma}^*(\varphi)) = 2t - 1$ and

$$\int_{\boldsymbol{\sigma}} \varphi = \int_{[0,1]^2} = \int_0^1 \int_0^1 (2t-1)\,\mathrm{d}s\,\mathrm{d}t = \int_0^1 (2t-1)\,\mathrm{d}t = 0,$$

assuming that the reader knows how to compute multivariate integrals. □

[2] Note that $\tilde{\Phi}_1$ and Φ_1 defined in the previous section are different functions.

We have seen in Chapter 13 that

$$\varphi = \varphi_{\mathbf{F}} = P(x,y,z)\,\mathrm{d}y \wedge \mathrm{d}z + Q(x,y,z)\,\mathrm{d}z \wedge \mathrm{d}x + R(x,y,z)\,\mathrm{d}x \wedge \mathrm{d}y \in \Omega^2(\mathbb{R}^3)$$

corresponds to the vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\mathbf{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z)).$$

If we set $\mathrm{d}\mathbf{A} = (\mathrm{d}y \wedge \mathrm{d}z, \mathrm{d}z \wedge \mathrm{d}x, \mathrm{d}x \wedge \mathrm{d}y)$, then we often write

$$\int_\sigma \varphi = \int_{\boldsymbol{\sigma}} \mathbf{F} \cdot \mathrm{d}\mathbf{A} = \iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{A},$$

where $S = \boldsymbol{\sigma}(V) = \{\boldsymbol{\sigma}(s,t) \mid (s,t) \in V\}$ is orientable. In that case, the **surface integral** (also known as the **flux integral**) of φ over $\boldsymbol{\sigma}$ is

$$\int_{\boldsymbol{\sigma}} \varphi = \int_V \boldsymbol{\sigma}^*(\varphi) = \pm \int_V \mathbf{F}(\boldsymbol{\sigma}) \cdot \left[\frac{\partial \boldsymbol{\sigma}}{\partial s} \times \frac{\partial \boldsymbol{\sigma}}{\partial t}\right] \mathrm{d}m$$

(the $\pm$ comes from the surface orientation).

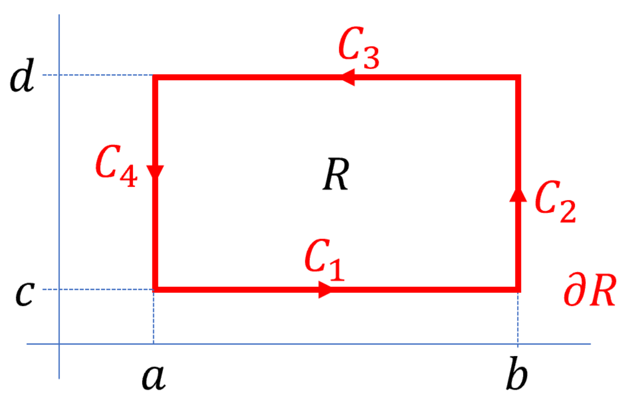
Interpretation of the Surface Integral Suppose a surface S parameterized by $\boldsymbol{\sigma}$ is "dropped" into a fluid whose flow is governed by the vector field $\mathbf{F}$. Then the **flux** of the fluid through S is given by $\int_{\boldsymbol{\sigma}} \Phi_2 \circ \mathbf{F} = \int_{\boldsymbol{\sigma}} \varphi_{\mathbf{F}}$.

14.3 Green's Theorem

Consider a **rectangle** $R = [a,b] \times [c,d] \subseteq \mathbb{R}^2$ and let ∂R be its **boundary**:

$$\partial R = ([a,b] \times \{c\}) \cup (\{b\} \times [c,d]) \cup ([a,b] \times \{d\}) \cup (\{a\} \times [c,d]),$$

together with the **induced orientation**, chosen so that as we travel ∂R, along the direction given by the orientation, the surface R falls to the left, as shown below.



Theorem 194 (Green's Theorem for a Rectangle)
Let $R=[a,b]\times[c,d]\subseteq\mathbb{R}^2$ (with the induced orientation) and $\omega\in\Omega^1(U)$, where $R\subseteq U\subseteq_O\mathbb{R}^2$. Then

$$\int_{\mathbf{R}} d\omega=\int_{\partial\mathbf{R}}\omega,$$

where $\mathbf{R}:R\to U$ and $\partial\mathbf{R}:\partial R\to U$ are the identity functions.

Proof: write $\omega=P(x,y)\,\mathrm{d}x+Q(x,y)\,\mathrm{d}y\in\Omega^1(U)$. We have seen that

$$\mathrm{d}\omega=\left(\frac{\partial Q(x,y)}{\partial y}-\frac{\partial P(x,y)}{\partial x}\right)\,\mathrm{d}x\wedge\mathrm{d}y$$

and

$$\begin{aligned}
\int_{\mathbf{R}}\mathrm{d}\omega&=\int_R\mathbf{R}^*(\mathrm{d}\omega)=\int_R\tilde{\Phi}_2^{-1}(\mathbf{R}^*(\mathrm{d}\omega))\,\mathrm{d}m=\int_R\left(\frac{\partial Q(x,y)}{\partial y}-\frac{\partial P(x,y)}{\partial x}\right)\,\mathrm{d}m\\
&=\int_a^b\int_c^d\left(\frac{\partial Q(x,y)}{\partial y}-\frac{\partial P(x,y)}{\partial x}\right)\,\mathrm{d}y\,\mathrm{d}x=\int_a^b\int_c^d\frac{\partial Q(x,y)}{\partial y}\,\mathrm{d}y\,\mathrm{d}x-\int_a^b\int_c^d\frac{\partial P(x,y)}{\partial x}\,\mathrm{d}y\,\mathrm{d}x\\
&=\int_c^d\int_a^b\frac{\partial Q(x,y)}{\partial y}\,\mathrm{d}x\,\mathrm{d}y-\int_a^b\int_c^d\frac{\partial P(x,y)}{\partial x}\,\mathrm{d}y\,\mathrm{d}x,\quad\text{by Fubini's theorem (see Chapter 21)}\\
&=\int_c^d(Q(b,y)-Q(a,y))\,\mathrm{d}y-\int_a^b(P(x,d)-P(x,c))\,\mathrm{d}x\\
&=\int_a^bP(x,c)\,\mathrm{d}x+\int_c^dQ(b,y)\,\mathrm{d}y+\int_b^aP(x,d)\,\mathrm{d}x+\int_d^cQ(a,y)\,\mathrm{d}y\\
&=\int_{[a,b]}P(t,c)\,\mathrm{d}m+\int_{[c,d]}Q(b,t)\,\mathrm{d}m-\int_{[a,b]}P(t,d)\,\mathrm{d}m-\int_{[c,d]}Q(a,t)\,\mathrm{d}m.
\end{aligned}$$

Now write $\partial\mathbf{R}=\mathbf{C}_1+\mathbf{C}_2+\mathbf{C}_3+\mathbf{C}_4$, where

$$\begin{aligned}
&\mathbf{C}_1:[a,b]\to\mathbb{R}^2,\ \mathbf{C}_1(t)=(t,c);\quad &&\mathbf{C}_3:[a,b]\to\mathbb{R}^2,\ \mathbf{C}_3(t)=(b+a-t,d);\\
&\mathbf{C}_2:[c,d]\to\mathbb{R}^2,\ \mathbf{C}_2(t)=(b,t);\quad &&\mathbf{C}_4:[c,d]\to\mathbb{R}^2,\ \mathbf{C}_4(t)=(a,d+c-t).
\end{aligned}$$

According to Proposition 189,

$$\begin{aligned}
&\int_{\partial\mathbf{R}}\omega=\int_{\mathbf{C}_1}\omega+\int_{\mathbf{C}_2}\omega+\int_{\mathbf{C}_3}\omega+\int_{\mathbf{C}_4}\omega\\
&=\int_{[a,b]}\tilde{\Phi}_1^{-1}(\mathbf{C}_1^*(\omega))+\int_{[c,d]}\tilde{\Phi}_1^{-1}(\mathbf{C}_2^*(\omega))+\int_{[a,b]}\tilde{\Phi}_1^{-1}(\mathbf{C}_3^*(\omega))+\int_{[c,d]}\tilde{\Phi}_1^{-1}(\mathbf{C}_4^*(\omega)),\\
&=\int_{[a,b]}\Big[P(t,c)\cdot1+Q(t,c)\cdot0\Big]\,\mathrm{d}m+\int_{[a,b]}\Big[P(b+a-t,d)\cdot(-1)+Q(b+a-t,d)\cdot0\Big]\,\mathrm{d}m\\
&+\int_{[c,d]}\Big[P(b,t)\cdot0+Q(b,t)\cdot1\Big]\,\mathrm{d}m+\int_{[c,d]}\Big[P(a,d+c-t)\cdot0+Q(a,d+c-t)\cdot(-1)\Big]\,\mathrm{d}m
\end{aligned}$$

so

$$\begin{aligned}\int_{\partial\mathbf{R}}\omega &= \int_{[a,b]} P(t,c)\,\mathrm{d}m + \int_{[c,d]} Q(b,t)\,\mathrm{d}m - \int_{[a,b]} P(b+a-t,d)\,\mathrm{d}m - \int_{[c,d]} Q(a,d+c-t)\,\mathrm{d}m \\ &= \int_{[a,b]} P(t,c)\,\mathrm{d}m + \int_{[c,d]} Q(b,t)\,\mathrm{d}m + \int_{[b,a]} P(s,d)\,\mathrm{d}m + \int_{[d,c]} Q(a,s)\,\mathrm{d}m \\ &= \int_{[a,b]} P(t,c)\,\mathrm{d}m + \int_{[c,d]} Q(b,t)\,\mathrm{d}m - \int_{[a,b]} P(t,d)\,\mathrm{d}m - \int_{[c,d]} Q(a,t)\,\mathrm{d}m\end{aligned},$$

which completes the proof. ■

This is a remarkable result: integrating a derivative on a rectangle is equivalent to integrating the antiderivative on the rectangle's boundary. As it happens, it is not specific to rectangles.[3]

Theorem 195 (Green's Theorem)
Let $K \subseteq_K \mathbb{R}^2$, and assume that ∂K can be given the induced orientation. If

$$\omega = P(x,y)\,dx + Q(x,y)\,dy \in \Omega^1(U)$$

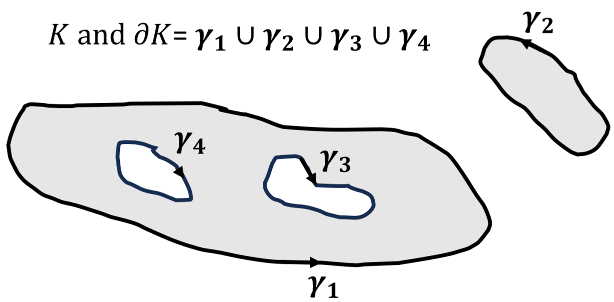
for $K \subseteq U \subseteq_O \mathbb{R}^2$, then

$$\int_{\mathbf{K}} d\omega = \int_{\partial\mathbf{K}} \omega,$$

where $\mathbf{K} : K \to \mathbb{R}^2$ and $\partial\mathbf{K} : \partial K \to \mathbb{R}^2$ are identity functions.

