Chapter 17

Compact Spaces

In Chapter 9, we discussed **compactness** in the context of metric spaces. In this chapter, we discuss the notion from a **topological perspective**.

17.1 Compactness

A **covering of a space** X is a family $\mathfrak{F}$ of subsets of X such that

$$\bigcup_{F \in \mathfrak{F}} F = X.$$

A subset Y of X is **covered by a family** $\mathfrak{F}$ if

$$Y \subseteq \bigcup_{F \in \mathfrak{F}} F.$$

We say that $\mathfrak{F}$ is an **open covering** when every $F \in \mathfrak{F}$ is open. A sub-collection $\mathcal{A} \subseteq \mathfrak{F}$ that still covers X is called a **sub-covering** of X.

> **Examples** (Coverings and Sub-coverings)
>
> 1. Consider the sets $\mathfrak{F} = \{(a, b) \mid a < b \in \mathbb{R}\}$, $\mathcal{A} = \{(a, b) \mid a < b \in \mathbb{Q}\}$ and $\xi = \{(n-1, n+1) \mid n \in \mathbb{Z}\}$. Then $\mathfrak{F}$ is an open covering of $\mathbb{R}$, $\mathcal{A}$ and ξ are sub-coverings, but ξ has no proper sub-covering.
>
> 2. The collection $\mathfrak{F} = \{[a, b) \mid a < b \in \mathbb{R}\}$ is an open covering of $\mathbb{R}_l$. $\square$

A space X is **compact** if every open covering of X contains a finite sub-covering. A subspace C of X is **compact in** X if every open covering of C contains a finite sub-covering.[1]

[1] This definition seems rather straightforward, on the face of it, but it is the culmination of a rather long and arduous process, with dead ends and wrong turns – we will look into some of these in the coming pages.

Examples (COMPACT SPACES)

1. $\mathbb{R}$ is not compact, since the covering ξ from the previous example contains no proper sub-covering, hence no finite sub-covering.

2. Let $X = \{0\} \cup \left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{N} \right\}$. Any open covering of X will contain a neighbourhood of 0, say V_0. For some N, we have $\frac{1}{n} \in V_0$ for all $n > N$. For each n such that $1 \leq n \leq N$, pick V_n from the open covering such that $\frac{1}{n} \in V_n$. Then $\{V_0, V_1, \ldots, V_n\}$ is a sub-covering, so X is compact. □

It takes some practice to get the hang of the definition.

Theorem 228
Every closed subset C of a compact set X is compact.

Proof: suppose $\{U_\alpha\}_\alpha$ is an open covering of C. As C is closed, $X \setminus C$ is open and $\{U_\alpha\}_\alpha \cup \{X \setminus C\}$ is an open covering of X. As X is compact, there exists a finite sub-covering of X, say $\{U_{\alpha_i}\}_{i=1}^n$. If $X \setminus C = U_{\alpha_j}$ for some j, discard U_{α_j}. The remaining $\{U_{\alpha_i}\}_{i(\neq j)=1}^n$ is a finite sub-covering of C. In the other case, the finite sub-covering of X is clearly a finite sub-covering of C. Hence C is compact. ■

In general, the converse is not true (see example 1 on p. 407). However, it holds for a broad class of spaces.

Theorem 229
If X is Hausdorff, every compact subset of X is closed.

Proof: let Y be a compact subset of X. As X is Hausdorff, if $x \notin Y$, for each $y \in Y$, there is two disjoint neighbourhoods U_y of y, V_y of x. Then $\{U_y\}_{y \in Y}$ is an open covering of Y. But Y is compact so there is a finite sub-covering, say $\{U_{y_i}\}_{i=1}^n$.

