Chapter 15

General Topology Concepts

In this chapter, we begin the study of general topology, which extends the concepts of analysis to **general spaces** (on which metrics may not necessarily be definable). We start by presenting the basic concepts and definitions of topology: **open sets**, **bases**, **separation axioms**, **continuity**, and **homeomorphisms**, and we present a few examples of frequently-encountered **topologies**: order, box, subspace, product, and quotient.

15.1 Basic Definitions

Let X be a set. A **topology $\mathfrak{T}$ on X** is a collection of subsets of X.[1] such that

1. $\varnothing, X \in \mathfrak{T}$;
2. if $U_1, \ldots, U_n \in \mathfrak{T}$, then $\bigcap_{i=1}^{n} U_i \in \mathfrak{T}$;
3. if $\{U_\alpha\}_{\alpha \in \mathcal{A}} \in \mathfrak{T}$, then $\bigcup_{\alpha \in \mathcal{A}} U_\alpha \in \mathfrak{T}$.

The ordered pair $(X, \mathfrak{T})$ is a **topological space**. The sets $U \in \mathfrak{T}$ are called the **open sets of** X. If U is an open set in X containing x, we say that U is a **neighbourhood of x in X**.

Examples: The following collections are topologies on X.

1. $\mathfrak{T} = \wp(X)$ is the **discrete topology** on X.
2. $\mathfrak{T} = \{\varnothing, X\}$ is the **indiscrete topology** on X.
3. If $X = \mathbb{R}$, $\mathfrak{T} = \{A \mid A = \text{ union of open intervals in } \mathbb{R}\}$ is the **standard topology on $\mathbb{R}$.**

[1] Or a subset $\mathfrak{T}$ of the power set $\wp(X)$.

4. If X is a metric space, $\mathfrak{T} = \{A \mid A \text{ is open in } X \text{ under the metric}\}$ is the **metric topology** on X.

5. $\mathfrak{T} = \{A \mid X \setminus A \text{ is finite}\} \cup \{\varnothing\}$ is the **finite complement topology** on X.

6. $\mathfrak{T} = \{A \mid X \setminus A \text{ is countable}\} \cup \{\varnothing\}$ is the **countable complement topology** on X. □

Let $\mathfrak{T}_1$ and $\mathfrak{T}_2$ be two topologies on a set X. If $\mathfrak{T}_1 \subseteq \mathfrak{T}_2$, then $\mathfrak{T}_2$ is **finer** than $\mathfrak{T}_1$ and $\mathfrak{T}_1$ is **coarser** than $\mathfrak{T}_2$. Obviously, the discrete topology is finer than all other topologies on X.

If $\mathfrak{T}_1 \subsetneq \mathfrak{T}_2$, then $\mathfrak{T}_2$ is **strictly finer** than $\mathfrak{T}_1$ and $\mathfrak{T}_1$ is **strictly coarser** than $\mathfrak{T}_2$. The collection of all topologies on a set X and the inclusion relation form a **poset**, but that will not be that important for us.

A **basis** $\mathfrak{B}$ for a topology is a family of subsets of X such that

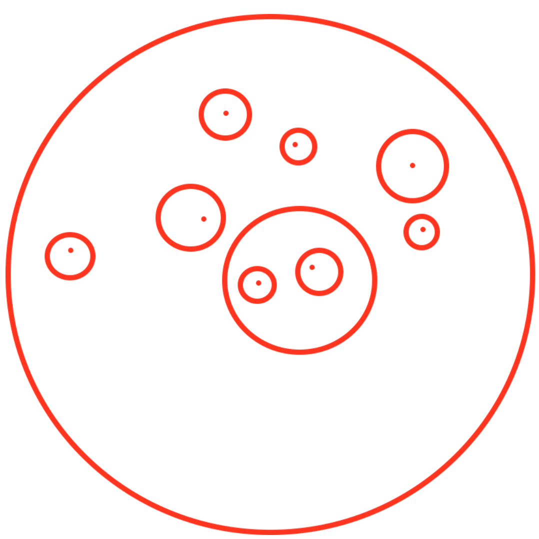
1. if $x \in X$, then there exists $B \in \mathfrak{B}$ such that $x \in B$;[2]

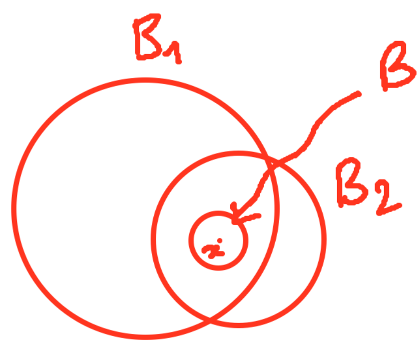
2. if $B_1, B_2 \in \mathfrak{B}$ and $x \in B_1 \cap B_2$, then there exists $B \in \mathfrak{B}$ such that $x \in B \subseteq B_1 \cap B_2$.

The **topology generated by the basis** $\mathfrak{B}$ is

$$\mathfrak{T}(\mathfrak{B}) = \left\{ \bigcup_{B \in \mathfrak{B}'} B \;\middle|\; \mathfrak{B}' \subseteq \mathfrak{B} \right\}.$$

We illustrate conditions 1 (left), 2 (right) for the standard topology on $\mathbb{R}^2$ below.



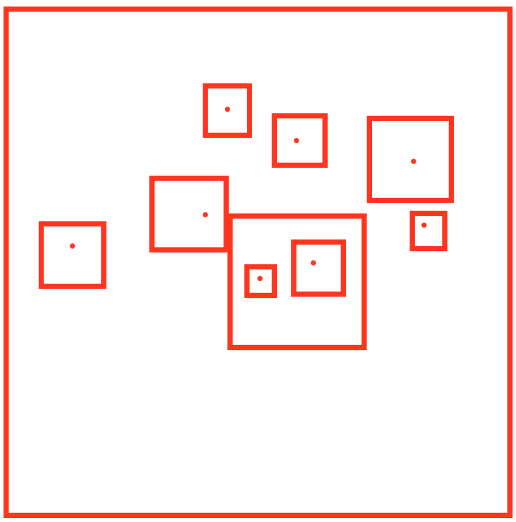


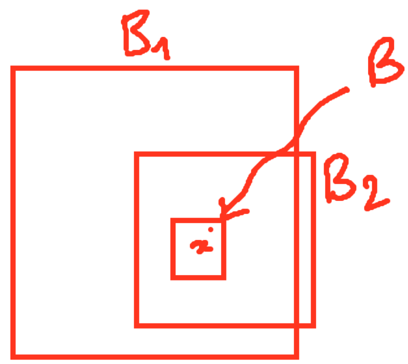
[2]Note that for a given x, the set B need not be unique.

Examples

1. The standard topology on $\mathbb{R}$ has the **open intervals** as a basis.
2. Let $X = \mathbb{R}^2$, $\mathfrak{B}_1$ be the set of all open discs in X, and $\mathfrak{B}_2$ the set of all open squares. Then $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are bases. □

We illustrate conditions 1 (left), 2 (right) for the ℓ_1 topology on $\mathbb{R}^2$.





Theorem 201
Suppose that $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are bases for topologies $\mathfrak{T}_1$ and $\mathfrak{T}_2$, respectively. Then $\mathfrak{T}_1$ is finer than $\mathfrak{T}_2$ if and only if for each $B_2 \in \mathfrak{B}_2$ and any $x \in B_2$, there exists $B_1 \in \mathfrak{B}_1$ such that $x \in B_1 \subseteq B_2$.

Proof: suppose $\mathfrak{T}_1$ is finer than $\mathfrak{T}_2$. Then $B_2 \in \mathfrak{T}_1$ exists $B \in \mathfrak{T}_1$ such that $x \in B \subseteq B_2$. Then, since $\mathfrak{B}_1$ is a basis for $\mathfrak{T}_1$, there exists $B_1 \in \mathfrak{B}_1$ such that $x \in B_1 \subseteq B \subseteq B_2$.

Conversely, let $B \in \mathfrak{B}_2$ and $x \in B$. Then there exists $B_x \in \mathfrak{B}_1$ such that $x \in B_x \subseteq B$, so

$$B = \bigcup_{x \in B} B_x,$$

and $B \in \mathfrak{T}_1$. But any $B_2 \in \mathfrak{T}_2$ is a union of open sets B, so $\mathfrak{T}_2 \subseteq \mathfrak{T}_1$. ■

In the preceding example (second item), it is possible to fit a square inside any circle and vice-versa, and so $\mathfrak{T}(\mathfrak{B}_1) = \mathfrak{T}(\mathfrak{B}_2)$.

A **sub-basis** for a topology on a set X is a collection $\mathfrak{S}$ of subsets of X such that for each $x \in X$, there exists $S \in \mathfrak{S}$ with $x \in \mathfrak{S}$ (note that this means that $X = \bigcup_{S \in \mathfrak{S}} S$).

Examples

1. Let X be a set. Then $\mathfrak{S} = \{x \mid x \in X\}$ is a sub-basis for the discrete topology and $\mathfrak{S}' = \{\varnothing, X\}$ is a sub-basis for the indiscrete topology.

