Chapter 16

Connected Spaces

In Chapter 9, we discussed **connectedness** and **path-connectedness** in the context of metric spaces. In this chapter, we discuss how these notions extend to **general topological spaces**.

16.1 Connected Sets

A **separation** of a space X is a pair of **disjoint** non-empty open sets U and V such that $X = U \cup V$. Note that both U and V are open and closed. When no separation of X exists, we say that X is **connected**. Alternatively, X is connected if the only sets that are closed and open in X are $\varnothing$ and X.

> **Example:** let $X = [1,2] \cup [3,4]$ be a subspace of $\mathbb{R}$. $U = [1,2]$ is closed in X as $U = X \cap [1,2]$ and $[1,2]$ is closed in $\mathbb{R}$. But $U = X \cap (0.5, 3.5)$, so $U \subseteq_O X$. Consequently, X is not connected. □

In general, a subspace $Y \subseteq X$ is connected if it is connected in the **subspace topology**.

> **Theorem 218**
> *A separation of a subset Y is a pair of non-empty subsets A and B whose union is Y and such that $\overline{A} \cap B = \varnothing$ and $A \cap \overline{B} = \varnothing$.*
>
> **Proof:** Suppose A and B satisfy the conditions of the theorem. Then
>
> $$\overline{A} \cap Y = \overline{A} \cap (A \cup B) = (\overline{A} \cap A) \cup (\overline{A} \cap B) = A \cup \varnothing = A,$$
>
> and A is closed in the subspace topology on Y (i.e., relatively closed). Similarly, B is relatively closed, so A and B are relatively open in Y. Consequently, A and B form a separation of Y.

Conversely, suppose A and B are a separation of Y. Then A is relatively closed and so $A = \overline{A} \cap Y$. Then

$$\overline{A} \cap B = \overline{A} \cap Y \cap B = A \cap B = \varnothing.$$

Similarly $A \cap \overline{B} = \varnothing$. ■

If $Y \subseteq X$ is a connected set, and U and V is a separation of X, then $Y \subseteq U$ or $Y \subseteq V$.[1]

Theorem 219
If $\{C_\alpha\}_{\alpha \in A}$ is a family of connected sets such that $\bigcap_\alpha C_\alpha \neq \varnothing$, then $\bigcup_\alpha C_\alpha$ is connected.

Proof: suppose $x \in \bigcap_\alpha C_\alpha$. If U and V is a separation of $\bigcup_\alpha C_\alpha$, then either $x \in U$ or $x \in V$. Without loss of generality, let $x \in U$. Let $\alpha \in A$. Since C_α is connected, either $C_\alpha \subseteq U$ or $C_\alpha \subseteq V$. But $x \in C_\alpha$, so $C_\alpha \subseteq U$. Then $\bigcup_\alpha C_\alpha \subseteq U$. Hence

$$\left(\bigcup_\alpha C_\alpha \right) \cap V \subseteq U \cap V = \varnothing.$$

As $\bigcup_\alpha C_\alpha = U \cup V$, this means that $V = \varnothing$, which is a contradiction since U and V form a separation. Consequently, there could be no such separation to start with, and $\bigcup_\alpha C_\alpha$ is connected. ■

Connectedness behaves well with respect to the closure of a set, as we can see below.

Theorem 220
If A is connected, and $A \subseteq B \subseteq \overline{A}$, then B is connected.

Proof: if U and V forms a separation of B, then $A \subseteq U$ or $A \subseteq V$. Without loss of generality, suppose $A \subseteq U$. Then $V = B \cap V \subseteq \overline{A} \cap V \subseteq \overline{U} \cap V = \varnothing$, by Theorem 218. But $V \neq \varnothing$ as U and V form a separation of B. Thus there cannot be a separation of B and B is connected. ■

As mentioned in Chapter 9, connectedness is a **topological property**.

Theorem 221
Let $f : X \to Y$ be a continuous function. If X is connected, $f(X)$ is connected.

Proof: suppose that $f(X)$ is not connected. Let U and V form a separation of $f(X)$. Then $f^{-1}(U)$ and $f^{-1}(V)$ form a separation of X and X is not connected. ■

[1]Otherwise $U \cap Y$ and $V \cap Y$ would form a separation of Y.