Now, write

$$V = \bigcap_{i=1}^n V_{y_i}, \quad \text{and} \quad U = \bigcup_{i=1}^n U_{y_i}.$$

Then V is a neighbourhood of x such that

$$V \cap Y \subseteq V \cap U = \bigcup_{i=1}^n (V \cap U_{y_i}) = \varnothing.$$

Hence we can fit an open set V around every $x \notin Y$, which means $X \setminus Y$ is open and Y is closed. ■

Note that we have in fact proven the following result.

Corollary
If X is Hausdorff, and C is a compact subset of X, then for $x \notin C$, there exists disjoint open sets U, V such that $x \in V$ and $C \subseteq U$.

What can we say when the spaces are not Hausdorff? Depends on the situation, actually.

Examples

1. If $X = \{a, b\}$ has the indiscrete topology, then every subset of X is compact. In particular, $\{a\}$ is compact. However, it is not closed since $\{b\}$ is not open.

2. In $\mathbb{R}$ with the finite complement topology, every subset is compact. Indeed, let C be a subset of $\mathbb{R}$, with open covering $\mathfrak{F}$. For $F \in \mathfrak{F}$, F covers C for at most a finite number of points, say $\{c_i\}_{i=1}^n$. Pick $F_i \in \mathfrak{F}$ such that $c_i \in F_i$ for all i. Then $\{F, F_1, \dots, F_n\}$ covers C, and so C is compact.

In the topology of the last example, even the open sets are compact. This does not contradict Theorem 229 since $\mathbb{R}$ is not Hausdorff in the finite complement topology. As it happens, compactness is a **topological notion**.

Theorem 230
The continuous image of a compact set $C \subseteq X$ by $f : X \to Y$ is compact.

Proof: let $\mathfrak{F}$ be an open covering of $f(C)$. By continuity, $\{f^{-1}(F)\}_{F\in\mathfrak{F}}$ is an open covering of C. So there is a finite sub-covering, say $\{f^{-1}(F_1), \dots, f^{-1}(F_n)\}$, as C is compact, and

$$f(C) \subseteq f\left(\bigcup_{i=1}^{n} f^{-1}(F_i)\right) = \bigcup_{i=1}^{n} f\left(f^{-1}(F_i)\right) \subseteq \bigcup_{i=1}^{n} F_i.$$

Then $\{F_1, \dots, F_n\}$ covers $f(C)$, and so $f(C)$ is compact. ■

There are all sorts of results about compact spaces and continuous functions.

Theorem 231
If X is compact, Y is Hausdorff, and $f : X \to Y$ is a continuous bijection, then f is a homeomorphism.

Proof: let C be a closed subset of X. As X is compact, C is compact. Since $f : X \to Y$ is continuous, $f(C)$ is compact in Y and thus closed in Y, as Y is Hausdorff. So f is closed. As f is a continuous bijection, f is a homeomorphism. ■

As we had done with connectedness, we would like to show that finite products of compact spaces are compact.[2] To do this we will need the following lemma.

Lemma 232 (Tube Lemma)
If Y is compact and N is an open set in $X \times Y$ which contains the slice $\{x_0\} \times Y$, then there exists a neighbourhood W of x_0 such that $W \times Y \subseteq N$.

Proof: if $y \in Y$, then $(x_0, y) \in N$. As N is open, there exists two neighbourhoods U_y of x_0 and V_y of y such that $U_y \times V_y \subseteq N$. Repeating this process for all $y \in Y$ yields an open covering $\{V_y\}_{y \in Y}$ of Y. As Y is compact, there is a finite sub-covering, say $\{V_{y_1}, \ldots, V_{y_n}\}$, with $U_{y_i} \times V_{y_i} \subseteq N$ for all $1 \leq i \leq n$. Let

$$W = \bigcap_{i=1}^{n} U_i.$$

then W is open in X as it is a finite intersection of open sets. Furthermore, $x_0 \in W$ as $x_0 \in U_{y_i}$ for all $1 \leq i \leq n$. Now, let $(x, y) \in W \times Y$. There is a j such that $y \in V_{y_j}$. As $x \in W$, $x \in U_{y_j}$. Then $(x, y) \in U_j \times V_j \subseteq N$, so $W \times Y \subseteq N$. ■

We now have all the machinery to prove the following result.