2. Either of the following sets of semi-finite intervals form a sub-basis for the standard topology on $\mathbb{R}$:

$$\begin{aligned} \mathfrak{S} &= \{(a, +\infty) \mid a \in \mathbb{R}\} \cup \{(-\infty, b) \mid b \in \mathbb{R}\} \\ \mathfrak{S}' &= \{(a, +\infty) \mid a \in \mathbb{R}\}. \end{aligned}$$

A basis $\mathfrak{B}$ can be built from a sub-basis $\mathfrak{S}$ by adding to it **all finite intersections** of its elements. Indeed, $B_1, B_2 \in \mathfrak{B} \implies B_1 \cap B_2 \in \mathfrak{B}$ if

$$\mathfrak{B} = \mathfrak{S} \cup \left\{ \bigcap_{i=1}^{n} S_i \,\middle|\, S_i \in \mathfrak{S} \right\}.$$

Example: consider $X = \mathbb{R}$ and $\mathfrak{B} = \{[a, b) \mid a, b \in \mathbb{R}\}$. Then,

$$[a, b) \cap [c, d) = \begin{cases} \varnothing & \text{if } b \leq c \\ [a, b) & \text{if } b \geq c, a \geq c, b \leq d \\ [c, d) & \text{if } b \geq c, a \leq c, b \geq d \\ [c, b) & \text{if } b \geq c, a \leq c, b \leq d \\ [a, d) & \text{if } b \geq c, a \geq c, b \geq d \end{cases}$$

The set $\mathfrak{B}$ is a basis for some topology $\mathfrak{T}'$ on $\mathbb{R}$. We compare $\mathfrak{T}'$ with the standard topology $\mathfrak{T}$ on $\mathbb{R}$ and show that the two topologies are not equal. Suppose $(a, b) \in \mathfrak{T}$. Then, for any $x \in (a, b)$, we get $[x, b) \in \mathfrak{B}$ and $[x, b) \subset (a, b)$. Hence $(a, b) \in \mathfrak{T}'$, and $\mathfrak{T} \subseteq \mathfrak{T}'$, i.e. $\mathfrak{T}'$ is finer than $\mathfrak{T}$.

However, the inclusion is not reversed, which is to say, $[a, b[\notin \mathfrak{T}$. If it were, since $a \in [a, b[$, there would exist (c, d) such that $a \in (c, d) \subseteq [a, b)$, but this is impossible. Thus $\mathfrak{T} \subsetneq \mathfrak{T}'$, i.e. $\mathfrak{T}'$ is strictly finer than $\mathfrak{T}$.

The topology $\mathfrak{T}'$ on $\mathbb{R}$ is the **lower limit topology**, denoted by $\mathbb{R}_l$. □

Let X be a set with a **total order** $\mathcal{R}$. By definition,

1. for every $x, y \in X$, if $x \neq y$, then $x\mathcal{R}y$ or $y\mathcal{R}x$;
2. there is no $x \in X$ such that $x\mathcal{R}x$, and
3. for every $x, y, z \in X$, if $x\mathcal{R}y$ and $y\mathcal{R}z$, then $x\mathcal{R}z$.

We usually write $x < y$ instead of $x\mathcal{R}y$.

It is possible to generalize the concept of an **interval** by writing

$$(a,b) = \{x \in X \mid a < x < b\}, \quad [a,b] = \{x \in X \mid a \leq x \leq b\},$$

and so on.

The **order topology** on X is generated by the basis $\mathfrak{B}$ having as elements intervals of the following forms:

1. (a,b), for $a < b$;
2. $[\perp, b)$, if $\perp$ is a **smallest element** of X ($\perp \leq a$ for all $a \in X$), and
3. $(a, \top]$, if $\top$ is a **greatest element** of X ($\top \geq b$ for all $b \in X$).

Examples

1. The order topology on $\mathbb{R}$ is the standard topology on $\mathbb{R}$, as $\mathbb{R}$ has no lowest or greatest element (all basis elements are of the form (a,b), for $a < b$).
2. In the order topology on $\mathbb{N}$, every point is open as

   $$\{1\} = [1,2) \qquad \text{and } \{n\} = (n-1, n+1) \qquad \text{for } n > 1.$$

   Hence the order topology on $\mathbb{N}$ is the discrete topology on $\mathbb{N}$.
3. Let $X = \{0\} \cup \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$. Then

   $$\{1\} = (1/2, 1] \quad \text{and} \quad \left\{\frac{1}{n}\right\} = \left(\frac{1}{n+1}, \frac{1}{n-1}\right) \quad \text{for } n > 1.$$

   But any open set containing 0 will contain a basic set of the form $[0, \frac{1}{N})$, with $\frac{1}{N+1} \in [0, \frac{1}{N})$. Hence $\{0\}$ is not open, and the order topology on X is not discrete. □

15.2 Box and Subspace Topologies

Suppose X and Y are topological spaces. Consider the family of subsets of $X \times Y$ given by

$$\mathfrak{B} = \{U \times V \mid U \subseteq_O X, V \subseteq_O Y\},$$

where $A \subseteq_O X$ stands for $S \in \mathfrak{T}$ ("*A is an open subset of X in the topology on X*").

As $X \subseteq_O X$ and $Y \subseteq_O Y$, we have $X \times Y \in \mathfrak{B}$, and so every element of $X \times Y$ lies in (at least) one element of $\mathfrak{B}$.

Now suppose $U_1 \times V_1, U_2 \times V_2 \in \mathfrak{B}$. As $U_1 \cap U_2 \subseteq_O X$ and $V_1 \cap V_2 \subseteq_O Y$, we have

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2) \in \mathfrak{B}.$$

This means that $\mathfrak{B}$ is a basis for a topology on $X \times Y$, which we call the **box product topology on** $X \times Y$.

Two mappings come with this topology:

$$\pi_1 \mid X \times Y \to X \quad \text{and} \quad \pi_2 \mid X \times Y \to Y,$$

defined by $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$. These mappings are called the **projections onto the first and second coordinates**; we have

$$U \times V = (U \times Y) \cap (X \times V) = \pi_1^{-1}(U) \cap \pi_2^{-1}(V),$$

where

$$\pi_1^{-1}(U) = \{(x, y) : \pi_1(x, y) \in U\} \quad \text{and} \quad \pi_2^{-1}(V) = \{(x, y) : \pi_2(x, y) \in V\}.$$

The set $\mathfrak{S} = \{\pi_1^{-1}(U) \mid U \subseteq_O X\} \cup \{\pi_2^{-1}(V) \mid V \subseteq_O Y\}$ is thus a sub-basis of the box product topology on $X \times Y$.

> **Example:** if $X = Y = \mathbb{R}$, the box product topology on $\mathbb{R}^2$ is the standard topology on $\mathbb{R}^2$ (and is also the same as the ℓ_1 and ℓ_2 topologies on $\mathbb{R}^2$). □

Suppose $Y \subseteq X$, where X is a topological space. For each $V \subseteq_O X$, we define $U = V \cap Y$ to be an open set in Y. This creates a topology on Y.

1. $\varnothing, Y \subseteq_O Y$ since $\varnothing = \varnothing \cap Y$ and $Y = X \cap Y$, and $\varnothing, X \subseteq_O X$.
2. Suppose $U_\alpha \subseteq_O Y$. Then $\exists V_\alpha \subseteq_O X$ such that $U_\alpha = V_\alpha \cap Y$. But

$$\bigcup_\alpha V_\alpha \subseteq_O X \quad \text{and} \quad \bigcup_\alpha U_\alpha = \left(\bigcup_\alpha V_\alpha\right) \cap Y \Longrightarrow \bigcup_\alpha U_\alpha \subseteq_O Y.$$

3. Suppose $U_i \subseteq_O Y$, for $1 \leq i \leq n$. Then $\exists V_i \subseteq_O X$ such that $U_i = V_i \cap Y$, for $1 \leq i \leq n$. But

$$\bigcap_{i=1}^n V_i \subseteq_O X \quad \text{and} \quad \bigcap_{i=1}^n U_i = \left(\bigcap_{i=1}^n V_i\right) \cap Y \Longrightarrow \bigcap_{i=1}^n U_i \subseteq_O Y.$$

This topology on Y is called the **subspace topology on** Y **relative to** X. The open sets in Y are called **relatively open**; they are not always open in X.

Theorem 202
Suppose Y is a subspace of X and $\mathfrak{B}$ is a basis for the topology on X. Then $\mathfrak{B}_Y = \{U \cap Y \mid U \in \mathfrak{B}\}$ is a basis for the subspace topology.

Proof: let $V = U \cap Y$ and suppose $y \in V$ and $U \subseteq_O X$. Let $B \in \mathfrak{B}$ such that $y \in B \subseteq U$. Hence $y \in B_Y = B \cap Y \subseteq U \cap Y$, and so $\mathfrak{B}_Y$ is a basis for the subspace topology on Y. ■

Some examples will help to solidify the concepts.

Examples

1. Let $X = \mathbb{R}$ and $Y = \mathbb{Q}$. A basic open set of Y is a set of the form $B = (a, b) \cap \mathbb{Q}$, where $a, b \in \mathbb{R}$. Note that B contains no interval of real numbers. Hence, no open set of $\mathbb{Q}$ can be open in $\mathbb{R}$.