Theorem 222 If X and Y are connected spaces, so is $X \times Y$.

Proof: if $x \in X$, the function $i_x : Y \to X \times Y$ defined by $i_x(y) = (x, y)$ is continuous. Then $i_x(Y) = \{x\} \times Y$ is connected. Similarly, $i_y(X) = X \times \{y\}$ is connected for all $y \in Y$. Then

$$i_x(Y) \cap i_y(X) = \{(x, y)\} \neq \varnothing$$

for all $y \in Y$. Then $C_y = i_y(X) \cup i_x(Y) = (X \times \{y\}) \cup (\{x\} \times Y)$ is connected for all $y \in Y$. Now

$$\bigcap_{y \in Y} C_y = \{x\} \times Y = i_x(Y) \neq \varnothing,$$

so $\bigcup_{y \in Y} C_y = X \times Y$ is connected. ■

As a result, any finite product of connected sets is connected. What about an infinite product of connected sets?

Theorem 223
Let $\{X_\alpha\}_{\alpha \in A}$ be a collection of connected sets. Then $\prod_\alpha X_\alpha$ is connected in the product topology.

Proof: if $\prod_\alpha X_\alpha = \varnothing$, then the theorem is trivially true, so let $b = (b_\alpha)_\alpha \in \prod_\alpha X_\alpha$. For each finite set $\{\alpha_1, \ldots, \alpha_n\}$ of A, consider the space

$$X(\alpha_1, \ldots, \alpha_n) = \left\{ (x_\alpha)_{\alpha \in A} \,\middle|\, \begin{matrix} x_\alpha = b_\alpha \text{ if } \alpha \notin \{\alpha_1, \ldots, \alpha_n\} \\ x_\alpha \in X_\alpha \text{ if } \alpha \in \{\alpha_1, \ldots, \alpha_n\} \end{matrix} \right\},$$

which is homeomorphic to $X_{\alpha_1} \times \cdots \times X_{\alpha_n}$, and so connected. Let $\mathfrak{B}$ be the collection of all finite subsets of A. Note that $b \in X(\alpha_1, \ldots, \alpha_n)$ for all $\{\alpha_1, \ldots, \alpha_n\} \in \mathfrak{B}$, hence

$$b \in \bigcap_{\{\alpha_1, \ldots, \alpha_n\} \in \mathfrak{B}} X(\alpha_1, \ldots, \alpha_n) \neq \varnothing.$$

Thus $Y = \bigcup_{\mathfrak{B}} X(\alpha_1, \ldots, \alpha_n)$ is connected. We show that $\overline{Y} = \prod_\alpha X_\alpha$. Since Y is connected, $\overline{Y}$ is connected and the theorem is proven.

Let $x = (x_\alpha)_\alpha \in \prod_\alpha X_\alpha \neq b$ and let V be a basic neighbourhood of x. Then $V = \prod_\alpha V_\alpha$, where $V_\alpha = X_\alpha$ for all but a finite number of open sets V_{α_i}, $1 \leq i \leq n$.

Define $y = (y_\alpha)_\alpha$ by

$$y_\alpha = \begin{cases} b_\alpha, & \text{if } \alpha \neq \alpha_i \text{ for all } 1 \leq i \leq n, \\ x_\alpha, & \text{if } \alpha = \alpha_i \text{ for some } 1 \leq i \leq n. \end{cases}$$

Then, $y_\alpha = b_\alpha \in V_\alpha = X_\alpha$ for $\alpha \notin \{\alpha_1, \ldots, \alpha_n\}$, and $y_\alpha = x_\alpha \in V_\alpha$ for $\alpha \in \{\alpha_1, \ldots, \alpha_n\}$. Hence $y_\alpha \in V_\alpha$ for all α and $y \in V$.

But, by construction, $y \in X(\alpha_1, \ldots, \alpha_n) \subseteq Y$, so $y \in V \cap Y \neq \varnothing$. As $b \neq x$, we get $y \neq x$, and x is a limit point of Y, that is $x \in \overline{Y}$. Consequently, $\overline{Y} = \prod_\alpha X_\alpha$. ■

In the usual topology, $\mathbb{R}$ has some useful properties, some of which can be extended to general spaces. A **linear continuum**. for isntance, is an ordered set X in which the following hold:

i. if $x < y \in X$, there exists $z \in X$ such that $x < z < y$;

ii. any non-empty set $A \subset X$ with an upper bound has a least upper bound.