Theorem 233
If X and Y are compact, then $X \times Y$ is compact.

Proof: let $\mathfrak{F}$ be an open covering for $X \times Y$. For each $x \in X$ we get a finite sub-covering of $\{x\} \times Y$ from $\mathfrak{F}$, say $F(x)_1, \ldots F(x)_n$. Let N be the open set $N = \bigcup_{i=1}^{n} F(x)_i$. By the Tube Lemma, there is a neighbourhood W_x of x in X such that $W_x \times Y \subseteq N$. Repeating this procedure for all $x \in X$, we get that $\{W_x\}_{x \in X}$ is an open covering of X. But X is compact, so there is a finite sub-covering $\{W_{x_1}, \ldots, W_{x_m}\}$. For each of these W_{x_i}, there were n_i corresponding sets $F(x_i)_j$ in $\mathfrak{F}$. Define

$$\mathfrak{F}' = \{F(x_i)_j \mid 1 \leq i \leq m, 1 \leq j \leq n_i\}.$$

$\mathfrak{F}'$ is a finite open collection, with $\sum_{i=1}^{m} n_i$ elements. For any $(x, y) \in X \times Y$, $x \in W_{x_i}$ for some i. Then $(x, y) \in W_{x_i} \times Y$ and $(x, y) \in F(x_i)_j$ for some j, so

$$X \times Y \subseteq \bigcup_{i=1}^{m} \bigcup_{j=1}^{n_i} F(x_i)_j.$$

Thus $\mathfrak{F}'$ is a finite sub-covering of $X \times Y$ from $\mathfrak{F}$, and so $X \times Y$ is compact. ■

[2] It is not a simple matter to generalize to arbitrary products of compact spaces. This will be the content of chapter 19.

Du to the complementary nature of open and closed sets, it is also possible to express compactness in term of closed sets. A family $\mathfrak{F}$ of sets has the **finite intersection property** whenever

$$\bigcap_{i=1}^{n} F_i \neq \varnothing$$

for any selection $F_i \in \mathfrak{F}$, $1 \leq i \leq n$.

Theorem 234
A space X is compact if and only if every family $\{F_\alpha\}_\alpha$ of closed subsets of X having the finite intersection property has a non-void intersection, that is, $\bigcap_\alpha F_\alpha \neq \varnothing$.

Proof: we make the following three remarks: $\{X \setminus F_\alpha\}_\alpha$ is an open family if and only if $\{F_\alpha\}_\alpha$ is a closed family;

$$\bigcup_{\alpha}(X \setminus F_\alpha) = X \iff \bigcap_{\alpha} F_\alpha = \varnothing$$

and

$$\bigcup_{i=1}^{n}(X \setminus F_{\alpha_i}) = X \iff \bigcap_{i=1}^{n} F_{\alpha_i} = \varnothing$$

for any selection $F_i \in \mathfrak{F}$, $1 \leq i \leq n$. The theorem is easily proved using the contrapositive statement and the three remarks. ■

There is another version of this theorem:

Theorem 234 (Reprise)
A space X is compact if and only if for every family $\mathcal{A}$ of subsets of X satisfying the finite intersection property, the intersection $\bigcap_{A\in\mathcal{A}} \overline{A}$ is not empty.

Proof: left as an exercise. ■

As an easy corollary we get the following result.

Corollary 235 Let X be a compact space, and suppose

$$C_1 \supseteq C_2 \supseteq \cdots \supseteq C_n \supseteq \cdots$$

is a nested sequence of closed sets. Then

$$\bigcap_{n\in\mathbb{N}} C_n \neq \varnothing.$$

Interest in compact spaces arose when we realized that there was something special about intervals in the usual topology on $\mathbb{R}$ that made the max/min theorem come out as it did.