Proof: we only provide a sketch. Green's theorem for a rectangle can be shown to apply to unions of rectangles where each pair shares at most an edge: if the rectangles do not share edges, then the result is obvious – if they do share edges, then the induced orientation ensures that the shared edges are traversed one way for one rectangle, and the other way for another, meaning that their contribution to the integral will cancel out and only the outside boundary counts.

We can write any compact set K as a (potentially infinite) union of such rectangles $\{R_n\}$; Green's theorem holds in the limit. ■



[3]It's not even specific to $\mathbb{R}^2$, as we shall see shortly.

The classical version of Green's theorem is

$$\iint_K \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y = \oint_{\partial K} P\,\mathrm{d}x + Q\,\mathrm{d}y.$$

Let $K \subseteq_K \mathbb{R}^2$ have a boundary with the induced orientation. By definition, we have

$$\mathrm{d}(x\,\mathrm{d}y) = \mathrm{d}x \wedge \mathrm{d}y = \mathrm{d}(-y\,\mathrm{d}x) \Longrightarrow \mathrm{d}x \wedge \mathrm{d}y = \mathrm{d}\left(\frac{1}{2}(-y\,\mathrm{d}x + \mathrm{d}y)\right) := \mathrm{d}\omega.$$

Thus, according to Green's theorem,

$$\text{Area}(K) = \iint_K \mathrm{d}A = \int_K 1 \cdot \mathrm{d}m = \int_{\mathbf{K}} \mathrm{d}\omega = \int_{\partial\mathbf{K}} \omega = \frac{1}{2}\int_{\partial\mathbf{K}} -y\,\mathrm{d}x + x\,\mathrm{d}y.$$

Example: what is the area of the ellipse

$$K = \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\right\}, \quad a,b > 0?$$

Solution: let $\boldsymbol{\gamma} : [0,2\pi] \to \mathbb{R}^2$ be defined by $\boldsymbol{\gamma}(t) = (a\cos t, b\sin t)$; then $\boldsymbol{\gamma}$ is a parameterization of $\partial K = \boldsymbol{\gamma}([0,2\pi])$, and so

$$\begin{aligned}
\text{Area}(K) &= \frac{1}{2}\int_{\partial\mathbf{K}} -y\,\mathrm{d}x + x\,\mathrm{d}y = \frac{1}{2}\int_{[0,2\pi]} \tilde{\Phi}_1^{-1}(\boldsymbol{\gamma}^*(\omega)) \\
&= \frac{1}{2}\int_{[0,2\pi]} P(\boldsymbol{\gamma}(t))\gamma_1'(t)\,\mathrm{d}t + Q(\boldsymbol{\gamma}(t))\gamma_2'(t)\,\mathrm{d}t \\
&= \frac{1}{2}\int_{[0,2\pi]} P(a\cos t, b\sin t)(-a\sin t)\,\mathrm{d}t + Q(a\cos t, b\sin t)(b\cos t)\,\mathrm{d}t \\
&= \frac{1}{2}\int_0^{2\pi} \Big[(-b\sin t)(a\sin t) + (a\cos t)(b\cos t)\Big]\,\mathrm{d}t = \frac{1}{2}\int_0^{2\pi} ab\,\mathrm{d}t = \pi ab,
\end{aligned}$$

which we could have derived by viewing ellipses as generalized circles, but it's nice to be able to do it analytically. □

A subset $X \subseteq \mathbb{R}^n$ is **simply connected**, denoted $\pi_1(X) \simeq 1$, if X is connected and if each loop in X is **homotopic** to a single point, which is to say that each loop in X can be **deformed continuously** to a single point (see Chapter 20 for more on this topic).[4]

Example: the connected component bounded by $\boldsymbol{\gamma}_2$ in the image on the previous page is simply connected; the connected component bounded by $\boldsymbol{\gamma}_1 \cup \boldsymbol{\gamma}_3 \cup \boldsymbol{\gamma}_4$ isn't.

[4]Roughly speaking, X is simply connected if its interior contains no "hole".

Corollary 196
Let $U \subseteq_O \mathbb{R}^2$ be simply connected. If $\omega \in \Omega^1(U)$ is closed, then ω is exact.

Proof: according to Theorem 194, for any rectangle $R \subseteq U$, we have

$$\int_{\partial \mathbf{R}} \omega = \int_{\mathbf{R}} \mathrm{d}\omega;$$

since ω is closed, then $\mathrm{d}\omega = 0$, so that $\int_{\partial \mathbf{R}} \omega = 0$.

For a fixed $\mathbf{x}_0 \in U$ and for all $\mathbf{x} \in U$, there is a piecewise $\mathcal{C}^1$ path $\boldsymbol{\gamma}_{\mathbf{x}}$ connecting $\mathbf{x}_0$ to $\mathbf{x}$ that is made up of horizontal and vertical segments in U.

We would like to define $f(\mathbf{x}) = \int_{\boldsymbol{\gamma}_{\mathbf{x}}} \omega$, so that $\mathrm{d}f = \omega$ (as in the proof of Theorem 192). But this is only possible if f is well-defined, meaning that $f(\mathbf{x})$ takes on the same value independently of the piecewise $\mathcal{C}^1$ path $\boldsymbol{\gamma}_{\mathbf{x}}$ taken from $\mathbf{x}_0$ to $\mathbf{x}$, as long as it is a path of horizontal and vertical segments.

If $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ are two such paths, then $\boldsymbol{\gamma}_1 - \boldsymbol{\gamma}_2$ enclose a region made up of contiguous rectangles, say $R_1 \cup \cdots \cup R_k$. According to Green's theorem for rectangles,

$$\int_{\mathbf{R}_1 \cup \cdots \cup \mathbf{R}_k} \mathrm{d}\omega = \int_{\mathbf{R}_1} \mathrm{d}\omega + \cdots + \int_{\mathbf{R}_k} \mathrm{d}\omega = \int_{\partial \mathbf{R}_1} \omega + \cdots + \int_{\partial \mathbf{R}_k} \omega = \int_{\boldsymbol{\gamma}_1 - \boldsymbol{\gamma}_2} \omega = \int_{\boldsymbol{\gamma}_1} \omega - \int_{\boldsymbol{\gamma}_2} \omega.$$

Since ω is closed in U, the left hand-side of that string of equations is 0, so that $\int_{\boldsymbol{\gamma}_1} \omega = \int_{\boldsymbol{\gamma}_2} \omega$. Thus f is well-defined and the proof is complete. ■

The condition that U be simply connected is necessary: if

$$\omega = \frac{-y}{x^2 + y^2}\,\mathrm{d}x + \frac{x}{x^2 + y^2}\,\mathrm{d}y \in \Omega^1(U = \mathbb{R}^2 \setminus \{\mathbf{0}\}),$$

then we have

$$\mathrm{d}\omega = \left(\frac{\partial}{\partial x} \left(\frac{x}{x^2 + y^2} \right) - \frac{\partial}{\partial y} \left(\frac{-y}{x^2 + y^2} \right) \right) \mathrm{d}x \wedge \mathrm{d}y = \left(\frac{-x^2 + y^2}{(x^2 + y^2)^2} - \frac{y^2 - x^2}{(x^2 + y^2)^2} \right) \mathrm{d}x \wedge \mathrm{d}y = 0.$$

If $\boldsymbol{\gamma} : [0, 2\pi] \to \mathbb{R}^2$ defined by $\boldsymbol{\gamma}(t) = (\cos t, \sin t) \in U$ is a parameterization of the unit circle, we have

$$\int_{\boldsymbol{\gamma}} \omega = \int_{[0,2\pi]} \tilde{\Phi}_1^{-1}(\boldsymbol{\gamma}^*(\omega)) = \int_0^{2\pi} \mathrm{d}t = 2\pi \neq 0 = \int_{\mathbf{B}^1} \mathrm{d}\omega,$$

and so ω cannot be exact in U since the 3rd statement in Theorem 192.3 does not hold. The only fly in the ointment is that $U = \mathbb{R}^2 \setminus \{\mathbf{0}\}$ is not simply connected.

14.4 Surfaces and Orientable Surfaces in $\mathbb{R}^3$

It is fairly easy (?) to parameterize areas in $\mathbb{R}^2$, but the addition of a 3rd dimension can complicate matters to some extent (especially when it comes to their boundaries).

There are 3 classical ways to describe a plane $S \subseteq \mathbb{R}^3$.

- The **implicit** approach requires a **normal vector n** to S and a point $P_0 \in S$:

$$S = \{\mathbf{v} \in \mathbb{R}^3 \mid (\mathbf{v} - P_0) \cdot \mathbf{n} = 0\} = \{(x, y, z) \mid \underbrace{ax + by + cz}_{=F(x,y,z)} - \underbrace{(ax_0 + by_0 + cz_0)}_{=d} = 0\}.$$

- The **explicit** approach views the plane as the **graph** of a function: as $\mathbf{n} = (a, b, c) \neq \mathbf{0}$, we may assume that $c \neq 0$.[5] Then we have $cz = d - ax - by$, so that

$$z = \frac{d - ax - by}{c} = f(x, y), \quad f : \mathbb{R}^2 \to \mathbb{R}^2,$$

  and we have $F(x, y, f(x, y)) = 0$ and $S = \{(x, y, f(x, y)) \mid (x, y) \in \mathbb{R}^2\}$.

- Finally, in the **parametric** approach, let $\mathbf{v}_1, \mathbf{v}_2 \in S_0$ be linearly independent, where

$$S_0 = \{(x, y, z) \mid F(x, y, z) = ax + by + cz = 0\};$$

  hence $S_0 = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. If $\mathbf{v}_0 \in S$, we have $S = \mathbf{v}_0 + S_0$. Let $\mathbf{g} : \mathbb{R}^2 \to \mathbb{R}^3$, defined by $g(s, t) = \mathbf{v}_0 + s\mathbf{v}_1 + t\mathbf{v}_2$; then $\mathbf{g}(\mathbb{R}^2) = S$ and so g is a **parameterization** of S.

These approaches generalize to non-planar surfaces. A subset $S \subseteq \mathbb{R}$ is a **surface** in $\mathbb{R}^3$ if one of the three following equivalent conditions hold.[6]

- **Explicit description:** $\forall \mathbf{p} \in S$, $\exists W_{\mathbf{p}} \subseteq_0 \mathbb{R}^3$ and $f : \pi_{x,y}(W_{\mathbf{p}}) \subseteq \mathbb{R}^2 \to \mathbb{R}$ smooth such that $S \cap W_{\mathbf{p}} = \text{Graph}(f)$.

- **Implicit description:** $\forall \mathbf{p} \in S$, $\exists W_{\mathbf{p}} \subseteq_0 \mathbb{R}^3$ and $\mathbf{F} : W_{\mathbf{p}} \to \mathbb{R}^3$ smooth such that

$$S \cap W_{\mathbf{p}} = \mathbf{F}^{-1}(\mathbf{0}) = \{\mathbf{w} \in W_{\mathbf{p}} \mid \mathbf{F}(\mathbf{w}) = \mathbf{0}\}$$

  and $\det(D\mathbf{F}) \neq 0$ on $S \cap W_{\mathbf{p}}$.