2. Let $X = \mathbb{R}$ and $Y = [0, 1]$. A basic open set of Y is a set of the form $B = (a, b) \cap [0, 1]$, where $a, b \in \mathbb{R}$. If $0 \leq a < b \leq 1$, the relatively open sets of Y will be open in $\mathbb{R}$. The basic sets in Y are the sets of the form $[0, b)$, $(a, 1]$, and (a, b), and the subspace topology on Y is the order topology.

3. Let $X = \mathbb{R}$ and $Y = \{-1\} \cup \left\{\frac{1}{n}\right\}_{n\in\mathbb{N}}$. In this case, the subspace topology is discrete. Indeed,
$$\{-1\}=(-3/2, -1/2) \cap Y, \qquad \left\{\frac{1}{n}\right\}=\left(\frac{1}{n+1/2}, \frac{1}{n-1/2}\right) \cap Y.$$

4. Let $X = \mathbb{R}$ and $Y = \{0\} \cup \left\{\frac{1}{n}\right\}_{n\in\mathbb{N}}$. In this case, the subspace topology is not discrete. Indeed, while
$$\left\{\frac{1}{n}\right\} = \left(\frac{1}{n+1/2}, \frac{1}{n-1/2}\right) \cap Y,$$
we have $\{0\} \neq (a, b) \cap Y$ for all $a < b \in X$. □

15.3 Dual Definitions and Separation Axioms

It is possible to define all the notions of topology in terms of **closed** sets, instead of open sets. Let X be a set. A **topology** $\mathfrak{T}$ **on** X is a collection of subsets of X such that

1. $\varnothing, X \in \mathfrak{T}$;
2. if $C_1, \ldots, C_n \in \mathfrak{T}$, then $\bigcup_{i=1}^n C_i \in \mathfrak{T}$;
3. if $\{C_\alpha\}_{\alpha\in\mathcal{A}} \in \mathfrak{T}$, then $\bigcap_{\alpha\in\mathcal{A}} C_\alpha \in \mathfrak{T}$.

The ordered pair $(X, \mathfrak{T})$ is a **topological space.** The sets $C \in \mathfrak{T}$ are called the **closed sets of** X. In general, a set V is closed in X, denoted by $V \subseteq_C X$, if and only if its **complement is open in** X.

Using this definition, it is easy to prove the following propositions.

Proposition 203
Let Y be a subspace of X. A set A is closed in Y if and only if it is the intersection of a closed set in X with Y.

Proof: left as an exercise. □

Proposition 204
Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.

Proof: left as an exercise. □

Again, let's take a look at some examples.

Examples

1. Let $X = \mathbb{R}$. Then $[a, b]$ is **closed** in $\mathbb{R}$ for all $a < b$.
2. Let $X = \mathbb{R}$. The set $[0, 1[$ is **neither open nor closed** in $\mathbb{R}$ with the standard topology.
3. If X has the discrete topology, then every set is **both open and closed**, since every set is the union of open singletons, and the complement of every set is also the union of open singletons.
4. Let $X = \{a, b, c, d\}$ be a set with 4 distinct elements. Define a topology on X by

$$\mathfrak{T} = \{\varnothing, \{a, b\}, \{c, d\}, X\}.$$

All sets which are open **are also closed, and vice-versa**; the topology is not discrete as $\{b, c\}$ **is neither open nor closed.** □

The **closure** of a set A in X is the smallest closed set containing A, usually denoted by $\overline{A}$. Obviously, $A \subseteq \overline{A}$. By definition, we have

$$\overline{A} = \bigcap_{A \subseteq C \subseteq_C X} C.$$

If $A \subseteq_C X$, then $A = \overline{A}$, as $\overline{A} \subseteq A$. Thus, A is closed if and only if $A = \overline{A}$.

Similarly, the **interior** of a set A in X is the largest open set contained in A, usually denoted by A°. Obviously, $A^\circ \subseteq A$. By definition, we also have

$$A^\circ = \bigcup_{V \subseteq A, V \subseteq_O X} V.$$

If $A \subseteq_O X$, then $A = A^\circ$, as $A^\circ \subseteq A$. Thus A is open if and only if $A = A^\circ$.

Examples

1. The closure of $(0,1)$ in $\mathbb{R}$ is $[0,1]$.
2. Let $X = \mathbb{R}$ and $A = \mathbb{Q}$. Then $A^\circ = \varnothing$ and $\overline{A} = \mathbb{R}$. □

The result from the last example follows from Theorem 206.

Theorem 205
Let A be a subset of X. Then $x \in \overline{A}$ if and only if every neighbourhood V of x has a non-empty intersection with A.

Proof: we show that $x \notin \overline{A}$ if and only if there is a neighbourhood V of x such that $A \cap V = \varnothing$. Suppose $x \notin \overline{A}$. Then there is a closed set C containing A with $x \notin C$. Let $V = X \backslash C \subseteq_O X$. Then $x \in V$ and $A \cap V \subseteq C \cap V = \varnothing$, so $A \cap V = \varnothing$.

Conversely, suppose there is a neighbourhood V of x such that $A \cap V = \varnothing$. Let $C = X \setminus V \subseteq_C X$. Then $A \subseteq C$ and $\overline{A} \subseteq C$, as C is closed. But $V \cap C = \varnothing$, so $x \notin C$ and thus $x \notin \overline{A}$. ■

Let A be a subset of X. A point $a \in X$ is a **limit point** of A if every neighbourhood of a contains a point of A different from a, i.e. $a \in \overline{A \setminus \{a\}}$.

Examples

1. Let $X = \mathbb{R}$ and $A = \{\frac{n}{n+1} \mid n \in \mathbb{N}\}$. Then $\{1\}$ is a limit point of A, and $\overline{A} = A \cup \{1\}$, according to Theorem 206.
2. Let X be a set with the indiscrete topology. For any non-empty subset A of X and any point $a \in X$, a is a limit point of A as long as $A \neq \{a\}$. For instance, Let $X = \{a, b\}$ with topology $\mathfrak{T} = \{\varnothing, X\}$. If $A = \{b\}$, then a is a limit point of A. Indeed, the only neighbourhood of a is X, and $A \cap X = \{b\} \neq \varnothing$.

We've alluded to it a few times already, so now it's time for Theorem 206.

> **Theorem 206**
> If A' is the set of all limit points of A, then $\overline{A} = A \cup A'$.
>
> **Proof:** if $x \in A \cup A'$, then $x \in A$ or $x \in A'$. In the first case, $x \in A \subseteq \overline{A}$. In the other, every neighbourhood of x contains a point of A. Thus $x \in \overline{A}$.
>
> Conversely, suppose $x \in \overline{A}$. Either $x \in A$ or $x \notin A$. It is sufficient to show that if $x \notin A$, then $x \in A'$. If $x \notin A$, every neighbourhood of x meets A in at least one point other than x. But $x \notin A$, so $x \in A'$. ■

We have the following corollary.

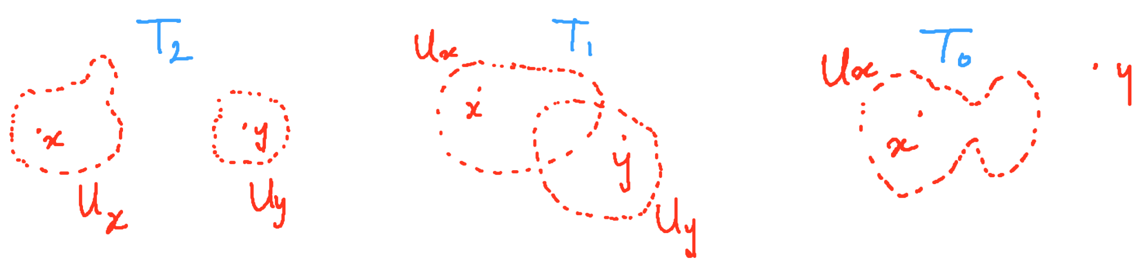
> **Corollary 207**
> *A is closed in X if and only if $A' \subseteq A$.*
>
> **Proof:** left as an exercise.

To avoid degenerate situations like the one found in the preceding example (which is to say, that any point could be the limit point of all non-singleton subsets in the indiscrete topology), we introduce the notion of **separation axioms**.

A space X is:

1. T_2 or **Hausdorff** if for every pair $x \neq y \in X$, there exist disjoint neighbourhoods U_x of x and U_y of y;
2. T_1 if for every pair $x \neq y \in X$, there exist neighbourhoods U_x of x and U_y of y such that $y \notin U_x$ and $x \notin U_y$;
3. T_0 if for every pair $x \neq y \in X$, there exist a neighbourhood U of either x or y that misses the other.[3]

Note that every T_2 space is T_1, and every T_1 space is T_0, but that there are T_0 spaces that are not T_1, and T_1 spaces that are not T_2; the conditions are illustrated below.



[3]Other separation axioms will be discussed at a later stage. In their studies, many topologists are only interested in spaces that are at least Hausdorff.

Theorem 208
If X is Hausdorff and $x \in X$ is a limit point of $A \subseteq X$, then every neighbourhood of x contains infinitely many points of A.

Proof: let x be a limit point of A and V be a neighbourhood of x. Since X is a T_2 space, its singletons are closed sets. Indeed, let $x \in X$. For all $y \neq x \in X$, there exist neighbourhoods U_x of x and U_y of y such that $x \notin U_y$ and $y \notin U_x$ (the T_1 condition holds for T_2 spaces). Then

$$X \setminus \{x\} = \bigcup_{y \in Y} U_y$$

is open in X and $\{x\}$ is closed; if x has a neighbourhood V such that $A \cap V$ is finite,

$$A \cap V = \{a_1, \ldots, a_n\}$$

must be closed, being the finite union of closed sets.