Theorem 236 If X has the order topology, where the ordering has the least upper bound property, then each interval

$$[a,b] = \{x \mid a \leq x \leq b\}$$

is compact.

Proof: the proof is similar to that of the Heine-Borel theorem (see Proposition 125 in Chapter 9, and Theorem 237). This means that the key point of the proof is the least upper bound property, and not the metric. ■

The first step in the process was a generalization of intervals to $\mathbb{R}^n$.

Theorem 237 (Heine-Borel Theorem – Reprise)
In the usual topology, the compact sets of $\mathbb{R}^n$ are exactly the closed and bounded sets.

Proof: since $\mathbb{R}^n$ is Hausdorff, any of its compact subset is closed. If C is compact in $\mathbb{R}^n$, it can be covered by

$$\{(-m,m)^n \mid m \in \mathbb{N}\}.$$

But C is compact, so it has a finite sub-covering and there exists $M \in \mathbb{N}$ such that $C \subseteq (-M,M)^n$. Thus C is bounded.

Conversely, suppose that C is a closed bounded set. Then, there exists $M \in \mathbb{N}$ such that $C \subseteq [-M,M]^n$. But $[-M,M]^n$ and C is a finite product of the compact spaces $[-M,M]$, and so is itself compact. C is then compact since it is a closed subset of a compact set. ■

Note that this result need not hold for a general metric space (where boundedness may not be defined, for instance), as we shall see shortly.[3]

Theorem 238 (Maximum and Minimum Value Theorem)
Let C be a compact subset of X, and suppose $f : X \to Y$ is continuous, where Y has a (total) order topology. Then f is bounded on C and actually attains its bounds there.

Proof: as C is compact and f is continuous, $f(C)$ is compact. If f does not have a largest value on C, then, for each $a \in C$, there exists $a' \in C$ such that $f(a) < f(a')$. For any $y \in Y$, denote $(-\infty, y) = \{z \in Y \mid z < y\}$.

[3]This is the reason for the less-than-intuitive definition of compactness currently in use.

Then

$$\{(-\infty, f(a))\}_{a\in C}$$

is an open covering of $f(C)$. But $f(C)$ is compact, so there exists a finite subcovering, say

$$\{(-\infty, f(a_i))\}_{i=1}^{n}.$$

Let $a_0 \in C$ be the a_i that maximizes $f(a_i)$. Since $f(a_0) \in f(C)$, $f(a_0) \in (-\infty, f(a_j))$ for some j, which means that $f(a_0) < f(a_j) \leq f(a_0)$, a contradiction, since $x \not< x$ in Y. Hence f has a largest value on C. The proof that f has a smallest value on C is similar. ■

This result is the generalization to topological spaces of one of the fundamental results of analysis (see Theorem 33 in Chapter 3.)

Metric Spaces (Reprise)

Let us revisit metric spaces from the vantage point of topology. If d is a metric on a space X, the basic open sets in X are the open balls

$$B_d(a, r) = \{x \in X \mid d(a, x) < r\}.$$

The topology generated by these basic sets is called the **metric topology on** X.[4]

Let us suppose that the metrics d and d' generate the topologies $\mathfrak{T}$ and $\mathfrak{T}'$ on X. $\mathfrak{T}$ is finer than $\mathfrak{T}'$ whenever $B_{d'}(a, r')$ is open in $\mathfrak{T}$ for all $a \in X$, $r' \in \mathbb{R}^+$, and so whenever there exists $r \in \mathbb{R}^+$ such that

$$B_d(a, r) \subseteq B_{d'}(a, r').$$

Example: let d be the Euclidean metric on $\mathbb{R}^2$ and d' be defined on $\mathbb{R}^2$ by $d'((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$. Then

$$B_d(0, 1) = \{(x, y) \mid x^2 + y^2 < 1\},\ B_{d'}(0, 1) = \{(x, y) \mid -1 < x < 1 \text{ and } -1 < y < 1\},$$