A rather tedious, but not very difficult, argument ([Munkres, , p.153] shows that linear continua are connected, and that rays and intervals are connected subsets in a linear continuum. As $\mathbb{R}$ is a linear continuum, it is connected. The next result is a generalization of a very important theorem from analysis (see Theorem 35, Chapter 3).

Theorem 224 (Intermediate Value Theorem)
Suppose $f : X \to Y$ is continuous and Y has the order topology for some ordering $<$. If X is connected and $a, b \in X$ are such that $f(a) < f(b)$, then for any $y \in Y$ such that $f(a) < y < f(b)$, there exists $x \in X$ such that $f(x) = y$.

Proof: let $A = \{z \in Y : z > y\}$ and $B = \{z \in Y : z < y\}$. Then $A, B \subseteq_O Y$, and, as f is continuous, $f^{-1}(A), f^{-1}(B) \subseteq_O X$. Furthermore, $f^{-1}(A) \cap f^{-1}(B) = \varnothing$, $a \in f^{-1}(B)$ and $b \in f^{-1}(A)$. Since X is connected, $X \neq f^{-1}(A) \cup f^{-1}(B)$ (otherwise, $f^{-1}(A)$ and $f^{-1}(B)$ would form a separation of X).

Hence, there exists $x \in X \setminus (f^{-1}(A) \cup f^{-1}(B))$. As $f(x) \notin A$ and $f(x) \notin B$, $f(x) = y$. ■

If $x \in X$, the (connected) **component** of x in X, denoted C_x is the union of all connected sets containing x. It is connected as the intersection of all these sets contain x. As C_x is connected, $\overline{C_x}$ is connected and so $\overline{C_x} \subseteq C_x$. Then the component C_x is closed in X; if X has a finite number of components, each component is also open.

We can define an equivalence relation on X as follows: xRy if and only if there is a connected set containing both x and y.

Then:

1. for all $x \in X$, xRx;
2. if xRy, then yRx, and
3. if xRy and yRx, then xRz.

The **equivalence class** of x is simply the (connected) component of x in X.

Examples (COMPONENTS)

1. Let $X = [1,2) \cup (3,4)$ be a subspace of X. Then X has two components, $[1,2)$ and $(3,4)$.
2. Let $x \in \mathbb{Q}$. Then the component of x is $\{x\}$ as the only connected subsets of $\mathbb{Q}$ are one-point sets. When all the components of X are singletons, we say that the space X is **totally disconnected**. □

16.2 Path-Connectedness

A **path** in a space X is a continuous map $p : [0,1] \to X$. Throughout, we denote $[0,1]$ by I. If $p(0) = a$ and $p(1) = b$, we say that p is a **path from** a **to** b, a is the **initial point** of p, while b is the **terminal point** of p. A space X is **path-connected** if for any pair of points $a, b \in X$, there is a path p from a to b.

Proposition 225
A path-connected space X is connected.

Proof: Suppose A, B were a separation of X. Let $a \in A$ and $b \in B$. As X is path-connected, there is a path p from a to b. But $p(I)$ is connected in X as I is connected, so $p(I) \subseteq A$ or $p(I) \subseteq B$. But $p(0) \in A$ and $p(1) \in B$, a contradiction. Hence X is connected. ■

We have already discussed paths in Chapter 14.

Examples (PATHS AND PATH-CONNECTEDNESS)

1. Let $a \in X$. The map $p_a : I \to X$ defined by $p_a(t) = a$ is a path, the **constant path at** a.
2. For $n > 1$, $\mathbb{R}^n \setminus \{0\}$ is path-connected. Let $a, b \in \mathbb{R}^n \setminus \{0\}$. Define $S_{a,b}$ to be the circle with diameter $\overline{ab}$. If $0 \notin S_{a,b}$, then either of the semi-circles form a path from a to b in $\mathbb{R}^n \setminus \{0\}$. If $0 \in S_{a,b}$, it can only lie on one of the semi-circles. Then the other semi-circle gives the desired path.