- **Parametric description:** $\forall \mathbf{p} \in S$, $\exists W_{\mathbf{p}} \subseteq_0 \mathbb{R}^3$ and a smooth injection $\mathbf{g} : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ such that $\text{rank}(D\mathbf{g}(\mathbf{x})) = 2$ for all $\mathbf{x} \in U$ and such that $\mathbf{g}^{-1} : S \cap W_{\mathbf{p}} \to U$ is continuous. In that case, we say that $\mathbf{g}$ is a local parameterization of S.

In the latter case, the challenge is usually to find the "right" $\mathbf{g}$.

[5] Change the variable representation, if necessary.

[6] The equivalence of the conditions is a consequence of the **implicit function theorem**.

Examples

1. Consider the unit sphere $S \subseteq \mathbb{R}^3$.

   - **Implicit descriptions:** $S = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$
   - **Explicit description:**
     a) If $W_1^+ = \{(x, y, z) \mid z > 0\}$, $V_1 = \pi_{x,y}(W_1^+) = \{(x, y) \mid x^2 + y^2 < 1\}$, and $f_1^+ : V_1 \to \mathbb{R}$ is given by $f_1^+(x, y) = \sqrt{1 - x^2 - y^2} = z$, then $S \cap W_1^+$ is the northern hemisphere.
     b) If $W_1^- = \{(x, y, z) \mid z < 0\}$, $V_1 = \pi_{x,y}(W_1^-) = \{(x, y) \mid x^2 + y^2 < 1\}$, and $f_1^- : V_1 \to \mathbb{R}$ is given by $f_1^-(x, y) = -\sqrt{1 - x^2 - y^2} = z$, then $S \cap W_1^-$ is the southern hemisphere.
     c) If $W_2^+ = \{(x, y, z) \mid y > 0\}$, $V_2 = \pi_{x,z}(W_2^+) = \{(x, z) \mid x^2 + z^2 < 1\}$, and $f_2^+ : V_2 \to \mathbb{R}$ is given by $f_2^+(x, z) = \sqrt{1 - x^2 - z^2} = y$, and so on.
   - **Parameteric description:** consider $\mathbf{g} : (0, \pi) \times (-\pi, \pi) \to \mathbb{R}^3$ defined by
     $$\mathbf{g}(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) = (x, y, z).$$
     Then
     $$D\mathbf{g}(\theta, \varphi) = \begin{pmatrix} \cos\theta\cos\varphi & -\sin\theta\sin\varphi \\ \cos\theta\sin\varphi & \sin\theta\cos\varphi \\ -\sin\theta & 0 \end{pmatrix}.$$
     It is an exercise to show that $\operatorname{rank}(D\mathbf{g}(\theta, \varphi)) = 2$ for all (θ, φ). Furthermore, $\mathbf{g}$ is injective over $U = (0, \pi) \times (-\pi, \pi)$. Indeed, if $(\theta, \varphi), (\theta', \varphi') \in U$ and $\mathbf{g}(\theta, \varphi) = \mathbf{g}(\theta', \varphi')$, then:
     - $\cos\theta = \cos\theta' \implies \theta = \theta'$;
     - $\sin\theta\cos\varphi = \sin\theta\cos\varphi \implies \cos\varphi = \cos\varphi'$;
     - $\sin\theta\sin\varphi = \sin\theta\sin\varphi \implies \sin\varphi = \sin\varphi'$.
     - the last two equations yield $\varphi = \varphi'$ over $(-\pi, \pi)$.

     Finally, we show that that $\mathbf{g}^{-1} : \mathbf{g}(U) \to U$ defined by $\mathbf{g}(x, y, z) = (\theta, \varphi)$ is continuous. Since $z = \cos\theta$< then $\theta = \arccos z$, which is continuous. Since $-\pi/2 < \varphi/2 < \pi/2$, we have $\cos(\varphi/2) \neq 0$, and we can write
     $$\tan\frac{\varphi}{2} = \frac{\sin\theta\sin\varphi}{\sin\theta + \sin\theta\cos\varphi} = \frac{y}{\sqrt{1 - z^2} + x},$$
     whence
     $$\varphi = 2\arctan\left(\frac{y}{\sqrt{1 - z^2} + x}\right),$$
     which is also continuous.

   But $C = \{(x, 0, z) \mid x^2 + z^2 = 1, x \leq 0\} \subseteq S$, so we have $\mathbf{g}(U) = S \setminus C$, and so $\mathbf{g}$ is a **local** parametrization of S – it is impossible to get all of S with $\mathbf{g}$.

2. Consider the infinite cone $S : z^2 = x^2 + y^2$, $z \geq 0$.

   - **Implicit description:** $S = \{(x, y, z) \mid |x^2 + y^2 - z^2 = 0\}$
   - **Explicit description:** If $f : \mathbb{R}^2 \to \mathbb{R}$ is given by $f(x, y) = x^2 + y^2$, then $S = \{(x, y, f(x, y)) \mid (x, y) \in \mathbb{R}^2\}$
   - **Parameteric description:** consider $\mathbf{g} : U = (0, 2\pi) \times (0, a) \to \mathbb{R}^3$ defined by

     $$\mathbf{g}(\varphi, r) = (r \cos \varphi, r \sin \varphi, r).$$

     We can show that $D\mathbf{g}$ is of full rank on U, that $\mathbf{g}$ is injective on U, and that $\mathbf{g}^{-1}$ is continuous on U (see exercises).

     Finally, if $C_0 = \{(x, 0, z) \mid a > x - z \geq 0\}$, then

     $$\mathbf{g}(U) = \{(x, y, z) \mid x^2 + y^2 = z^2 < a^2\} \setminus C_0;$$

     the parameterization is local. $\square$

In both examples, the local parameterization covers the surface entirely, except for a set of **measure** (area) zero (see Chapter 21) – the missing pieces do not contribute to the integrals.

A subset $S \subseteq \mathbb{R}^3$ is a **surface with a boundary** in $\mathbb{R}^3$ if for at least some point $\mathbf{p} \in S$, there is a $W_{\mathbf{p}} \subseteq_O \mathbb{R}^3$ and a parameterization $\mathbf{g} : U \to \mathbb{R}^3$ such that $\mathbf{g}(U) = V = W_{\mathbf{p}} \cap S$ and $U \subseteq_0 \mathbb{R}^2_+$. We write $\mathbf{p} \in \partial S$ if $\mathbf{p} = \mathbf{g}(\mathbf{u})$ for some $\mathbf{u} \in \partial\mathbb{R}^2_+ = \{(x, y) \mid y = 0\}$.

Examples

1. Consider the surface S which is the northern hemisphere of the unit sphere in $\mathbb{R}^3$. Let $\mathbf{p}$ be a point of S which is not on the equator: $\exists \mathbf{0} \in U \subseteq_O \mathbb{R}^2$ and a local parameterization $\mathbf{g} : U \to \mathbb{R}^3$ such that $\mathbf{g}(\mathbf{0}) = \mathbf{p}$ and $\mathbf{g}(U) \subseteq S$. For a point $\mathbf{p}$ on the equator, we can find $\mathbf{0} \in U' \subseteq_O \mathbb{R}^2_+$ and a local parameterization $\mathbf{g}' : U' \to \mathbb{R}^3$ such that $\mathbf{g}'(\mathbf{0}) = \mathbf{p}$ and $\mathbf{g}'(U') \subseteq S$. Thus ∂S is the equator.

2. A pair of trousers S is a "surface" in $\mathbb{R}^3$; the boundary ∂S consists of the top of the waistband and the bottom of the two leg openings.

3. The ellipsoid

   $$S = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \right\}$$

   is a surface without a boundary. $\square$

In the last example, there is a sense in which the **volume**

$$V=\left\{(x,y,z)\in\mathbb{R}^3 \;\middle|\; \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\leq 1\right\}$$

(which is not the same as the surface S) DOES have a "boundary", namely $\partial V=S$. In general, if S is a $m-$**dimensional object**, its boundary should be a $m-1-$**dimensional object**.

14.5 Integral of a Form on an Orientable Surface

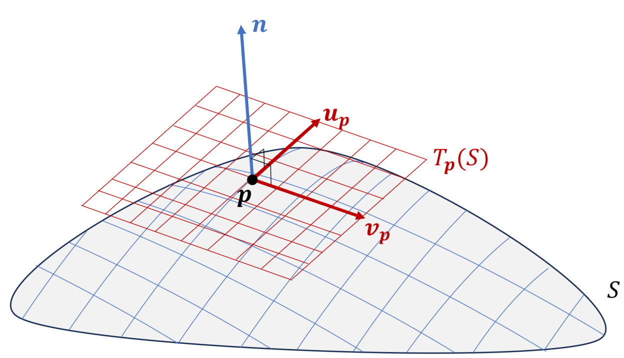
We have seen that we can induce an orientation on the boundary of planar regions; can we orient surfaces as well? Let $\mathcal{E}=\{\mathbf{e}_1,\ldots,\mathbf{e}_n\}$ and $\mathcal{E}=\{\mathbf{f}_1,\ldots,\mathbf{f}_n\}$ be two bases of $\mathbb{R}^n$, and let P be the **change of basis matrix** from $\mathcal{E}$ to $\mathcal{F}$. We say that $\mathcal{E}$ and $\mathcal{F}$ have the **same orientation** if $\det(P)>0$ and that they have **opposite orientation** if $\det(P)<0$.

Examples

1. In $\mathbb{R}^2$, if $\mathcal{E}=\{(1,0),(0,1)\}$ and $\mathcal{F}_\alpha=\{(\cos\alpha,\sin\alpha),(-\sin\alpha,\cos\alpha)\}$, the change of basis matrix is $P=\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}$ and $\det P=1$, so $\mathcal{E}$ and $\mathcal{F}_\alpha$ have the same orientation. □
2. In $\mathbb{R}^2$, if $\mathcal{E}=\{(1,0),(0,1)\}$ and $\mathcal{F}=\{(1,0),(0,-1)\}$, then $P=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ and $\det P=-1$, so $\mathcal{E}$ and $\mathcal{F}$ have opposite orientations. □

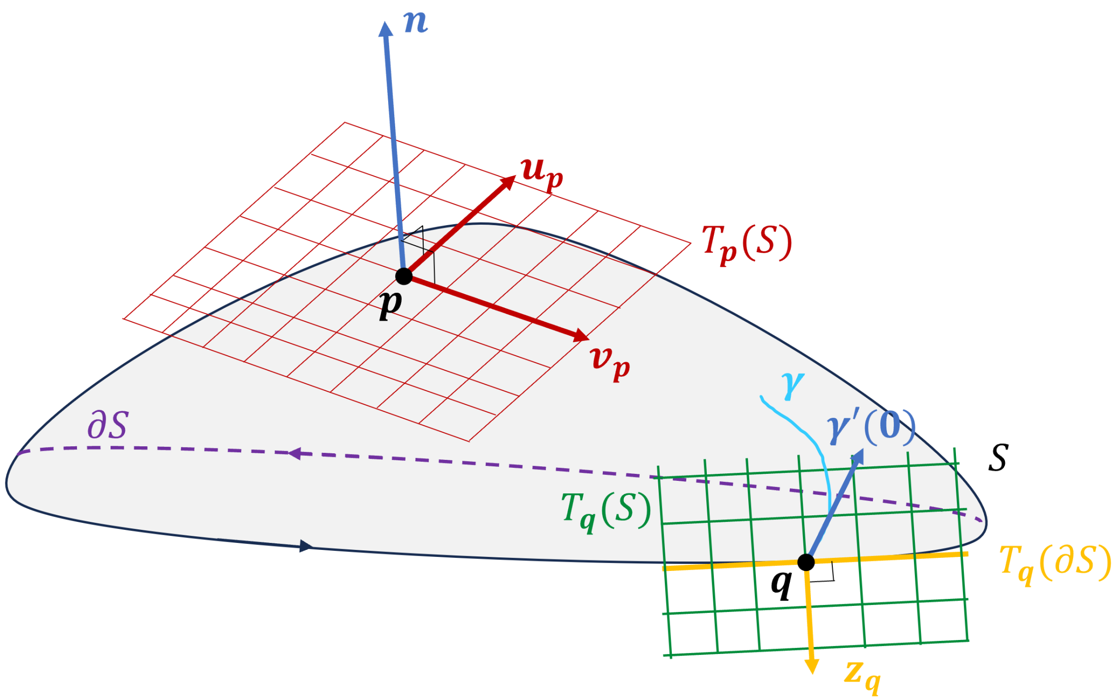
By convention, the orientation of the **canonical basis** of $\mathbb{R}^n$ is taken to be positive.