Let $W = V \setminus (A \cap V)$. If $x \in W$, then W is a neighbourhood of x such that $W \cap A = \varnothing$, which contradicts x being a limit point of A. Hence $x \in A \cap V$. After reordering if necessary, suppose $x = a_1$. Then

$$W_1 = V \setminus \{a_2, \ldots, a_n\}$$

is a neighbourhood of x such that $W_1 \cap A = \{a_1\} = \{x\}$, so that x cannot be a limit point of A. By *reductio ad absurdum*, $A \cap V$ is infinite. ■

Hausdorff spaces are particularly well-behaved with respect to toplogies.

Theorem 209
Every simply ordered set is T_2 in the order topology. The product of two T_2 spaces is T_2. A subspace of a T_2 space is T_2.

Proof: left as an exercise. ■

15.4 Continuity and Homeomorphisms

Suppose that X and Y are topological spaces. A function $f : X \to Y$ is **continuous** if $f^{-1}(V)$ is open in X whenever V is open in Y.[4]

Theorem 210
Let $f : X \to Y$. If $\mathfrak{B}$ is a basis for the topology of Y, then f is continuous if and only if $f^{-1}(B) \subseteq_O X$ for every $B \in \mathfrak{B}$.

[4]Similarly, if $\mathfrak{S}$ is a sub-basis for Y, then f is continuous if and only if $f^{-1}(S) \subseteq_O X$ for all $S \in \mathfrak{S}$.

Proof: if f is continuous, $f^{-1}(B) \subseteq_O X$ for all $B \in \mathfrak{B}$ since such $B \subseteq_O Y$. Conversely, suppose $f^{-1}(B)$ is open for all $B \in \mathfrak{B}$. Let $V = \bigcup_{i \in I} B_i$ be an open subset of Y. Then

$$f^{-1}(V) = f^{-1}\left(\bigcup_{i \in I} B_i\right) = \bigcup_{i \in I} f^{-1}(B_i)$$

is open in X since all $f^{-1}(B_i)$ is open in X for all $i \in I$. ■

Continuous functions are to topology what linear maps are to linear algebra.

Examples

1. If X and Y are metric spaces and $f : X \to Y$ is continuous with respect to the metrics in the usual sense, it is continuous in the topological sense.

2. For a product space $X \times Y$, the projections π_1, π_2 are continuous. Indeed, $\pi_1^{-1}(U) = U \times Y, \pi_2^{-1}(V) = X \times V \subseteq_O X \times Y$ when $U \subseteq_0 X, V \subseteq_O Y$.

3. For each $b \in Y$, the **inclusion map** $i_b : X \to X \times Y$ defined by $i_b(x) = (x, b)$ is continuous. Indeed, let $U \times V$ be a basic neighbourhood in $X \times Y$. Then

$$i_b^{-1}(U \times V) = \begin{cases} \varnothing, & b \notin V, \\ U, & b \in V, \end{cases}$$

   which is open in X. Thus the inclusion map is continuous.

4. For any X, the identity map id $: X \to X$ is continuous when X has the same topology as a domain as it has as a range.

5. The function id $: \mathbb{R} \to \mathbb{R}_l$ is not continuous. Indeed, let $[a, b)$ be an open set in $\mathbb{R}_l$. Then $\text{id}^{-1}([a, b)) = [a, b)$ is not open in $\mathbb{R}$, so id is not continuous. The function id $: \mathbb{R}_l \to \mathbb{R}$ is continuous, however. Let (a, b) be a basic open set in $\mathbb{R}$. Then $\text{id}^{-1}(a, b) = (a, b) = \bigcup_{n \in \mathbb{N}} [a + 1/n, b)$ is open in $\mathbb{R}_l$, so id is continuous.

6. Let $f : X \to Y$ and $g : Y \to Z$ be continuous functions. Then $g \circ f : X \to Z$ is a continuous function. Indeed, let $U \subseteq_O Z$. Then $V = g^{-1}(U) \subseteq_O Y$ since g is continuous, and $f^{-1}(V) \subseteq_O X$ as f is continuous. Then

$$(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) = f^{-1}(V)$$

   is open in X and $g \circ f$ is continuous. □

There are other ways to verify if a function is continuous.

Theorem 211
Let $f : X \to Y$. The following statements are equivalent:

1. *f is continuous;*
2. *for any $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$;*
3. *if C is closed in Y, then $f^{-1}(C)$ is closed in X.*

Proof:

1. $\implies$ 2.: If $x \in \overline{A}$, then every neighbourhood of x contains a point of A. If V is a neighbourhood of $f(x)$ then $f^{-1}(V)$ is open in X and $x \in f^{-1}(V)$. As x is a limit point of A, there exists $a \in A$ with $a \in f^{-1}(V)$ and $f(a) \in V$, so $f(a) \in V \cap f(A)$. But this just means that $f(x)$ is a limit point of $f(A)$, so $f(x) \in \overline{f(A)}$, that is $f(\overline{A}) \subseteq \overline{f(A)}$.

2. $\implies$ 3.: If C is closed in Y, then $C = \overline{C}$. Let $A = f^{-1}(C)$ then $A \subseteq \overline{A}$ and
$$f(\overline{A}) = \overline{f(A)} = \overline{f(f^{-1}(C))} \subseteq \overline{C} = C.$$
Then $\overline{A} \subseteq f^{-1}(C)$ so $f^{-1}(C)$ is closed.

3. $\implies$ 1.: If $f^{-1}(C)$ is closed whenever C is closed, then if V is open in Y, $Y \setminus V$ is closed in Y, so $f^{-1}(Y \setminus V)$ is closed in X. But
$$f^{-1}(Y \setminus V) = f^{-1}(Y) \setminus f^{-1}(V) = X \setminus f^{-1}(V),$$
so $f^{-1}(V)$ is open. Hence f is continuous. ■

A **homeomorphism** $f : X \to Y$ is a bijection for which both f and the inverse function $g : Y \to X$ are continuous. We say that X and Y are **homeomorphic** when there is a homeomorphism $f : X \to Y$. [5]

Examples

1. Let $X = \mathbb{R}$, $Y = (0, \infty)$. The function $f : X \to Y$, defined by $f(x) = e^x$ is continuous. The inverse function $g : Y \to X$ defined by $g(y) = \ln y$ is also continuous. Both these functions are bijections, so $\mathbb{R}$ and $(0, \infty)$ are homeomorphic in the standard topology.

[5] Homeomorphisms play the same role for topological spaces as isomorphisms play for groups. Consequently, homeomorphism of spaces is an equivalence relation on the 'set' of topological spaces. Homeomorphic spaces are **identical** from the point of view of topology.

2. The bijections $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ and $\arctan : \mathbb{R} \to (-\pi/2, \pi/2)$ are both continuous, so $\mathbb{R}$ is homeomorphic to $(-\pi/2, \pi/2)$.

3. The continuous bijections $f : (a, b) \to (c, d)$ and $g : (c, d) \to (a, b)$,

$$f(x) = c + \frac{d-c}{b-a}(x-a) \quad \text{and} \quad g(y) = a + \frac{b-a}{d-c}(y-c),$$

are inverses of one another, so (a, b) is homeomorphic to (c, d). □

The continuous function $f : X \to Y$ is an **embedding** of X into Y if the map $g : X \to f(X)$ defined by $g(x) = f(x)$ is a homeomorphism when $f(X)$ has the subspace topology.

Examples

1. For $b \in Y$, the **inclusion map** $i_b : X \to X \times Y$, $x \mapsto (x, b)$, is an embedding.

2. Let $A \subseteq X$. The inclusion map $\iota : A \to X$, $a \mapsto a$, is an embedding. □

Continuous functions enjoy a whole slew of attractive properties.

Theorem 212
Let X, Y, Z be top. spaces, and $V_\alpha \subseteq_O X$, $A_i \subseteq_C X$.

1. *Constant functions are continuous.*

2. *The inclusion function $\iota : A \subseteq X \to X$ is continuous.*

3. *If $f : X \to Y$ is continuous, then the restriction function $f|_A$ for all subsets $A \subseteq X$ is continuous.*

4. *If $f : X \to Y$ is continuous, then $f : X \to Z$ is continuous, assuming that $f(X) \subseteq Z$ and either $Z \subseteq Y$ or $Y \subseteq Z$.*

5. *If $X = \bigcup V_\alpha$ and the restriction $f|_{V_\alpha} : V_\alpha \to Y$ is continuous for each α, then $f : X \to Y$ is continuous.*

6. *If $X = \bigcup_{i=1}^{n} A_i$ and the restriction $f|_{A_i} : A_i \to Y$ is continuous for each $1 \leq i \leq n$, then $f : X \to Y$ is continuous.*

Proof: left as an exercise. ■

As a special case of Theorem 212, we get the following result.