3. Any **convex** subset C of $\mathbb{R}^n$ is connected. Indeed, let $a, b \in C$ and define a path $p : I \to X$ by

$$p(t) = (1-t)a + tb = t(b-a) + a.$$

Then p is continuous, $p(0) = a$ and $p(1) = b - a + a = b$. Hence C is path-connected, so connected.

4. $\mathbb{R} \setminus \{0\}$ is not connected, as $(-\infty, 0)$, $(0, \infty)$ is a separation. Let $n > 1$. Then $\mathbb{R}^n \setminus \{0\}$ and $\mathbb{R} \setminus \{0\}$ are not homeomorphic. But this actually means that $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}$. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ was a homeomorphism. Then $f(\mathbb{R}^n \setminus \{0\}) = \mathbb{R} \setminus \{f(0)\}$ would be the continuous image of a connected set, so should be connected. But it clearly isn't, so there can be no homeomorphism.

5. Let $A = \{(x, y) \mid x = ny, n \in \mathbb{N}, 0 \leq x \leq 1\}$. Graphically, A represents the union of lines through the origin of slopes 1, $\frac{1}{2}$, $\frac{1}{3}$, $\ldots$, restricted to $I \times I$. A is connected, as it is clearly path-connected. Let $X = A \cup \{(1, 0)\}$. Then X is connected since $A \subseteq X \subseteq \overline{A}$. We show that X is not path-connected by showing that there is no path in X from $b = (1, 0)$ to any point of A. As a result, connected spaces need not be path connected.

   Suppose $p : I \to X$ is a path with $p(0) = b$ and let V be a neighbourhood of b, excluding $(0, 0)$. Let $t_0 \in p^{-1}(b)$. As p is continuous, there exists a basic (hence connected) neighbourhood U of t_0 such that $p(U) \subseteq V$. If $t_1 \in U$ and $p(t_1) \neq b$, then $p(t_1)$ lies on $x = ny$ for some $n \in \mathbb{N}$. Write

$$W_1 = \left\{(x, y) : x < \left(n + \frac{1}{2}\right) y\right\} \cap V$$

   and

$$W_2 = \left\{(x, y) : x > \left(n + \frac{1}{2}\right) y\right\} \cap V.$$

   Then W_1 and W_2 forms a separation of V. Thus $p(U) \subseteq W_1$ or $p(U) \subseteq W_2$. But $t_0 \in U$, so $b = p(t_0) \in p(U)$ and $b = (1, 0) \in W_2$. Then $p(U) \subseteq W_2$. However $p(t_1) \in W_1$. So there can be no such t_1 and $p(U) = \{b\}$. Consequently, $p^{-1}(b) = I$, as it is non-empty and both open and closed in I. So p is the constant path p_b, and no point in A can be reached from b. □

It is possible to define another relation on X: xPy if there is a path in X from x to y.

1. For all $x \in X$, xPx as there is a path $p : I \to X$ defined by $p(t) = x$ for all $t \in I$;

2. if xPy there is a path $p : I \to X$ such that $p(0) = x$ and $p(1) = y$. Then, yPx as there is a path $q : I \to X$ defined by $q(t) = p(1-t)$.

3. if xPy and yPx there are paths $p, q : I \to X$ such that $p(1) = q(0) = y$, $p(0) = x$ and $q(1) = z$. Then xPz as there is a path $r = p.q : I \to X$ defined by

$$r(t) = (p.q)(t) = \begin{cases} p(2t) & \text{if } t \in [0, 1/2], \\ q(2t-1) & \text{if } t \in [1/2, 1]. \end{cases}$$

So P is an equivalence relation. The equivalence class of x is the **path component** of x in X. A path component need not be closed. Consider the space X from example 5 on p. 399. The subset A is a path component of X, but A is not closed in X since $(1,0) \in \overline{A}$ but $(1,0) \notin A$.

16.3 Local (Path) Connectedness

A space X is **locally (path) connected** if for each $x \in X$, every neighbourhood V_x of x contains a (path) connected neighbourhood of x. The following examples show that local (path) connectedness and (path) connectedness are independent properties.