Let $S\subseteq\mathbb{R}^3$ be a surface. For all $\mathbf{p}$, let $T_{\mathbf{p}}(S)\subseteq\mathbb{R}^3$ denote the **tangent plane to** S **at p**. By definition, $T_{\mathbf{p}}(S)\simeq\mathbb{R}^2=\text{Span}(\mathbf{u_p},\mathbf{v_p})$, $\mathbf{n}\perp T_{\mathbf{p}}(S)$, as below. We say that S is **orientable** if it is possible to **continuously** select a basis $\{\mathbf{u_p},\mathbf{v_p}\}$ of $T_{\mathbf{p}}(S)$ as $\mathbf{p}\in S$ varies **continuously**.[7]

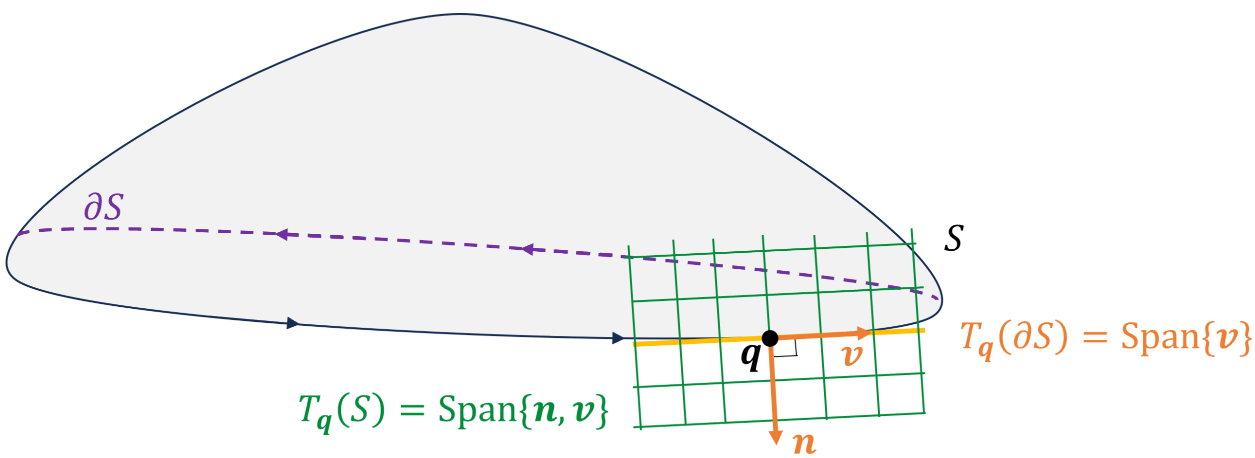


[7] Importantly, not every surface is orientable (such as a Möbius strip or a Klein bottle, for instance).

Let $S \subseteq \mathbb{R}^3$ be a compact surface with boundary ∂S. Let $\mathbf{q} \in \partial S$ and define $T_{\mathbf{q}}(\partial S) \subseteq T_{\mathbf{q}}(S)$ to be the $1-$dimensional line tangent to ∂S at $\mathbf{p}$. Pick $\alpha > 0$ and let $\boldsymbol{\gamma} : [0,t) \to S$ be a $\mathcal{C}^1$ path on S with $\boldsymbol{\gamma}(0) = \mathbf{q}$. Pick a $\mathbf{z_q} \in T_{\mathbf{q}}(S)$ such that $\mathbf{z_q} \perp T_{\mathbf{q}}(\partial S)$ and the angle between $\mathbf{z_q}$ and $\boldsymbol{\gamma}'(0) \in T_{\mathbf{q}}(S)$ is greater than a right angle. We say that $\mathbf{z_q}$ points to the **exterior of** S, whereas $-\mathbf{z_q}$ points to the **interior of** S.



The boundary ∂S is **orientable** when for all $\mathbf{q} \in \partial S$, the orientation of $T_{\mathbf{q}}(\partial S)$ is given by a vector $\mathbf{v}$ such that the orientation of $T_{\mathbf{q}}(S)$ is given by the basis $\{\mathbf{n}, \mathbf{v}\}$, where $\mathbf{n}$ is normal to $T_{\mathbf{q}}(\partial S)$ and points towards the exterior of S.



At any point of the boundary, the cross-product $\mathbf{n} \times \mathbf{v}$ (in that order) points towards the **positive orientation of the surface** S (the direction given by the **right-hand rule**).

Recall that if $U \subseteq_0 \mathbb{R}^2$ and $\omega = P(x,y)\,\mathrm{d}x \wedge \mathrm{d}y \in \Omega^2(\mathbb{R}^2)$ where P is integrable over U (see Chapter 21 for details), then

$$\int_{\mathbf{U}} \omega = \int_U P\,\mathrm{d}m, \quad \text{where} \quad \mathbf{U} : U \to \mathbb{R}^2 \equiv \text{identity on } U.$$

Let $W \subseteq_O \mathbb{R}^3$, U a Borel[8] subset of $\mathbb{R}^2$, $U \subseteq_O U$, $U \subseteq_O \mathbb{R}^2$ with Area$(U - U_0) = 0$ and let $\boldsymbol{\varphi} : U \to W$ be such that $\boldsymbol{\varphi}|_{U_0} = \boldsymbol{\varphi}_0 : U_0 \to W$ is $\mathcal{C}^1$. If $\omega \in \Omega^2(W)$, then

$$\int_{\boldsymbol{\varphi}} \omega = \int_{\boldsymbol{\varphi}_0} \omega.$$

This is well-defined, as we can see below. Let $U_0', \boldsymbol{\varphi}_0'$ be objects that satisfy the same properties as $U_0, \boldsymbol{\varphi}_0'$. Denote $\boldsymbol{\varphi}_0^*(\omega) = P_0(x,y)\,\mathrm{d}x \wedge \mathrm{d}y$ and $\boldsymbol{\varphi}'^*_0(\omega) = P_0'(x,y)\,\mathrm{d}x \wedge \mathrm{d}y$. We must show that

$$\int_U P\,\mathrm{d}m = \int_{U'} P'\,\mathrm{d}m.$$

Write $U_0'' = U_0 \cap U_0'$; we have $P_0 = P_0'$ on U_0'' and

$$U_0 \setminus U_0'' = U_0 \cap (U_0')^c \subseteq U \cap (U_0')^c = U \setminus U_0'.$$

Thus,

$$\text{Area}(U_0 \setminus U_0'') \leq \text{Area}(U_0 \setminus U_0') = 0.$$

Similarly, Area$(U_0' \setminus U_0'') = 0$, and so

$$\int_{U_0} P_0\,\mathrm{d}m = \int_{U_0''} P_0\,\mathrm{d}m = \int_{U_0''} P_0'\,\mathrm{d}m = \int_{U_0'} P_0'\,\mathrm{d}m.$$

> **Example:** let $\omega = xz^2\,\mathrm{d}y \wedge \mathrm{d}z + yx^2\,\mathrm{d}x \wedge \mathrm{d}y + zy^2\,\mathrm{d}x \wedge \mathrm{d}y \in \Omega^2(\mathbb{R}^3)$ and set $a > 0$. We consider the function $\mathbf{\Phi} : [0,\pi] \times [0, 2\pi) \to \mathbb{R}^3$ defined by
>
> $$(\theta, \varphi) \mapsto a(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta);$$
>
> $\mathbf{\Phi}$ is a parameterization in spherical coordinates of the surface
>
> $$S_a = \{(x,y,z) \mid x^2 + y^2 + z^2 = a^2\}.$$
>
> Let $U = [0,\pi] \times [0,2\pi)$ and $U_0 = (0,\pi) \times (0,2\pi)$; then $\mathbf{\Phi}_0 = \mathbf{\Phi}|_{U_0}$ is $\mathcal{C}^1$. Since Area$(U \setminus U_0) = 0$, we have
>
> $$\int_{\mathbf{\Phi}} \omega = \int_{U_0} \mathbf{\Phi}^*(\omega).$$

[8] For all intents and purposes, U is sufficiently "nice" (see Chapter 21).

We can show that

$$\mathbf{\Phi}^*(\omega) = a^5(\sin^3\theta\cos^2\theta + \sin^5\theta\cos^2\varphi\sin^2\varphi)\,\mathrm{d}\theta\wedge\mathrm{d}\varphi,$$

and so

$$\int_{\mathbf{\Phi}}\omega = \int_0^{\pi}\int_0^{2\pi} a^5(\sin^3\theta\cos^2\theta + \sin^5\theta\cos^2\varphi\sin^2\varphi)\,\mathrm{d}\theta\,\mathrm{d}\varphi = \frac{4}{5}\pi a^5.$$

For any (θ,φ), the basis $\{\frac{\partial\mathbf{\Phi}}{\partial\theta},\frac{\partial\mathbf{\Phi}}{\partial\varphi}\}$ defines the positive orientation on S_a *via* the right-hand rule; $\mathbf{\Phi}_0$ then defines a local parameterization of S_a up to a set of area 0. $\square$

If S is orientable in $\mathbb{R}^3$ and $\mathbf{\Phi}: U\to\mathbb{R}^3$, $\mathbf{\Psi}: V\to\mathbb{R}^3$ are two orientation-preserving parameterizations of S, let $\boldsymbol{\eta}: U\to V$ be the unique bijection such that $\mathbf{\Phi}=\mathbf{\Psi}\circ\boldsymbol{\eta}$. Then $\boldsymbol{\eta}$ is a diffeomorphism and $\forall\mathbf{u}\in U$,

$$D\mathbf{\Phi}(\mathbf{u}) = D\mathbf{\Psi}(\boldsymbol{\eta}(\mathbf{u}))D\boldsymbol{\eta}(\mathbf{u}).$$

Since $\{\frac{\partial\mathbf{\Phi}(\mathbf{u})}{\partial u_1},\frac{\partial\mathbf{\Phi}(\mathbf{u})}{\partial u_2}\}$ is a positive basis of $T_{\mathbf{\Phi}(\mathbf{u})}(S)$ and since $\{\frac{\partial\mathbf{\Psi}(\boldsymbol{\eta}(\mathbf{u}))}{\partial v_1},\frac{\partial\mathbf{\Psi}(\boldsymbol{\eta}(\mathbf{u}))}{\partial v_2}\}$ is a positive basis of $T_{\mathbf{\Phi}(\boldsymbol{\eta}(\mathbf{u}))}(S)$, both $D\mathbf{\Phi}(\mathbf{u})$ and $D\mathbf{\Psi}(\boldsymbol{\eta}(\mathbf{u}))$ transform the canonical basis of $\mathbb{R}^2$ into positive-orientation bases of $T_{\mathbf{\Phi}(\mathbf{u})}(S)$.