Lemma 213 (Pasting Lemma)
Suppose $X = A \cup B$ where A and B are closed sets. If $f : A \to Y$ and $g : B \to Y$ are such that $f(x) = g(x)$ for all $x \in A \cap B$, then the function $h : X \to Y$ defined by

$$h(x) = \begin{cases} f(x), & \text{if } x \in A\,, \\ g(x), & \text{if } x \in B \end{cases}$$

is continuous. The same holds if A and B are both open.

Proof: left as an exercise. ∎

Lemma 213 is extremely useful.

Examples

1. If $X = Y = \mathbb{R}$, let $A = [0, \infty)$ and $B = (-\infty, 0]$, and define $f : A \to Y$ by $f(x) = x$ and $g : B \to Y$ by $g(x) = -x$. Then $h(x) = |x|$ is continuous by Lemma 213.
2. Instead, take $B = (-\infty, 0)$ and define $f : A \to Y$ by $f(x) = x + 1$ and $g : B \to Y$ by $g(x) = x$. The function h obtained by Lemma 213 construction is not continuous as $h^{-1}(1/2, 3/2) = [0, 1/2)$. □

This last example shows that Lemma 213 does not hold if A and B are not both closed, or open.

Theorem 214
Let $f : X \to Y \times Z$. Then f is continuous if and only if the functions $\pi_1 f$ and $\pi_2 f$ are continuous.

Proof: if f is continuous then $\pi_1 f$ and $\pi_2 f$ are continuous since the projections are continuous. Conversely, suppose $\pi_1 f$ and $\pi_2 f$ are continuous. If $U \times V$ is a basic open set in $Y \times Z$, then

$$f^{-1}(U \times V) = (\pi_1 f)^{-1}(U) \cap (\pi_2 f)^{-1}(V),$$

which is open as $\pi_1 f$ and $\pi_2 f$ are continuous. Hence f is continuous. ∎

The following local formulation of continuity is sometimes useful in applications. A function $f : X \to Y$ is **locally continuous** at $x \in X$ if for any open set V with $f(x) \in V$, there is a neighbourhood U of x such that $f(U) \subseteq V$. A function $f : X \to Y$ is thus continuous if and only if it is locally continuous at every point of X, as can easily be verified.

15.5 Product Topology

Suppose $\{X_\alpha\}_{\alpha\in A}$ is a family of topological spaces, where A is an arbitrary **indexing set**.[6] Then

$$X = \prod_{\alpha\in A} X_\alpha$$

is the set of all maps $x : A \to \bigcup_{\alpha\in A} X_\alpha$ such that $x(\alpha) \in X_\alpha, \forall\alpha \in A$. We write x_α for $x(\alpha)$ and $x = (x_\alpha)_{\alpha\in A}$. This set X comes equipped with **projection mappings** π_α for each $\alpha \in A$, defined by $\pi_\alpha(x) = x_\alpha$ for all $x \in X$.

We can endow X with a topology by extending the box product topology to **arbitrary products**. A basic open set in this **box topology** is a set of the form $\prod_\alpha U_\alpha$, where $U_\alpha \subseteq_O X_\alpha$ for each $\alpha \in A$.

Alternatively, extending the topology obtained by the sub-basis

$$\mathfrak{S} = \bigcup_{\alpha\in A}\{\pi_\alpha^{-1}(V_\alpha) \mid V_\alpha \subseteq_O X_\alpha\}$$

to arbitrary products yields a topology called the **product topology** on X. The basic open sets in the product topology have the form $\prod_\alpha U_\alpha$, where $U_\alpha = X_\alpha$ **except in a finite number of cases** $U_{\alpha_i} \subseteq_O X_{\alpha_i}$, for $1 \le i \le n$.

Note that when A is finite, the box and product topologies coincide. Furthermore, the basic open sets in the product topology are open in the box topology, and so **the box topology is finer than the product topology**. But this inclusion is strict. For instance, $(-1,1)^\omega$ is open in $\mathbb{R}^\omega = \prod_{n\in\mathbb{N}} \mathbb{R}$ with the box topology, but it is not open in $\mathbb{R}^\omega$ with the product topology as this would imply that $\mathbb{R} \subseteq (-1,1)$.

Theorem 215
If $\mathfrak{B}_\alpha$ is a basis for the topology on X_α, then

$$\mathfrak{B} = \left\{\prod_{\alpha\in A} B_\alpha \,\middle|\, B_\alpha \in \mathfrak{B}_\alpha\right\}$$

is a basis for $\prod_\alpha X_\alpha$ in the box topology.

Proof: left as an exercise. ∎

Theorem 216
In both the box and product topologies, the product of subspaces is a subspace and the product of Hausdorff spaces is Hausdorff.

Proof: left as an exercise. ∎

[6] In particular, A may not be countable.

While the definition of the box topology might seem the more natural of the two generalizations to infinite products, there is at least one way in which the product topology is superior (and hence, preferable).

Theorem 217
Let $f : Y \to X = \prod_\alpha X_\alpha$ and and $f_\alpha = \pi_\alpha f$ for all α. When X is endowed with the product topology, f is continuous if and only if f_α is continuous for all α.

Proof: suppose f is continuous. The projections π_α are continuous. Indeed, pick α. Let V_α be a basic open of X_α. Then $W = \pi_\alpha^{-1}(V_\alpha) = \prod_\beta W_\beta$, where $W_\alpha = V_\alpha$ and $W_\beta = X_\beta$. But W is open in the product topology, so π_α is continuous. Thus, $f_\alpha = \pi_\alpha f$ is continuous for each α, being the composition of two continuous functions.

Conversely, suppose that f_α is continuous for all α. Let $\pi_\alpha^{-1}(U_\alpha)$ be a sub-basic subset of X. As $f_\alpha = \pi_\alpha f$ is continuous,

$$f^{-1}\left(\pi_\alpha^{-1}(U_\alpha)\right) = f_\alpha^{-1}(U_\alpha)$$

is open in Y, which is to say that f is continuous. ■

This result need not be true in the box topology.

Example: consider the function $f : \mathbb{R} \to \mathbb{R}^\omega$, defined by $f_n(x) = nx$ for all $x \in \mathbb{R}$. Each f_n is continuous on $\mathbb{R}$, and $f(x) = (nx)_{n\in\mathbb{N}}$. In the box topology, $(-1,1)^\omega \subseteq_O \mathbb{R}^\omega$. But $f_n^{-1}(-1,1) = (-1/n, 1/n)$ and $f^{-1}\left((-1,1)^\omega\right) = \{0\}$, which is not open in $\mathbb{R}$. Hence f is not continuous in the box topology. □

15.6 Quotient Topology

Let X be a topological space and $f : X \to Y$ be a **surjective** mapping. We make f continuous by defining a topology on Y through

$$V \subseteq_O Y \iff f^{-1}(V) \subseteq_O X.$$

That this defines a topology is clear:

1. $\varnothing \subseteq_O Y$ as $f^{-1}(\varnothing) = \varnothing \subseteq_O X$; $Y \subseteq_O Y$ as $f^{-1}(Y) = X \subseteq_O X$ since f is surjective.
2. If $U, V \subseteq_O Y$, then $f^{-1}(U), f^{-1}(V) \subseteq_O X$. But
$$f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V) \subseteq_O X \Longrightarrow \text{so } U \cap V \subseteq_O Y.$$
3. If $U_\alpha \subseteq_O Y$ for all α, then $f^{-1}(U_\alpha) \subseteq_O X$ for all α. But
$$f^{-1}\left(\bigcup_\alpha U_\alpha\right) = \bigcup_\alpha f^{-1}(U_\alpha) \subseteq_O X \Longrightarrow \text{so } \bigcup U_\alpha \subseteq_O Y.$$

This is the **quotient topology on** Y, and $f : X \to Y$ is a **quotient map**. Thus, any continuous map $f : X \to Y$ is a quotient map whenever it is a surjective. Note that a quotient map need not be open.

> **Example:** let $X = [0, 2]$ have the subspace topology from $\mathbb{R}$, and set $Y = \{a, b\}$ where $\mathfrak{T}_Y = \{\varnothing, \{a\}, Y\}$, and define $f : X \to Y$ by
>
> $$f(x) = \begin{cases} a, & \text{if } 0 \leq x < 1, \\ b, & \text{if } 1 \leq x \leq 2. \end{cases}$$
>
> As $f^{-1}(\{a\}) = [0, 1) \subseteq_O X$, f is continuous and a quotient map (as it is also surjective). However, $f(1, 2) = \{b\}$ is not open in Y, so f is not open. □