Examples (Local (Path) Connectedness)

1. The space X from example 5 on p. 399 is connected but not locally connected, since the only connected neighbourhood of $(1,0)$ is X.
2. The space $X = (0,1) \cup (2,3)$ is locally connected and locally path-connected, but it is clearly neither connected nor path connected.
3. Let $Y = X \cup S$, where X is the space from example 5 on p. 399 and S is an arc joining $(1,0)$ to $(1,1)$ without meeting any other point of X. Then X is path connected, but it is not locally path-connected. Indeed, the neighbourhood $V = B((1,0), 1/2) \cap Y$ contains no path-connected neighbourhood.

There is a simple characterization of locally connected spaces.

Theorem 226
A space X is locally connected if and only if the components of each open subset V of X are open.

Proof: if X is locally path-connected and $V \subseteq_O X$, let C be a component of V. If $x \in V$, there is a connected neighbourhood U of x where $U \subseteq V$. As C is a maximal connected set, $U \subseteq C$ and C is open.

Conversely, suppose the components of open subsets are open. If V is a neighbourhood of x, let U be the component of x in V. Then U is a connected neighbourhood of x lying in V, so X is locally connected. ■

A similar theorem holds for locally path-connected spaces. We finish this section with the following result.

> **Theorem 227**
> *If X is a locally path-connected space, then the components and path components of X coincide.*
>
> **Proof:** If $x \in X$, there is a component C and a path component D of x. Since D is connected, $D \subseteq C$. By the previous theorem, $D \subseteq_O C$. If $y \in C \setminus D$, then there exists a path-connected neighbourhood V of y such that $V \subseteq C$. Then $V \cap D = \varnothing$. Otherwise $y \in D$ since there would be a path from x to y. Hence $y \in V \subseteq C \setminus D$ and $C \setminus D \subseteq_O C$. Then D is closed and open in C. Since C is connected, either $D = \varnothing$ or $D = C$. But $x \in D$, so $D = C$. ■

16.4 Solved Problems

1. Let A and B be connected subsets of a space X. For each of the following condition, either prove it to be sufficient to ensure that $A \cup B$ be connected or provide a counter-example to show that $A \cup B$ need not be connected:

   a) $\overline{A} \cap \overline{B} \neq \varnothing$;
   b) $\overline{A} \cap B \neq \varnothing$ and $A \cap \overline{B} \neq \varnothing$;
   c) $\overline{A} \cap B \neq \varnothing$ or $A \cap \overline{B} \neq \varnothing$.

   Solution:

   a) Let $X = \mathbb{R}$, $a \in \mathbb{R}$, $A = (-\infty, a)$ and $B = (a, +\infty)$. Then $\overline{A} = (-\infty, a]$, $\overline{B} = [a, +\infty)$ and $\overline{A} \cap \overline{B} = \{a\} \neq \varnothing$, but $A \cap B = \varnothing$, so $A \cup B$ is not connected. The condition is not sufficient.

   (b and c) Let $Y = A \cup B$. By a theorem seen in class, a separation of Y is a pair of non-empty subsets W and Z of Y such that $\overline{W} \cap Z = \varnothing$, $W \cap \overline{Z} = \varnothing$ and $Y = W \cup Z$. By hypothesis (in both cases), A and B can not form a separation of Y. Now suppose W and Z formed a separation of Y. Since A and B are connected, each of W and Z must contain exactly one of A and B, say $A \subseteq W$ and $B \subseteq Z$.[2] Since W and Z are disjoint, and $W \cup Z \subseteq A \cup B$, we get $W \subseteq A$ and $Z \subseteq B$, and so W and Z can not form a separation of Y, which is a contradiction. Hence, in both cases, $A \cup B$ is connected. □

2. Let X be locally path-connected. Show that every connected open set in X is path-connected.

   Proof: If $U = \varnothing$, the statement is vacuously true. So suppose $U \neq \varnothing$ is an open connected set in X. Since $U \subseteq_O X$, and X is locally path-connected, then, for every

[2] The only other possibility is that Y lies in one of W xor Z, which would make the other subset empty, and so W and Z could not form a separation of Y.

$x \in U$, there exists $V_x \subseteq_O X$ such that $x \in V_x \subseteq U$ and V_x is path-connected. Now, pick $z \in U$, define V to be the path component of U containing z and let $Y = U - V$. Since X is locally path-connected, V is open in X. Note that

$$\left(\bigcup_{y \in Y} V_y \right) \cap V = \varnothing;$$

otherwise, there would be a $y \in Y \cap V$, a contradiction. Hence we have $Y = \bigcup_{y \in Y} V_y$ and $Y \subseteq_O X$ since $V_y \subseteq_O X$ for all $y \in Y$.