In that case, $D\boldsymbol{\eta}(\mathbf{u})$ preserves the orientation of $\mathbb{R}^2$ and $\det(D\boldsymbol{\eta}(\mathbf{u}))>0$ for all $\mathbf{u}$.

If $\omega\in\Omega^2(\mathbb{R}^3)$, we have $\mathbf{\Phi}^*(\omega)=a(u_1,u_2)\,\mathrm{d}u_1\wedge\mathrm{d}u_2$, $\mathbf{\Psi}^*(\omega)=b(v_1,v_2)\,\mathrm{d}v_1\wedge\mathrm{d}v_2$ for $a\in\Omega^0(U)$ and $b\in\Omega^0(V)$. Since $\mathbf{\Phi}=\mathbf{\Psi}\circ\boldsymbol{\eta}$, we have

$$\mathbf{\Phi}^*(\omega) = a\,\mathrm{d}u_1\wedge\mathrm{d}u_2 = \boldsymbol{\eta}^*(\mathbf{\Psi}^*(\omega)) = \boldsymbol{\eta}^*(b\,\mathrm{d}v_1\wedge\mathrm{d}v_2) = (b\circ\boldsymbol{\eta})\det(D\boldsymbol{\eta})\,\mathrm{d}u_1\wedge\mathrm{d}u_2.$$

Thus, according to the change of variable theorem (see Chapter 21), we have

$$\begin{aligned}\int_U\mathbf{\Phi}^*(\omega) &= \int_{\mathbf{\Phi}}\omega = \int_U a\,\mathrm{d}u_1\,\mathrm{d}u_2 = \int_U(b\circ\boldsymbol{\eta})\det(D\boldsymbol{\eta})\,\mathrm{d}u_1\,\mathrm{d}u_2 = \int_U(b\circ\boldsymbol{\eta})|\det(D\boldsymbol{\eta})|\,\mathrm{d}u_1\,\mathrm{d}u_2\\ &= \int_V b\,\mathrm{d}v_1\,\mathrm{d}v_2 = \int_V\mathbf{\Psi}^*(\omega) = \int_{\mathbf{\Psi}}\omega.\end{aligned}$$

We have then proven the following result.

Theorem 197
Under the hypotheses outlined above, the integrability of ω with respect to $\mathbf{\Phi}$ and the value of $\int_{\mathbf{\Phi}}\omega$ depend only on ω and the surface $S=\mathbf{\Phi}(U)$.

We say that $\omega\in\Omega^2(\mathbb{R}^3)$ is integrable over $S\subseteq\mathbb{R}^3$ if ω is integrable with respect to a parameterization $\mathbf{\Phi}$ of S and we write $\int_S\omega=\int_{\mathbf{\Phi}}\omega$.

14.6 Area of a Surface and Flux Integral

In an exercise from the previous chapter, we saw that if $\mathbf{u}, \mathbf{v}, \mathbf{n} \in \mathbb{R}^3$ are such that $\mathbf{u}$ and $\mathbf{v}$ are not parallel, $\mathbf{n} \perp \mathbf{u}, \mathbf{v}$ with $\|\mathbf{n}\| = 1$ and

$$\varphi = n_1\,\mathrm{d}y \wedge \mathrm{d}x + n_2\,\mathrm{d}z \wedge \mathrm{d}x + n_3\,\mathrm{d}x \wedge \mathrm{d}y \in \Lambda^2(\mathbb{R}^3),$$

then $\varphi(\mathbf{u}, \mathbf{v})$ represents the **signed area** of the parallelogram bound by $\mathbf{u}$ and $\mathbf{v}$. Thus:

- if $\mathbf{n} = \frac{\mathbf{u}\times\mathbf{v}}{\|\mathbf{u}\times\mathbf{v}\|}$, then Area $= \varphi(\mathbf{u}, \mathbf{v})$;
- if $\mathbf{n} = -\frac{\mathbf{u}\times\mathbf{v}}{\|\mathbf{u}\times\mathbf{v}\|}$, then Area $= -\varphi(\mathbf{u}, \mathbf{v})$.

Let $S \subseteq \mathbb{R}^3$ be an orientable surface, and let $\mathbf{n} : S \to \mathbb{R}^3$ be the **vector field of unit vectors normal to S, pointing towards the exterior of S.**[9]

> **Example:** consider the sphere of radius $a > 0$ centered at the origin:
>
> $$S_a = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 - a^2 = F(x, y, z) = 0\}.$$
>
> Then $\nabla F(x, y, z) = (2x, 2y, 2z) \perp S_a$ and points towards the exterior of S_a for all $(x, y, z) \in S_a$, so we could pick
>
> $$\mathbf{n}(x, y, z) = \frac{\nabla F(x, y, z)}{\|\nabla F(x, y, z)\|}. \qquad \square$$

The **area differential** $\boldsymbol{\sigma} = n_1\,\mathrm{d}y \wedge \mathrm{d}x + n_2\,\mathrm{d}z \wedge \mathrm{d}x + n_3\,\mathrm{d}x \wedge \mathrm{d}y \in \Omega^2(\mathbb{R}^3)$ is such that $\boldsymbol{\sigma} : \mathbb{R}^3 \to \Lambda^2(\mathbb{R}^3)$. According to the preceding discussion, for all $\mathbf{s} \in S \subseteq \mathbb{R}^3$, and for all $\mathbf{u}, \mathbf{v} \in T_{\mathbf{s}}(S)$, we have

$$\boldsymbol{\sigma}(\mathbf{s})(\mathbf{u}, \mathbf{v}) = \text{signed area of parallelogram bound by } \mathbf{u} \text{ and } \mathbf{v}.$$

Using the above notation, we then have the following result.

> **Proposition 198**
> *For an orientable surface $S \subseteq \mathbb{R}^3$, let $\boldsymbol{\sigma} \in \Omega^2(\mathbb{R}^3)$ be the area differential of S. Then the signed area of S is given by $\int_S \omega$.*

We sometimes used the following formulation:

$$\text{Signed Area}(S) = \iint_{U_0} \left\| \frac{\partial \boldsymbol{\sigma}}{\partial s} \times \frac{\partial \boldsymbol{\sigma}}{\partial t} \right\| \mathrm{d}s\,\mathrm{d}t,$$

where $\boldsymbol{\Phi} : U_0 \to \mathbb{R}^3$ is a parameterization of S.

[9] In other words, we can find a continuous mapping $\mathbf{s} \in S \mapsto \{\mathbf{u}(\mathbf{s}), \mathbf{v}(\mathbf{s})\}$, where $\{\mathbf{u}(\mathbf{s}), \mathbf{v}(\mathbf{s})\} \in T_{\mathbf{s}}(S)$ defines the orientation of S, so that $\{\mathbf{n}(\mathbf{s}), \mathbf{u}(\mathbf{s}), \mathbf{v}(\mathbf{s})\}$ forms a basis of $\mathbb{R}^3$ with **positive orientation.**

Example: consider the unit sphere

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 - 1 = F(x, y, z) = 0\}.$$

The outward normal vector field $\mathbf{n} : S \to \mathbb{R}^3$ is given by

$$\mathbf{n}(x, y, z) = \frac{\nabla F(x, y, z)}{\|\nabla F(x, y, z)\|} = (x, y, z) \perp S.$$

The area differential of S is thus $\boldsymbol{\sigma} = x\, \mathrm{d}y \wedge \mathrm{d}x + y\, \mathrm{d}z \wedge \mathrm{d}x + z\, \mathrm{d}x \wedge \mathrm{d}y \in \Omega^2(\mathbb{R}^3)$. In order to calculate $\int_S \boldsymbol{\sigma}$, we use the following parameterization of S:

$$\mathbf{\Phi} : U_0 = [0, \pi] \times [0, 2\pi) \to \mathbb{R}^3, \quad \text{where} \quad \mathbf{\Phi}(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta),$$

and

$$\int_S \boldsymbol{\sigma} = \int_{\mathbf{\Phi}} \boldsymbol{\sigma} = \int_{U_0} \mathbf{\Phi}^*(\boldsymbol{\sigma}).$$

But $\mathbf{\Phi}^*(\boldsymbol{\sigma}) = (\sin^3\theta + \cos^2\theta\sin\theta)\, \mathrm{d}\theta \wedge \mathrm{d}\varphi$, so that

$$\int_{U_0} \mathbf{\Phi}^*(\boldsymbol{\sigma}) = \int_0^\pi \int_0^{2\pi} (\sin^3\theta + \cos^2\theta\sin\theta)\, \mathrm{d}\theta\, \mathrm{d}\varphi = 4\pi. \qquad \square$$

14.7 Stokes' Theorem

We finish this chapter (and this part of the course notes) with a generalization of Green's theorem, which we unfortunately present without proof.

Theorem 199 (STOKES' THEOREM)
Let $M \subseteq W \subseteq_O \mathbb{R}^n$ be a compact orientable manifold with orientable boundary ∂M such that $\dim(M) = p$*. If $\omega \in \Omega^{p-1}(W)$, then $\int_{\partial M} \omega = \int_M d\omega$.*

When $M = S \subseteq \mathbb{R}^3$ and $p = \dim(M) = 2$, then we usually write Stokes' theorem as

$$\int_S (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{A} = \oint_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

Corollary 200
Let $\partial M = \varnothing$ in Theorem 199. If $\phi \in \Omega^p(W)$ is exact, then $\int_M \phi = 0$.

Proof: since φ is exact, $\exists \eta \in \Omega^{p-1}(W)$ such that $\mathrm{d}\eta = \varphi$, so that

$$\int_M \varphi = \int_M \mathrm{d}\eta = \int_{\partial M} \eta = 0. \qquad \blacksquare$$

14.8 Solved Problems

Let's do some vector calculus!

1. Let $\mathbf{F}(x, y) = (xy, x - y)$ and C be the boundary of the triangle with vertices $(1, 0)$, $(-1, 0)$ and $(0, 1)$. Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$.