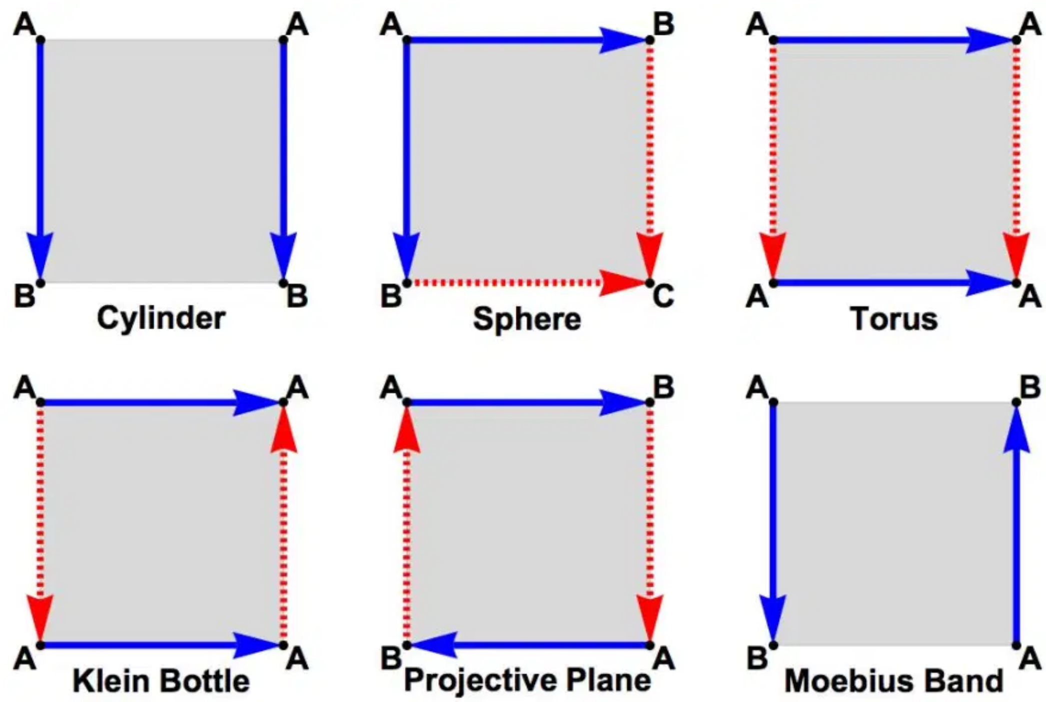
If $f : X \to Y$ is a quotient map, we define an **equivalence relation on** X by

$$x_1 \sim x_2 \iff f(x_1) = f(x_2).$$

Equivalence classes of $X/\sim$ are in $1-$to-1 correspondence with elements of Y; $X/\sim$ and Y are homeomorphic under the **identification topology**.

> **Examples:** in what follows, we set $X = I \times I$, where $I = [0, 1]$, with the usual subspace topology from $\mathbb{R}^2$.
>
> 1. The **cylinder** is defined via the following equivalence relation on X:
>
> $$(x, y) \sim (x, y') \iff y - y' \in \mathbb{Z}^2.$$
>
> 2. The **torus** is defined via the following equivalence relation on X:
>
> $$(x, y) \sim (x', y') \iff (x - x', y - y') \in \mathbb{Z}^2.$$
>
> 3. The **Möbius band** is defined via the following equivalence relation on X:
>
> $$(x, y) \sim (x', y') \iff x - x' \in \mathbb{Z} \text{ and } y + y' = 1.$$
>
> 4. The **Klein bottle** is defined via the following equivalence relation on X:
>
> $$(x, y) \sim (x', y') \iff (x - x' \in \mathbb{Z} \text{ and } y + y' = 1) \text{ or } (x = x' \text{ and } y - y' \in \mathbb{Z}).$$
>
> 5. The **projective plane** is defined via the following equivalence relation on X:
>
> $$(x, y) \sim (x', y') \iff (x - x' \in \mathbb{Z} \text{ and } y + y' = 1) \text{ or } (x + x' = 1 \text{ and } y - y' \in \mathbb{Z}).$$

The identification topologies of those spaces on X are shown below (from `thatsmaths.com`).



15.7 Solved Problems

1. Show that if $\mathfrak{B}$ is a basis for a topology on X, then the topology generated by $\mathfrak{B}$ is the intersection of all topologies on X that contain $\mathfrak{B}$. Prove the same if $\mathfrak{B}$ is a sub-basis.

   Proof: let $\mathfrak{B}$ be a basis, and suppose $\mathfrak{T}(\mathfrak{B})$ is the topology on X generated by $\mathfrak{B}$. We first show that $\mathfrak{T}(\mathfrak{B}) \subseteq \bigcap_{\mathfrak{B}\subseteq\mathfrak{T}} \mathfrak{T}$.

   Let $U \in \mathfrak{T}(\mathfrak{B})$. Then $U = \bigcup_{B\in\mathfrak{B}_U} B$, for some $\mathfrak{B}_U \subseteq \mathfrak{B}$. Let $\mathfrak{T}$ be any topology on X containing $\mathfrak{B}$. In particular, it also contains $\mathfrak{B}_U$, and

   $$\bigcup_{B\in\mathfrak{B}_U} B = U \in \mathfrak{T},$$

   since arbitrary unions of open sets in $\mathfrak{T}$ are open in $\mathfrak{T}$. But $\mathfrak{T}$ was arbitrary, so $U \in \bigcap_{\mathfrak{B}\subseteq\mathfrak{T}} \mathfrak{T}$, and $\mathfrak{T}(\mathfrak{B}) \subseteq \bigcap_{\mathfrak{B}\subseteq\mathfrak{T}} \mathfrak{T}$. Conversely, since $\mathfrak{T}(\mathfrak{B})$ is a topology on X containing $\mathfrak{B}$, then

   $$\bigcap_{\mathfrak{B}\subseteq\mathfrak{T}} \mathfrak{T} \subseteq \mathfrak{T}(\mathfrak{B}).$$

   Hence $\bigcap_{\mathfrak{B}\subseteq\mathfrak{T}} \mathfrak{T} = \mathfrak{T}(\mathfrak{B})$.

Now suppose $\mathfrak{B}$ is a sub-basis. The proof follows the same lines. The sole difference is that the topology on X generated by $\mathfrak{B}$ is

$$\mathfrak{T}(\mathfrak{B}) = \left\{ \bigcup_{\text{arbitrary}} \left(\bigcap_{\text{finite}} B_i \right) \middle| B_i \in \mathfrak{B} \right\}.$$

So we need only to verify that if $U \in \mathfrak{T}(\mathfrak{B})$, then $U \in \bigcap_{\mathfrak{B} \subseteq \mathfrak{T}} \mathfrak{T}$. Let

$$U = \bigcup_{\text{arb.}} \left(\bigcap_{\text{fin.}} B_i \right)$$

and $\mathfrak{T}$ be any topology on X containing $\mathfrak{B}$. Then $U \in \mathfrak{T}$ since arbitrary unions and finite intersections of open sets in $\mathfrak{T}$ are open in $\mathfrak{T}$.

The rest of the proof is identical to the above proof for when $\mathfrak{B}$ is a basis. ■

2. Show that the collection

$$\mathfrak{B} = \{[a, b) \mid a < b, a, b \in \mathbb{Q}\}$$

is a basis that generates a topology different from that of $\mathbb{R}_l$.

Proof: to show that $\mathfrak{B}$ is a basis, it suffices to show the second property, since $\mathbb{R} = \bigcup_{n \in \mathbb{N}} [-n, n)$. Let $[a, b)$ and $[c, d)$ belong to $\mathfrak{B}_2$. Then

$$[a, b) \cap [c, d) = \begin{cases} \varnothing & \text{if } b \leq c \\ [a, b) & \text{if } b \geq c, a \geq c, b \leq d \\ [c, d) & \text{if } b \geq c, a \leq c, b \geq d \\ [c, b) & \text{if } b \geq c, a \leq c, b \leq d \\ [a, d) & \text{if } b \geq c, a \geq c, b \geq d \end{cases},$$

where $a, b, c, d \in \mathbb{Q}$.

Thus, whenever $x \in [a, b) \cap [c, d)$, there exists an interval $I \in \mathfrak{B}$ such that $x \in I \subseteq [a, b) \cap [c, d)$. Hence B is a basis. Denote the topology on $\mathbb{R}$ generated by $\mathfrak{B}$ by $\mathfrak{T}$, and that of the lower limit topology on $\mathbb{R}$ by $\mathfrak{T}_l$. Clerly, $[\pi, 4) \in \mathfrak{T}_l$. Does it also belong to $\mathfrak{T}$?

If it does, we can then write

$$[\pi, 4) = \bigcup_{\alpha \in \mathcal{A}} [a_\alpha, b_\alpha),$$

for $a_\alpha, b_\alpha \in \mathbb{Q}$. But notice that each of the a_α must be greater than π. In particular, since $\pi \notin \mathbb{Q}$, each of the a_α must be strictly greater than π, since they are all rational. Hence, we can at best obtain

$$(\pi, 4) = \bigcup_{\alpha \in \mathcal{A}} [a_\alpha, b_\alpha),$$

if $a_\alpha, b_\alpha \in \mathbb{Q}$. Hence $[\pi, 4) \notin \mathfrak{T}$ and $\mathfrak{T}_l \neq \mathfrak{T}$. ■

3. Show that if Y is a subspace of X, and A is a subset of Y, then the subspace topology on A as a subspace of Y is the same as the subspace topology on A as a subspace of X.

   Proof: let U be open in the subspace topology on A as a subspace of X, and V be open in the subspace topology on A as a subspace of Y.

   Then, there exists $W \subseteq_O X$ and $Z \subseteq_O Y$ such that $U = A \cap W$ and $V = A \cap Z$. But if $Z \subseteq_O Y$, there exist $Z' \subseteq_O X$ such that $Z = Y \cap Z'$, and so $V = A \cap Y \cap Z'$.

   Since $A \subseteq Y$,

$$U = A \cap W = A \cap Y \cap W, \quad V = A \cap Y \cap Z' = A \cap Z',$$

   where W and Z' are open in X.