But U is connected, so either $V = \varnothing$ or $Y = \varnothing$. Since $z \in V$, we must have $Y = \varnothing$ and $U = V$. Hence U is path-connected. ■

3. Let X be an ordered set (with at least two elements) in the order topology. Show that if X is connected, then X is a linear continuum.

Proof: a linear continuum is an ordered set in which

i. if $x < y$, there exists z such that $x < z < y$;
ii. any non-empty set A with an upper bound has a least upper bound.

Define the upper open ray and the lower open ray at x by

$$\begin{aligned} \text{UR}(x) &= \{y \in X | y < x\} \\ \text{LR}(x) &= \{y \in X | x < y\} \end{aligned}$$

for all $x \in X$. In the order topology, $\text{UR}(x), \text{LR}(x) \subseteq_O X$ for all $x \in X$. Now let $x, y \in X$ be such that $x < y$, and suppose that there does not exist $z \in X$ such that $x < z < y$. Then $\text{UR}(y) \cap \text{LR}(x) = \emptyset$, and

$$\text{UR}(y) \cup \text{LR}(x) = X.$$

Hence $\text{UR}(y), \text{LR}(x)$ is a separation of X, a contradiction since X is connected, so there must exist a $z \in X$ such that $x < z < y$.

Now, let A be a subset of X with at least one upper bound. Define the sets

$$\begin{aligned} U &= \bigcup_{a \in A} \text{UR}(a) \\ V &= \bigcup_{\substack{w > a \\ \forall a \in A}} \text{LR}(w). \end{aligned}$$

By construction, both U and V are open, and $U \cap V = \emptyset$. Since X is connected, $U \cup V \neq X$, otherwise U and V would be a separation of X. Suppose $b, c \in X - (U \cup V)$. Then, either $b < c$, $c < b$ or $b = c$. If $b < c$, then $c > a$ for all $a \in A$. By i., there exists $w \in X$ such that $b < w < c$, and $c \in \text{LR}(w) \subseteq V$. Similarly, if $c < b$, $b \in V$. This leaves only the possibility that $b = c$, that is $X - (U \cup V) = \{b\}$. By construction, b is smaller than any upper bound of A, and it is greater (or equal) than any element of A, so it is the least upper bound of A. Hence, X is a linear continuum. ■

16.5 Exercises

1. Prepare a 2-page summary of this chapter, with important definitions and results.
2. Is the product of an arbitrary collection of connected spaces connected in the box topology?
3. Show that a space X is locally path-connected if and only if the path-connected components of each open subset V of X are open.
4. Let A be a connected subset of a space X. If $A \subseteq B \subseteq \overline{A}$, show that B is connected. Are the interior and the boundary of A necessarily connected? If either of these is connected, must A be connected? What if both of them are connected?
5. Let A be a subset of a locally connected space. Prove or disprove:
   a) If A is path-connected and $A \subseteq B \subseteq \overline{A}$, then B is path-connected.
   b) If A is open and connected, then A is path-connected.
   c) If A is open, the path components are open.
6. Let X be the subspace

$$X = \left\{ \frac{t}{1+t} e^{it} \;\middle|\; t \geq 0 \right\} \cup \{e^{i\pi}\}.$$

   Give detailed answers to the following:
   a) Is X connected?
   b) Is X locally connected?
   c) Is X path-connected?
   d) Is X locally path-connected?
7. Let $\mathfrak{T}$ and $\mathfrak{T}'$ be two topologies on a space X. If $\mathfrak{T}'$ is finer than $\mathfrak{T}$, does connectedness of X in one topology imply anything about its connectedness in the other?
8. If $|X|$ is infinite, show that X is connected in the finite complement topology.
9. If X_α is path-connected for each α, show that $\prod_\alpha X_\alpha$ is path-connected. If each X_α is also locally path-connected, show that $\prod_\alpha X_\alpha$ is also locally path-connected. Investigate what happens when each X_α is locally path connected, but not necessarily path-connected.