   Solution: the triangle is parameterized by

   $$C_1 : (t, 0), \quad -1 \le t \le 1, \quad C_2 : (1 - t, t), \quad 0 \le t \le 1, \quad C_3 : (-t, 1 - t), \quad 0 \le t \le 1.$$

   Thus, the line integral of interest is

   $$\begin{aligned} I &= \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot \mathrm{d}\mathbf{r} \\ &= \int_{-1}^{1} (t^2, t) \cdot (1, 0)\, \mathrm{d}t + \int_0^1 (t - t^2, 1 - 2t) \cdot (-1, 1)\ \mathrm{d}t + \int_0^1 (t^2 - t, -1) \cdot (-1, -1)\, \mathrm{d}t = 1. \end{aligned}$$

   Under the other orientation, the answer is -1. $\square$

2. Let $\mathbf{F}(x, y) = (2xe^{x^2} \sin y, e^{x^2} \cos y)$ and C be the path defined by $x(t) = t$, $y(t) = \frac{\pi}{2} t$, $0 \le t \le 1$.

   a) Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ directly.

   b) Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ using the fundamental theorem of line integrals.

   Solution:

   a) We have

   $$\begin{aligned} I &= \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_0^1 (2te^{t^2} \sin(\pi t/2), e^{t^2} \cos(\pi t/2)) \cdot (1, \pi/2)\, \mathrm{d}t \\ &= \int_0^1 e^{t^2} (2t \sin(\pi t/2) + \pi/2 \cos(\pi t/2))\, \mathrm{d}t = \left[e^{t^2} \sin(\pi t/2) \right]_0^1 = e. \end{aligned}$$

   b) Let $f(x, y) = e^{x^2} \sin y$. Then $\mathbf{F} = \nabla f$ and

   $$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = f(1, \pi/2) - f(0, 0) = e - 0 = e,$$

   according to the fundamental theorem of line integrals. $\square$

3. Compute $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$, if $\mathbf{F}(x, y) = (x^2 y, -xy)$ and $C = \{\mathbf{r}(t) = (t^3, t^4) \mid 0 \le t \le 1\}$.

   Solution: we have $\mathbf{r}'(t) = (3t^2, 4t^3)$. Thus,

   $$\begin{aligned} \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, \mathrm{d}t = \int_0^1 \mathbf{F}\left(t^3, t^4\right) \cdot (3t^2, 4t^3)\, \mathrm{d}t \\ &= \int_0^1 \left(t^{10}, -t^7\right) \cdot (3t^2, 4t^3)\, \mathrm{d}t = \int_0^1 \left(3t^{12} - 4t^{10}\right)\ \mathrm{d}t \\ &= \left[\frac{3t^{13}}{13} - \frac{4t^{11}}{11} \right]_0^1 = -\frac{19}{143}. \quad \square \end{aligned}$$

4. Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y,z)=(y+z,-x^2,-4y^2)$ and

$$C=\{\mathbf{r}(t)=(t,t^2,t^4)\mid 0\leq t\leq 1\}.$$

Solution: in this case, we have $\mathbf{r}'(t)=(1,2t,4t^3)$. Thus,

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_0^1 \mathbf{F}\left(t,t^2,t^4\right)\cdot(1,2t,4t^3)\,dt \\ &= \int_0^1 \left(t^2+t^4,-t^2,-4t^4\right)\cdot(1,2t,4t^3)\,dt = \int_0^1 \left(t^2-2t^3+t^4-16t^7\right)\,dt \\ &= \left[\frac{t^3}{3}-\frac{t^4}{2}+\frac{t^5}{5}-2t^8\right]_0^1 = -\frac{59}{30}. \qquad \square\end{aligned}$$

5. Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$ if $\mathbf{F}(x,y,z)=(\sin x,\cos y,xz)$ and

$$C=\{\mathbf{r}(t)=(t^3,-t^2,t)\mid 0\leq t\leq 1\}.$$

Solution: in this case, we have $\mathbf{r}'(t)=(3t^2,-2t,1)$. Thus,

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_0^1 \mathbf{F}\left(t^3,-t^2,t\right)\cdot(3t^2,-2t,1)\,dt \\ &= \int_0^1 \left(\sin(t^3),\cos(-t^2),t^4\right)\cdot(3t^2,-2t,1)\,dt = \int_0^1 \left(3t^2\sin(t^3)-2t\cos(-t^2)+t^4\right)\,dt \\ &= \left[-\cos(t^3)-\sin(t^2)+\frac{t^5}{5}\right]_0^1 = \frac{6}{5}-\cos(1)-\sin(1). \qquad \square\end{aligned}$$

6. Are $\mathbf{F}(x,y)=(ye^x+\sin y, e^x+x\cos y)$ and $\mathbf{F}(x,y)=(ye^{xy}+4x^3y, xe^{xy}+x^4)$ a conservative vector fields? If so, find their potential.

Solution: the vector field $\mathbf{F}$ is conservative if and only if

$$\frac{\partial F_1}{\partial y}=\frac{\partial F_2}{\partial x}.$$

Since

$$\begin{aligned}\frac{\partial F_1}{\partial y} &= \frac{\partial}{\partial y}\left(ye^x+\sin y\right)=e^x+\cos y \\ \frac{\partial F_2}{\partial x} &= \frac{\partial}{\partial x}\left(e^x+x\cos y\right)=e^x+\cos y\end{aligned}$$

the field is conservative. In this case, the potential f satisfies $\nabla f=\mathbf{F}$, that is

$$\begin{aligned}f_x(x,y) &= F_1(x,y)=ye^x+\sin y \\ f_y(x,y) &= F_2(x,y)=e^x+x\cos y\end{aligned}$$

whence

$$f(x,y)=\int f_x(x,y)dx=\int\left(ye^x+\sin y\right)dx=ye^x+x\sin y+k(y),$$

where $k(y)$ is a function of y. Substituting this function f in the equation for f_y, we have

$$f_y(x,y) = e^x + x\cos y + k'(y) = e^x + x\cos y;$$

the function $k(y)$ is a constant since the derivative in y is zero. Thus, the family of potential for **F** is $f(x,y) = ye^x + x\sin y + k$, $k \in \mathbb{R}$.

Since

$$\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}\left(ye^{xy} + 4x^3y\right) = e^{xy} + xye^{xy} + 4x^3$$
$$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}\left(xe^{xy} + x^4\right) = e^{xy} + xye^{xy} + 4x^3$$

the second field is conservative. In this case, the potential f satisfies $\nabla f = \mathbf{F}$, that is

$$f_x(x,y) = F_1(x,y) = ye^{xy} + 4x^3y$$
$$f_y(x,y) = F_2(x,y) = xe^{xy} + x^4$$

whence

$$f(x,y) = \int f_x(x,y)dx = \int \left(ye^{xy} + 4x^3y\right)dx = e^{xy} + x^4y + k(y),$$

where $k(y)$ is a function of y. Substituting this function f in the equation for f_y, we have

$$f_y(x,y) = xe^{xy} + x^4 + k'(y) = xe^{xy} + x^4;$$

the function $k(y)$ is a constant since the derivative in y is zero. Thus, the family of potential for **F** is $f(x,y) = e^{xy} + x^4y + k$, $k \in \mathbb{R}$. □

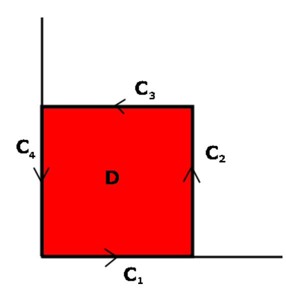
7. Find a potential for these vector fields, if one exists.

   a) $\mathbf{F}(x,y) = (2xy^3, 3x^2y + x)$;
   b) $\mathbf{F}(x,y) = (2xy^3 + y, 3x^2y + x)$;
   c) $\mathbf{F}(x,y) = (2xy, x^2 + 8y)$.

   Solution: a) and b) do not have potential functions, but $f(x,y) = x^2y + 4y^2$ is a potential function for c). □

8. Using the direct approach and Green's theorem, compute $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the square with vertices $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$, and $\mathbf{F}(x,y) = (x^2y, xy^3)$.

   Solution: the region is shown below.



Let C_1 be the segment from $(0,0)$ to $(1,0)$; C_2 the segment from $(1,0)$ to $(1,1)$; C_3 the segment from $(1,1)$ to $(0,1)$, and C_4 the segment from $(0,1)$ to $(0,0)$. Thus

$$\begin{aligned}
C_1 &= \{\mathbf{r}(t) = (t,0) \mid 0 \leq t \leq 1\} \\
C_2 &= \{\mathbf{r}(t) = (1,t) \mid 0 \leq t \leq 1\} \\
C_3 &= \{\mathbf{r}(t) = (1-t,1) \mid 0 \leq t \leq 1\} \\
C_4 &= \{\mathbf{r}(t) = (0,1-t) \mid 0 \leq t \leq 1\}
\end{aligned}$$

and

$$\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_{C_1} \mathbf{F}\cdot \mathrm{d}\mathbf{r} + \int_{C_2} \mathbf{F}\cdot \mathrm{d}\mathbf{r} + \int_{C_3} \mathbf{F}\cdot \mathrm{d}\mathbf{r} + \int_{C_4} \mathbf{F}\cdot \mathrm{d}\mathbf{r}.$$

We can show with ease that

$$\begin{aligned}
\int_{C_1} \mathbf{F}\cdot \mathrm{d}\mathbf{r} &= \int_0^1 (t^2(0), t(0)^3)\cdot(1,0)\,\mathrm{d}t = 0 \\
\int_{C_2} \mathbf{F}\cdot \mathrm{d}\mathbf{r} &= \int_0^1 (1^2(t), 1t^3)\cdot(0,1)\,\mathrm{d}t = \int_0^1 t^3\,\mathrm{d}t = \frac{1}{4} \\
\int_{C_3} \mathbf{F}\cdot \mathrm{d}\mathbf{r} &= \int_0^1 ((1-t)^2(1), (1-t)(1)^3)\cdot(-1,0)\,\mathrm{d}t = \int_0^1 -(1-t)^2\,\mathrm{d}t = -\frac{1}{3} \\
\int_{C_4} \mathbf{F}\cdot \mathrm{d}\mathbf{r} &= \int_0^1 (0^2(1-t), 0(1-t)^3)\cdot(0,-1)\,\mathrm{d}t = 0
\end{aligned}$$

so that

$$\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = 0 + \frac{1}{4} - \frac{1}{3} + 0 = -\frac{1}{12}.$$

Using Green's theorem instead, we have

$$\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\,\mathrm{d}A = \iint_D \left(y^3 - x^2\right)\,\mathrm{d}A,$$

where the region of integration D (in red) is bounded by the curve C, with the positive orientation. Since $D = \{(x,y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$, we have

$$\begin{aligned}
\iint_D \left(y^3 - x^2\right)\,\mathrm{d}A &= \int_0^1 \int_0^1 (y^3 - x^2)\,dy\,dx = \int_0^1 \left[\frac{y^4}{4} - x^2 y\right]_{y=0}^{y=1} dx \\
&= \int_0^1 \left(\frac{1}{4} - x^2\right) dx = \left[\frac{x}{4} - \frac{x^3}{3}\right]_0^1 = \frac{1}{4} - \frac{1}{3} = -\frac{1}{12}.
\end{aligned}$$

This completes the computations. $\square$

9. Compute $\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r}$, where C is the circle $x^2+y^2=4$ from $(2,0)$ to $(\sqrt{2},\sqrt{2})$, then along the segment from $(\sqrt{2},\sqrt{2})$ to the origin and finally along the segment from the origin to $(2,0)$ (with the positive orientation), for $\mathbf{F}(x,y) = (y^2 - x^2 y, xy^2)$.