   Hence U is open in the subspace topology on A as a subspace of Y, and V is open in the subspace topology on A as a subspace of X, and so the two topologies are equal. ■

4. If $\mathfrak{T}$ and $\mathfrak{T}'$ are topologies on X and $\mathfrak{T}'$ is strictly finer than $\mathfrak{T}$, what can you say about the corresponding subspace topologies on the subset Y of X?

   Solution: let $\mathfrak{T}_Y$ and $\mathfrak{T}'_Y$ be the subspaces topologies on a subset Y of X corresponding to $\mathfrak{T}$ and $\mathfrak{T}'$ respectively. It should be clear that $\mathfrak{T}'_Y$ is finer than $\mathfrak{T}_Y$. Indeed let $B = V \cap Y$ for some $V \in \mathfrak{T} \subsetneq \mathfrak{T}'$. Hence $B = V \cap Y$ for some $V \in \mathfrak{T}'$.

   Can we necessarily say that $\mathfrak{T}'_Y$ is strictly finer than $\mathfrak{T}$? Well, suppose all $U \in \mathfrak{T}'$ where $U \notin \mathfrak{T}$ are such that $U \cap Y = \varnothing$.[7] Then

$$A = Y \cap U = Y \cap \varnothing \in \mathfrak{T}_Y$$

   since $\varnothing$ is open in $\mathfrak{T}$.

   For all other $V \in \mathfrak{T}'$, we have $V \in \mathfrak{T}$, and so we have $A = V \cap Y \in \mathfrak{T}$. Hence, in this case $\mathfrak{T}_Y = \mathfrak{T}'_Y$. The following example shows that $\mathfrak{T}'_Y$ could be strictly finer than $\mathfrak{T}_Y$.

   Let $X = \mathbb{R}$ (as a set), $Y = (0,1)$ and suppose $\mathfrak{T}$ and $\mathfrak{T}'$ are the usual topology on $\mathbb{R}$ and the lower limit topology on $\mathbb{R}$, respectively.

   Then $[0.5, 1) \in \mathfrak{T}'_Y$, but it is not open in the usual subspace topology on Y since there is no interval (a,b) such that

$$[0.5, 1) = (0,1) \cap (a,b).$$

   In this case, $\mathfrak{T}'_Y$ is strictly finer than $\mathfrak{T}_Y$. Thus, the most we can say without more information is that $\mathfrak{T}'_Y$ is finer than $\mathfrak{T}_Y$. □

[7] For instance, let $X = \{a,b,c\}$, $Y = \{c\}$, $\mathfrak{T} = \{\varnothing, X\}$ and $\mathfrak{T}' = \{\varnothing, \{a,b\}, X\}$. Then $\mathfrak{T} \subsetneq \mathfrak{T}'$, and the only $U \in \mathfrak{T}'$ where $U \notin \mathfrak{T}$ is $U = \{a,b\}$, so $Y \cap U = \varnothing$.

5. Show that the projections $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open maps.

Proof: we show that π_1 is open, the proof that π_2 is open is similar. Let B be a basic open set in $X \times Y$. Hence $B = U \times V$, where U is open in X and V is open in Y. Then $\pi_1(B) = U$ is open in X. Now, any open W in $X \times Y$ is written

$$W = \bigcup_{\alpha \in \mathcal{A}} (U_\alpha \times V_\alpha),$$

where $U_\alpha \times V_\alpha$ is a basic open set for all $\alpha \in \mathcal{A}$. Now

$$\begin{aligned} \pi_1(W) &= \pi_1(\{(u,v) \in X \times Y | (u,v) \in U_\alpha \times V_\alpha \text{ for some } \alpha \in A\} \\ &= \{u \in X | u \in U_\alpha \text{ for some } \alpha \in A\} = \cup_{\alpha \in \mathcal{A}} U_\alpha, \end{aligned}$$

which is open in X, since it is an arbitrary union of open sets in X, so π_1 is open. ■

6. Show that X is Hausdorff if and only if the diagonal

$$\Delta = \{(x,x) | x \in X\}$$

is closed in $X \times X$.

Proof: since $\varnothing$ and any one point set are vacuously Hausdorff, and since their respective Δ are $\varnothing$ and X, which are closed sets in X, the result holds when $X = \varnothing$ and $X = \{*\}$. We can thus restrict ourselves to spaces X with at least two elements. For any such X, $X \times X \setminus \Delta \neq \varnothing$.

Suppose X is Hausdorff. We show that $X \times X \setminus \Delta$ is open in $X \times X$, and so that Δ is closed in $X \times X$.

Let $(x,y) \in X \times X \setminus \Delta$. Then $x \neq y$. So there exists two sets U_x, V_y (open in X) such that $x \in U_x$, $y \in V_y$ and $U_x \cap V_y = \varnothing$. Now $(x,y) \in U_x \times V_y$, which is open in $X \times X$. We show that $(U_x \times V_y) \cap \Delta = \varnothing$. Suppose

$$(z,z) \in (U_x \times V_y) \cap \Delta \neq \varnothing.$$

Then $z \in U_x$ and $z \in V_y$, so $z \in U_x \cap V_y$. But $U_x \cap V_y = \varnothing$, so there is no such (z,z). Hence, we can fit an open set around each $(x,y) \in X \times X \setminus \Delta$, and so $X \times X \setminus \Delta$ is open in $X \times X$.

Conversely, suppose Δ is closed in $X \times X$, and let $x, y \in X$ such that $x \neq y$. Then $(x,y) \in X \times X \setminus \Delta$, an open set of $X \times X$. Hence there exists a basic open set $U \times V$ of $X \times X$ such that

$$(x,y) \in U \times V \subseteq X \times X \setminus \Delta.$$

But $U \cap V = \varnothing$, otherwise there would exist $z \in X$ such that

$$(z,z) \in U \times V \nsubseteq X \times X \setminus \Delta.$$

Thus U, V are open subsets of X with $x \in U$, $y \in V$, and $U \cap V = \varnothing$, and so X is Hausdorff. ■

7. Let $A \subseteq X$, and let $f : A \to Y$ be continuous; let Y be Hausdorff. Show that if f may be extended to a continuous function $g : \overline{A} \to Y$, then g is uniquely determined.

   Proof: suppose f can be extended to g and h, as in the statement of the problem. Suppose $g \neq h$. Then, there exists $x_0 \in \overline{A} \setminus A = \partial A$ such that $g(x_0) \neq h(x_0)$, since $f = g|_A = h|_A$.

   But Y is Hausdorff, so $\exists U, V \subseteq_O Y$ such that $g(x_0) \in U$, $h(x_0) \in V$, and $U \cap V = \varnothing$.

   Since g and h are continuous, $g^{-1}(U), h^{-1}(V) \subseteq_O X$. Furthermore,

   $$x_0 \in g^{-1}(U) \cap h^{-1}(V) \subseteq_O X.$$

   As $x_0 \in \overline{A}$, there exists $a \neq x_0$ in A such that $a \in g^{-1}(U) \cap h^{-1}(V)$, and so $g(a) \in U$ and $h(a) \in V$. But $g(a) = h(a) = f(a)$ since $a \in A$, which yields $f(a) \in U \cap V$, a contradiction, as this set is supposed empty. Thus when f can be extended, it can be done uniquely. ■

8. If $f_1 : X_1 \to Y_1$, $f_2 : X_2 \to Y_2$ are continuous, show that $F : X_1 \times X_2 \to Y_1 \times Y_2$ is continous, where $F(x_1, x_2) = (f_1(x_1), f_2(x_2))$.

   Proof: the set $\mathfrak{B} = \{U \times V \mid U \subseteq_O Y_1, V \subseteq_O Y_2\}$ is a basis for the product topology on $Y_1 \times Y_2$. Then, it is enough to show that $F^{-1}(U \times V) \subseteq_O X_1 \times X_2$ for all $U \times V \in \mathfrak{B}$. But

   $$\begin{aligned} F^{-1}(U \times V) &= \{(x_1, x_2) \in X_1 \times X_2 \mid F(x_1, x_2) \in U \times V\} \\ &= \{(x_1, x_2) \in X_1 \times X_2 \mid f_1(x_1) \in U, f_2(x_2) \in V\} \\ &= \{(x_1, x_2) \in X_1 \times X_2 \mid x_1 \in f_1^{-1}(U), x_2 \in f_2^{-1}(V)\} = f_1^{-1}(U) \times f_2^{-1}(V). \end{aligned}$$

   But $f_1^{-1}(U) \subseteq_O X_1$ and $f_2^{-1}(V) \subseteq_O X_2$ since f_1 and f_2 are continuous, and so

   $$F^{-1}(U \times V) = f_1^{-1}(U) \times f_2^{-1}(V) \subseteq_O X_1 \times X_2$$

   in the product topology, which means that F is continuous. ■

9. Let $f : X \to Y$ be an onto mapping. For each of the properties T_1 and T_2, prove or disprove that if one of X, Y has the property, then so must the other when

   a) f is continuous;
   b) f is open;
   c) f is both open and continuous.