Solution: according to Green's theorem,

$$\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\,\mathrm{d}A = \iint_D \left(y^2 - 2y + x^2\right)\,\mathrm{d}A,$$

where the region D is bounded by the curve C, oriented positively. In polar coordinates,

$$D_{(r,\theta)} = \{(r,\theta) \mid 0 \leq r \leq 2, 0 \leq \theta \leq \frac{\pi}{4}\},$$

and $y^2 - 2y + x^2 = r^2 - 2r\sin\theta$, whence

$$\iint_D \left(y^2 - 2y + x^2\right)\, \mathrm{d}A = \int_0^{\pi/4}\int_0^2 (r^2 - 2r\sin\theta)r\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{\pi/4}\left[4 - \frac{16}{3}\sin\theta\right]\mathrm{d}\theta$$

$$= \left[4\theta + \frac{16}{3}\cos\theta\right]_0^{\pi/4} = \pi + \frac{8}{3}(\sqrt{2} - 2).$$

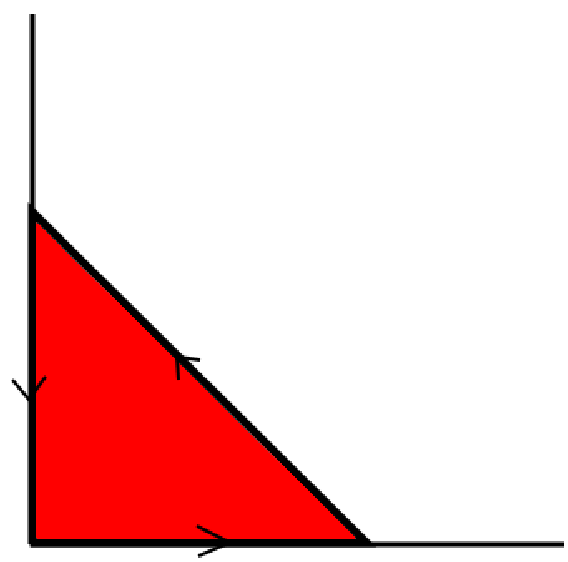
□

10. What is the work accomplished by the vector field $\mathbf{F}(x,y) = (x(x+y), xy^2)$ on a particle traveling along the $x-$axis from the origin to $(1,0)$, then from $(1,0)$ to $(0,1)$ along a straight line, and finally back to the origin along the $y-$axis?

Solution: the work in question is given by

$$W = \int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathrm{d}A = \iint_D \left(y^2 - x\right)\, \mathrm{d}A,$$

where the region D (in red) is bounded by the curve C, oriented positively.



Since $D = \{(x,y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1 - x\}$,

$$W = \iint_D \left(y^2 - x\right)\, \mathrm{d}A = \int_0^1\int_0^{1-x}(y^2 - x)\,dy\,dx = \int_0^1 \left[\frac{y^3}{3} - xy\right]_{y=0}^{y=1-x} dx$$

$$= \int_0^1 \left(\frac{(x-1)^3}{3} - x(1-x)\right)dx = \left[-\frac{1}{12}(1-x)^4 - \frac{x^2}{2} + \frac{x^3}{3}\right]_0^1 = -\frac{1}{12}. \quad \square$$

11. Let $\mathbf{F}(x,y,z) = (\frac{z}{2}, y, 2x)$ and S be the rectangle with vertices $(2,0,4)$, $(2,3,4)$, $(0,0,4)$ et $(0,3,4)$. Compute the surface integral $\iint_S \mathbf{F}\cdot d\mathbf{A}$.

    Solution: the region S is parameterized by

    $$x(s,t) = s,\ y(s,t) = t,\ z(s,t) = 4,\ (s,t) \in D : 0 \le s \le 2,\ 0 \le t \le 3$$

    Thus, $\mathbf{v}_s = (1,0,0)$, $\mathbf{v}_t = (0,1,0)$ and

    $$\mathbf{v}_s \times \mathbf{v}_t = (0,0,1).$$

    Restricted to S, the vector field takes the form

    $$\mathbf{F}(x(s,t), y(s,t), z(s,t)) = (2, t, 2s).$$

    The positive orientation of the surface S was not specified, so we select the upwards orientation as the positive orientation. Since $\mathbf{v}_s \times \mathbf{v}_t = (0,0,1)$ points upwards,

    $$I = \iint_S \mathbf{F}\cdot \mathrm{d}\mathbf{A} = \iint_D (2,t,2s)\cdot(0,0,1)\, ds\, \mathrm{d}t = \int_0^3 \int_0^2 2s\, ds\, \mathrm{d}t = 12. \qquad \square$$

12. Let $\mathbf{F}(x,y,z) = (x,y,z)$ and S be the surface defined by $z = -2x - 4y + 1$ in the first octant. Compute the surface integral $\iint_S \mathbf{F}\cdot d\mathbf{A}$.

    Solution: the region S is parameterized by

    $$x(s,t) = s,\ y(s,t) = t,\ z(s,t) = -2s - 4t + 1,\ (s,t) \in D : 0 \le t \le 1/4,\ 0 \le s \le 1/2 - 2t$$

    Thus, $\mathbf{v}_s = (1,0,-2)$, $\mathbf{v}_t = (0,1,-4)$ and

    $$\mathbf{v}_s \times \mathbf{v}_t = (2,4,1).$$

    Restricted to S, the vector field becomes

    $$\mathbf{F}(x(s,t), y(s,t), z(s,t)) = (s, t, -2s - 4t + 1).$$

    The positive orientation of S is still not specified, so we select the upwards orientation. Since $\mathbf{v}_s \times \mathbf{v}_t = (2,4,1)$ points upwards, we have

    $$\begin{aligned} I &= \iint_S \mathbf{F}\cdot \mathrm{d}\mathbf{A} = \iint_D (s,t,-2s-4t+1)\cdot(2,4,1)\, ds\, \mathrm{d}t \\ &= \int_0^{1/4}\int_0^{1/2-2t} 1\, ds\ \mathrm{d}t = \int_0^{1/4} (1/2 - 2t)\, \mathrm{d}t = \frac{1}{16}. \qquad \square \end{aligned}$$

13. Let $\mathbf{F}(x,y,z) = (-xz, -yz, z^2)$ and S be the surface $z^2 = x^2 + y^2$ lying above the plane $z = 0$ and below the plane $z = 1$. Compute $\iint_S \mathbf{F}\cdot d\mathbf{A}$.

    Solution: the region S is parameterized by

    $$x(s,t) = s,\ y(s,t) = t,\ z(s,t) = \sqrt{s^2 + t^2},\ (s,t) \in D : s^2 + t^2 \le 1.$$

Thus, $\mathbf{v}_s = (1, 0, \frac{s}{\sqrt{s^2+t^2}})$, $\mathbf{v}_t = (0, t, \frac{t}{\sqrt{s^2+t^2}})$ and

$$\mathbf{v}_s \times \mathbf{v}_t = (-s(s^2+t^2)^{-1/2}, -t(s^2+t^2)^{-1/2}, 1).$$

Restricted to S, the vector field becomes

$$\mathbf{F}(x(s,t), y(s,t), z(s,t)) = (-s\sqrt{s^2+t^2}, -t\sqrt{s^2+t^2}, s^2+t^2).$$

The positive orientation of S is once again not specified, we again select the upwards orientation as the positive one. Since $\mathbf{v}_s \times \mathbf{v}_t$ points upwards, we have

$$\begin{aligned} I &= \iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{A} = \iint_D (-s\sqrt{s^2+t^2}, -t\sqrt{s^2+t^2}, s^2+t^2) \cdot (\frac{-s}{\sqrt{s^2+t^2}}, \frac{-t}{\sqrt{s^2+t^2}}, 1)\, ds\, \mathrm{d}t \\ &= 2\iint_D (s^2+t^2)\, ds\, \mathrm{d}t. \end{aligned}$$

In polar coordinates, this last integral is easy to evaluate: $s = r\cos\theta$, $t = r\sin\theta$, $0 \le r \le 1$, $0 \le \theta \le 2\pi$:

$$I = 2\iint_D (s^2+t^2)\, ds\ \mathrm{d}t = 2\int_0^1 \int_0^{2\pi} (r^2\cos^2\theta + r^2\sin^2\theta) r\, \mathrm{d}\theta\, \mathrm{d}r = 2\int_0^1 \int_0^{2\pi} r^3\, \mathrm{d}\theta\, dr = \pi. \qquad \square$$

14. Let $\mathbf{F}(x,y,z) = (y,x,0)$ and S be the surface defined by $x^2+y^2 = 9$, $0 \le x \le 3$, $-3 \le y \le 3$, $1 \le z \le 2$. Compute the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{A}$.

    Solution: without a single computation, it is possible to determine that the flux must be zero. Why is that? $\square$

15. Let $\mathbf{F}(x,y,z) = (x,0,0)$ and let S be the surface parameterized by $x = e^p$, $y = \cos 3q$, $z = 6p$, $0 \le p \le 4$, $0 \le q \le \frac{\pi}{6}$. Compute the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{A}$.

    Solution: the region S is parameterized by

$$x(p,q) = e^p,\ y(p,q) = \cos(3q),\ z(p,q) = 6p,\ (s,t) \in D : 0 \le p \le 4,\ 0 \le q \le \frac{\pi}{6}.$$

Thus, $\mathbf{v}_p = (e^p, 0, 6)$, $\mathbf{v}_q = (0, -3\sin 3q, 0)$ and

$$\mathbf{v}_p \times \mathbf{v}_q = (18\sin(3q), 0, -3e^p\sin(3q)).$$

Restricted to S, the vector field becomes

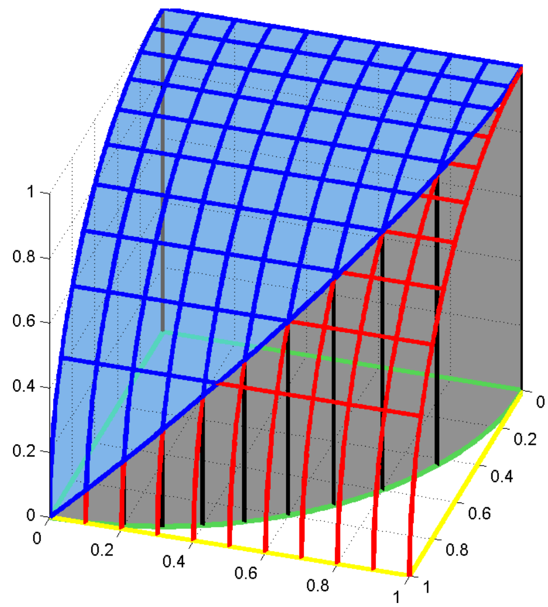
$$\mathbf{F}(x(p,q), y(p,q), z(p,q)) = (e^p, 0, 0).$$

Guess what, the surface orientation has not been specified, so we select the positive $x-$axis as a positive orientation. Since the first component of $\mathbf{v}_p \times \mathbf{v}_q$ is positive when $0 \le q \le \frac{\pi}{6}$, we have

$$\begin{aligned} I &= \iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{A} = \iint_D (e^p, 0, 0) \cdot (18\sin(3q), 0, -3e^p\sin(3q))\, dp\, dq \\ &= 18\int_0^{\pi/6} \int_0^4 e^p \sin(3q)\, dp\, dq = 6(e^4 - 1). \qquad \square \end{aligned}$$

16. What is the area of the piece S of the cylinder $x^2+z^2=a^2$ bounded by the surface of the cylinder $x^2+y^2=a^2$, where $a>0$?