   Solution: throughout, we assume that both X and Y have at least two elements – otherwise, all the statements are vacuously or trivially true. Recall that a space W is T_1 when, for each pair of distinct points $x, y \in W$, there exists U_x, V_y open sets in W such that $x \in U_x \not\ni y$ and $y \in V_y \not\ni x$.

a) f is continuous: $f^{-1}(U)$ is open in X whenever U is open in Y.

i. $\underline{X \text{ is } T_1}$. Let $X = \mathbb{R}$, $Y = \{a, b\}$ with the indiscrete topology and define the surjection $f : \mathbb{R} \to Y$ by $f(0) = a$ and $f(x) = b$ for all $x \neq 0$. Then $\mathbb{R}$ is T_1, since it is T_2, and f is continuous, since $f^{-1}(Y) = \mathbb{R}$ is open in $\mathbb{R}$, but Y is not T_1 since every neighbourhood of a contains b. So X is $T_1 \not\Rightarrow Y$ is T_1.

ii. $\underline{Y \text{ is } T_1}$. Let $X = \{a, b, c, d\}$ with $\mathfrak{T}_X = \{\varnothing, \{a, c\}, \{b, d\}, X\}$, $Y = \{a, b\}$ with the discrete topology and define the surjection $f : X \to Y$ by $f(a) = a$, $f(b) = b$, $f(c) = a$ and $f(d) = b$. Then f is continuous, since both $f^{-1}(\{a\}) = \{a, c\}$, $f^{-1}(\{b\}) = \{b, d\}$ lie in $\mathfrak{T}_X$, but X is not T_1 since every neighbourhood of a contains c. So Y is $T_1 \not\Rightarrow X$ is T_1.

iii. $\underline{X \text{ is } T_2}$. In the counter-example a)i., X is also T_2, but Y is not T_1, so it is certainly not T_2. Hence X is $T_2 \not\Rightarrow Y$ is T_2.

iv. $\underline{Y \text{ is } T_2}$. In the counter-example a)ii., Y is also T_2, but X is not T_1, so it is certainly not T_2. Hence Y is $T_2 \not\Rightarrow X$ is T_2.

b) f is open: $f(V)$ is open in Y whenever V is open in X.

i. $\underline{X \text{ is } T_1}$. See b)iii. X is $T_1 \not\Rightarrow Y$ is T_1.

ii. $\underline{Y \text{ is } T_1}$. In the counter-example a)ii., f is surjective, it is open since Y has the discrete topology, and Y is T_1. But X is not T_1. So Y is $T_1 \not\Rightarrow X$ is T_1.

iii. $\underline{X \text{ is } T_2}$. Let $X = \mathbb{R}$, $Y = \{a, b\}$ with the indiscrete topology, and define the surjection $f : \mathbb{R} \to Y$ by $f(x) = a$ whenever $x \in \mathbb{Q}$ and $f(x) = b$ whenever $x \notin \mathbb{Q}$. Then f is open. Indeed, any basic open set (a, b) contains both rational and irrational numbers, and so $f(a, b) = Y \subseteq_O Y$. Note that $\mathbb{R}$ is T_2, but Y is not T_2, as it is not even T_1. Thus, X is $T_2 \not\Rightarrow Y$ is T_2.

iv. $\underline{Y \text{ is } T_2}$. In the counter-example a)ii., f is surjective, it is open since Y has the discrete topology, and Y is T_2. But X is not T_2, as it is not T_1. Thus, Y is $T_2 \not\Rightarrow X$ is T_2.

c) f is both open and continuous: $f^{-1}(U)$ is open in X whenever U is open in Y and $f(V)$ is open in Y whenever V is open in X.

i. $\underline{X \text{ is } T_1}$. See b)iii. X is $T_1 \not\Rightarrow Y$ is T_1.

ii. $\underline{Y \text{ is } T_1}$. In the counter-example a)ii., f is surjective, it is open since Y has the discrete topology, it is continuous by definition and Y is T_1. But X is not T_1. Hence Y is $T_1 \not\Rightarrow X$ is T_1.

iii. $\underline{X \text{ is } T_2}$. In the counter-example b)iii., the function f is also continuous since Y has the indiscrete topology and $X = \mathfrak{T}_2$. But Y is not T_2 as it is not even T_1. Hence X is $T_2 \not\Rightarrow Y$ is T_2.

iv. $\underline{Y \text{ is } T_2}$. In the counter-example a)ii., f is surjective, it is open since Y has the discrete topology, it is continuous by definition and Y is T_2. But X is not T_2. As it is not even T_1. Hence Y is $T_2 \not\Rightarrow X$ is T_2.

And that's it, folks: T_1 and T_2 do not behave nicely with respect to continuous functions. $\square$

10. Show that the set A of all bounded sequences is both open and closed in the box topology on $\mathbb{R}^\omega$.

Proof: let A be the set

$$A = \{(x_n)_{n\in\mathbb{N}} \mid \exists M \in \mathbb{R} \text{ with } |x_n| < M \ \forall n \in \mathbb{N}\}.$$

We start by showing that $A \subseteq_O \mathbb{R}^\omega$.

Let $(x_n)_{n\in\mathbb{N}} \in A$. Then $\exists M \in \mathbb{R}$ such that $|x_n| < M$ for all $n \in \mathbb{N}$. Set

$$U_n = (x_n - 1, x_n + 1), \text{ for all } n \in \mathbb{N}.$$

Then

$$U = \prod_{n\in\mathbb{N}} (x_n - 1, x_n + 1)$$

is open in the box topology on $\mathbb{R}^\omega$, since $U_n \subseteq_O \mathbb{R}$ for all $n \in \mathbb{N}$. Clearly, $(x_n)_{n\in\mathbb{N}} \in U$, since $x_n \in U_n$ for all $n \in \mathbb{N}$. But $U \subseteq A$.

Indeed, suppose $(w_n)_{n\in\mathbb{N}} \in U$. Then $w_n \in U_n$ for all $n \in \mathbb{N}$ and so $x_n - 1 < w_n < x_n + 1$ for all $n \in \mathbb{N}$. But this means that

$$-M - 1 < x_n - 1 < w_n < x_n + 1 < M + 1$$

and $|w_n| < M + 1$ for all $n \in \mathbb{N}$. Hence $(w_n)_{n\in\mathbb{N}} \in A$ and $U \subseteq A$ so we conclude that $A \subseteq_O \mathbb{R}^\omega$.

We now show that $A \subseteq_C \mathbb{R}^\omega$. Suppose $(x_n)_{n\in\mathbb{N}} \in \overline{A}$, and let

$$V = \prod_{n\in\mathbb{N}} \left(x_n - \frac{1}{n}, x_n + \frac{1}{n} \right).$$

Then $(x_n)_{n\in\mathbb{N}} \in V \subseteq_O \mathbb{R}^\omega$ and there exists $(a_n)_{n\in\mathbb{N}} \in A$ such that $(a_n)_{n\in\mathbb{N}} \in V$. In that case, there exists $M \in \mathbb{R}$ such that $-M < a_n < M$ for all $n \in \mathbb{N}$. However, $a_n \in (x_n - \frac{1}{n}, x_n + \frac{1}{n})$ so that

$$a_n - \frac{1}{n} < x_n < a_n + \frac{1}{n}$$

for all $n \in \mathbb{N}$, and so

$$-M - 1 < a_n - \frac{1}{n} < x_n < a_n + \frac{1}{n} < M + 1$$

and $|x_n| < M + 1$ for all $n \in \mathbb{N}$.

Thus $(x_n)_{n\in\mathbb{N}} \in A$, and $\overline{A} \subseteq A$, which yields $A \subseteq_C \mathbb{R}^\omega$. ■

15.8 Exercises

1. Prepare a 2-page summary of this chapter, with important definitions and results.
2. Let Z be a subspace of Y. Are the subspace topologies on Z relative to X and Y the same?
3. Show directly that the box product topology on $\mathbb{R}^2$ is identical to the ℓ_1 and ℓ_2 topologies on $\mathbb{R}^2$.
4. Provide a proof of Results 203, 204, 207, 209, 212, 213, 215, and 216.
5. Show that a function which is locally continuous at every point is continuous, and *vice-versa*.
6. Provide the details for the homeomorphism examples of pp. 381-382.
7. Provide the details for the embedding examples of p. 382.
8. Provide the equivalence relation for the identification topology of the cylinder, the sphere, and the projective plane.
9. Show that the map $f : X \to Y$ is continuous if and only if $f(\overline{A}) \subseteq \overline{f(A)}$ for any subset A of X.
10. Let $f, g : X \to Y$ be continuous maps from a space X to a Hausdorff space Y. Prove that the set $C = \{x \mid f(x) = g(x)\}$ is closed in X.
11. Suppose that $f : X \to Y$ is a bijection. If $\mathfrak{B}$ is a basis for the topology on X, prove that f is a homeomorphism if and only if the collection $\{f(B) \mid B \in \mathfrak{B}\}$ is a basis for the topology on Y.
12. Show that the map $f : X \to Y$ is continuous if and only if $f(\overline{A}) \subseteq \overline{f(A)}$ for any subset A of X.
13. Let $f, g : X \to Y$ be continuous maps from a space X to a Hausdorff space Y. Prove that the set $C = \{x \mid f(x) = g(x)\}$ is closed in X.
14. Suppose that $f : X \to Y$ is a bijection. If $\mathfrak{B}$ is a basis for the topology on X, prove that f is a homeomorphism if and only if the collection $\{f(B) \mid B \in \mathfrak{B}\}$ is a basis for the topology on Y.