Solution: in the image below, the situation is illustrated in the first octant, for $a=1$: the cylinder $x^2+z^2=a^2$ appears in grey, the cylinder $x^2+y^2=a^2$ in red. The part of S in the first octant shows up in blue.



The surface S is parameterized by

$$x=p,\quad y=q,\quad z=\sqrt{a^2-p^2},\quad (p,q)\in\Omega$$

where Ω is the region of the $xy-$plane bounded by the green curve. Accordingly,

$$A(S)=8\iint_\Omega \|\mathbf{v}_p\times\mathbf{v}_q\|\,dq\,dp,$$

where

$$\mathbf{v}_p=\left(1,0,-\frac{p}{\sqrt{a^2-p^2}}\right),\quad \mathbf{v}_q=(0,1,0)$$

and

$$\mathbf{v}_p\times\mathbf{v}_q=\left(\frac{p}{\sqrt{a^2-p^2}},0,1\right),$$

whence

$$\|\mathbf{v}_p\times\mathbf{v}_q\|=\frac{a}{\sqrt{a^2-p^2}}.$$

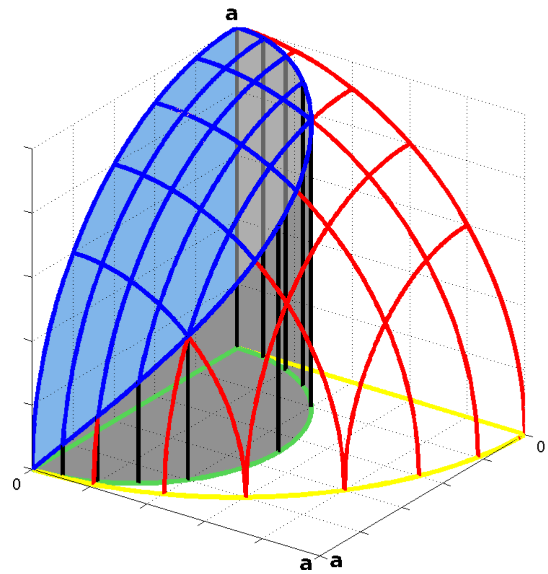
Thus,

$$A(S)=8\iint_\Omega \|\mathbf{v}_p\times\mathbf{v}_q\|\,dp\,dq=8\iint_\Omega\frac{a}{\sqrt{a^2-p^2}}\,dq\,dp=8\int_0^a\int_0^{\sqrt{a^2-p^2}}\frac{a}{\sqrt{a^2-p^2}}\,dq\,dp$$

$$=8\int_0^a\left[\frac{a}{\sqrt{a^2-p^2}}q\right]_0^{\sqrt{a^2-p^2}}dp=8a\int_0^a dp=8a^2.\qquad\square$$

17. What is the area of the piece S of the sphere $x^2+y^2+z^2=a^2$ bounded by the surface of the cylinder $x^2+y^2=ax$, where $a>0$?

Solution: in the image below, the situation is illustrated in the first octant, for $a=1$: the cylinder $x^2+y^2=ax$ appears in grey, the sphere $x^2+y^2+z^2=a^2$ in red. The part of S in the first octant shows up in blue.



The surface S is parameterized by

$$x=p, \quad y=q, \quad z=\sqrt{a^2-p^2-q^2}, \quad (p,q)\in\Omega$$

where Ω is the region of the $xy-$plane bounded by the green curve. Accordingly,

$$A(S)=4\iint_\Omega \|\mathbf{v}_p\times\mathbf{v}_q\|\,dq\,dp,$$

where

$$\mathbf{v}_p=\left(1,0,-\frac{p}{\sqrt{a^2-p^2-q^2}}\right), \quad \mathbf{v}_q=\left(0,1,-\frac{q}{\sqrt{a^2-p^2-q^2}}\right)$$

and

$$\mathbf{v}_p\times\mathbf{v}_q=\left(\frac{p}{\sqrt{a^2-p^2-q^2}},\frac{q}{\sqrt{a^2-p^2-q^2}},1\right), \quad \text{whence } \|\mathbf{v}_p\times\mathbf{v}_q\|=\frac{a}{\sqrt{a^2-p^2-q^2}}.$$

Thus,

$$A(S)=4\iint_\Omega \|\mathbf{v}_p\times\mathbf{v}_q\|\,dp\,dq=4\iint_\Omega \frac{a}{\sqrt{a^2-p^2-q^2}}\,dq\,dp=4\int_0^a\int_0^{\sqrt{ap-p^2}}\frac{a}{\sqrt{a^2-p^2-q^2}}\,dq\,dp$$

$$=4\int_0^a\left[a\arctan\left(\frac{q}{\sqrt{a^2-p^2-q^2}}\right)\right]_0^{\sqrt{ap-p^2}}dp=4a\int_0^a\arctan\left(\sqrt{\frac{p}{a}}\right)dp$$

$$=4a\left[(p+a)\arctan\left(\sqrt{\frac{p}{a}}\right)-\sqrt{ap}\right]_0^a=2a^2(\pi-2). \qquad \square$$

18. Let $\mathbf{F}(x,y,z) = (2x-y, x+4y, 0)$. Compute the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ using Stokes' theorem, when C is a circle of radius 10 centered at the origin

   a) in the plane $z=0$;
   b) in the plane $x=0$.

Solution: Since curl $\mathbf{F}(x,y,z) = (0,0,2)$, if C is oriented positively, we have

$$\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \iint_S \text{curl}\,\mathbf{F}\cdot \mathrm{d}\mathbf{A}$$

according to Stokes' Theorem.

a) We select the $xy-$plane region S parameterized by

$$\begin{aligned}&x = r\cos\theta,\ y = r\sin\theta,\ z=0,\\ &(s,t)\in D = \{0\leq r\leq 10,\ 0\leq\theta\leq 2\pi\}.\end{aligned}$$

Thus, $\mathbf{v}_r = (\cos\theta, \sin\theta, 0)$,

$$\mathbf{v}_\theta = (-r\sin\theta, r\cos\theta, 0)$$

and $\mathbf{v}_r\times\mathbf{v}_\theta = (0,0,r)$. The positive orientation has to be the upwards orientation. Since $\mathbf{v}_r\times\mathbf{v}_\theta$ points upwards when $r\geq 0$,

$$\begin{aligned}I &= \iint_S \text{curl}\,\mathbf{F}\cdot\mathrm{d}\mathbf{A}\\ &= \iint_S (0,0,2)\cdot(0,0,r)\,\mathrm{d}r\,\mathrm{d}\theta\\ &= \iint_D 2r\,\mathrm{d}r\,\mathrm{d}\theta = 2\int_0^{2\pi}\int_0^{10} r\,\mathrm{d}r\,\mathrm{d}\theta = 200\pi.\end{aligned}$$

b) We select the $yz-$plane region S parameterized by

$$x=0,\ y=r\cos\theta,\ z=r\sin\theta,\ (r,\theta)\in D: 0\leq r\leq 10,\ 0\leq\theta\leq 2\pi.$$

Thus, $\mathbf{v}_r = (0, \cos\theta, \sin\theta)$,

$$\mathbf{v}_\theta = (0, -r\sin\theta, r\cos\theta)$$

and $\mathbf{v}_r\times\mathbf{v}_\theta = (r,0,0)$. Independently of the orientation of S, we have

$$\begin{aligned}I &= \iint_S \text{curl}\,\mathbf{F}\cdot\mathrm{d}\mathbf{A}\\ &= \iint_D (0,0,2)\cdot(r,0,0)\,\mathrm{d}r\,\mathrm{d}\theta = \iint_S 0\,\mathrm{d}r\,\mathrm{d}\theta = 0.\square\end{aligned}$$

14.9 Exercises

1. Prepare a 2-page summary of this chapter, with important definitions and results.
2. Translate all of the solved problems of this section (and their solutions) into the language of differential forms.
3. If $\varphi : [a, b] \to [c, d]$ is a $\mathcal{C}^1$ diffeomorphism, show that $\varphi'(t) \neq 0$ for all $t \in [a, b]$.
4. Prove Proposition 189.
5. Flesh out the details in the proof of Green's theorem.
6. For the parametric description of the unit sphere $S \subseteq \mathbb{R}^3$, show that $\text{rank}(D\mathbf{g}(\theta, \varphi)) = 2$ for all (θ, φ).
7. For the parametric description of the cone $S \subseteq \mathbb{R}^3$, show that $\text{rank}(D\mathbf{g}(\varphi, r)) = 2$ for all (φ, r), that $\mathbf{g}$ is injective, and that $\mathbf{g}^{-1}$ is continuous.
8. Complete the calculations of the example on pp. 352-352.
9. Complete the calculations of the example on p. 355.
10. Consider the following classical mathematical results.

    Fundamental Theorem of Calculus: Let $f : [a, b] \to \mathbb{R}$ be R-int and $F : [a, b] \to \mathbb{R}$ be such that F is continuous on $[a, b]$, differentiable on (a, b) and $F'(x) = f(x)$ for all $x \in (a, b)$. Then $\int_a^b f = F(b) - F(a)$.

    Fundamental Theorem of Line Integrals: Let $U \subseteq_O \mathbb{R}^n$, $\phi : U \to \mathbb{R}$ be C^1 and L be a piecewise-C^1 path from A to B in U. Then $\int_L \nabla\phi(\mathbf{r}) \cdot d\mathbf{r} = \phi(B) - \phi(A)$.

    Green's Theorem: Let C be a positively oriented, piecewise smooth, simple closed curve in $\mathbb{R}^2$ and let D be the region bounded by C. If L and M are C^1 on an open region containing D, then

    $$\oint_C (L\,\mathrm{d}x + M\,\mathrm{d}y) = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right)\,\mathrm{d}A.$$

    Classical Stokes' Theorem: Let $S \subseteq \mathbb{R}^3$ be a compact surface with a piecewise-smooth boundary C. If $\mathbf{F} : S \to \mathbb{R}^3$ is C^1, then

    $$\int_S \text{curl}\,\mathbf{F} \cdot \mathrm{d}\mathbf{A} = \oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

    Divergence Theorem: Let $W \subseteq \mathbb{R}^3$ be a compact solid with a piecewise-smooth boundary ∂W. If $\mathbf{F} : W \to \mathbb{R}^3$ is C^1, then

    $$\iiint_W \text{div}\,\mathbf{F}\,\mathrm{d}V = \int_{\partial W} \mathbf{F} \cdot \mathrm{d}\mathbf{A}.$$

    Using the language of differential forms, explain why these five results are special instances of the same result.