$B_d(0, 1) \subseteq B_{d'}(0, 1)$ and $B_{d'}(0, \frac{1}{\sqrt{2}}) \subseteq B_d(0, 1)$. Generalizing to all open balls, one gets $\mathfrak{T} = \mathfrak{T}'$. □

[4]The collection of all open balls is a basis. Indeed, $x \in B_d(x, 1)$ for all $x \in X$. The empty set is a ball of radius 0. Suppose that $y \in B_d(x_1, r_1) \cap B_d(x_2, r_2) \neq \varnothing$. Then $y \in B_d(y, r) \subseteq B_d(x_1, r_1) \cap B_d(x_2, r_2)$, where

$$r = \frac{\min\{d(x_1, y) - r_1, d(x_2, y) - r_2\}}{2}.$$

Let X be a metric space with metric d. The **standard bounded metric** $\overline{d}$ on X is the metric defined by

$$\overline{d}(x,y) = \min\{d(x,y),1\}$$

(the only property that is not trivially true is the triangle inequality). For any ball $B_d(0,\varepsilon)$, put $\delta = \min\{\varepsilon, 1\}$. Then

$$B_{\overline{d}}(a,\delta) \subseteq B_d(a,\varepsilon).$$

For a ball $B_{\overline{d}}(a,\delta)$, if $\delta \leq 1$, then $B_{\overline{d}}(a,\delta) = B_d(a,\delta)$. If $\delta > 1$, $B_{\overline{d}}(a,\delta) = X$. This means that the topology generated by the bounded standard metric $\overline{d}$ on X is the same as that generated by the metric d. Consequently, we may assume that the metric d is bounded.

A space X is **metrizable** if there is a metric d on X where the metric topology on X coincides with the topology on X. This leads us to one of the fundamental differences between metric spaces and general topological spaces, a result which is simple to state, but whose proof is surprisingly sophisticated.[5]

Theorem 239
Any countable product of metrizable spaces is metrizable.

Proof: Suppose (X_n, d_n) is a metric space and d_n is the standard bounded metric on X_n for all $n \in \mathbb{N}$. Let $x, y \in X = \prod X_n$ and define

$$d(x,y) = \text{l.u.b.}\left\{\frac{d_n(x_n,y_n)}{n}\right\}_{n\in\mathbb{N}}$$

It is not hard to see that this defines a metric on X. We need to verify that the topology generated on X by d is that given by the product topology.

Suppose $U \subseteq X$ is open in the product topology. If $x = (x_n) \in U$, there is a basic set $\prod V_n$, where $V_n \subseteq_O X_n$, and $V_n = X_n$ for all but a finite number of n's, i.e for all $n > N$ for some N. Then there exists $\varepsilon_n > 0$ such that $B(x_n, \varepsilon_n) \subseteq V_n$ for all $n \in \mathbb{N}$. Let

$$\varepsilon = \min\left\{\frac{\varepsilon_n}{n}\right\}_{n=1}^{N}.$$

If $y = (y_n) \in B(x,\varepsilon)$, then $d(x,y) < \varepsilon$, so

$$\frac{d_n(x_n,y_n)}{n} \leq d(x,y) < \varepsilon \leq \frac{\varepsilon_n}{n}$$

for $1 \leq n \leq N$.

[5] As a reminder, the notations $A \subseteq_O X$, $A \subseteq_C X$, and $A \subseteq_K X$ are used respectively for A *is open in* X, A *is closed in* X, and A *is compact in* X.

Hence $d(x_n, y_n) < \varepsilon_n$, and so

$$y_n \in B(x_n, \varepsilon_n) \subseteq V_n$$

for $1 \leq n \leq N$ and $y_n \in X_n = V_n$ for all $n > N$, so $y \in \prod V_n$ and $B(x, \varepsilon) \subseteq \prod X_n$. Then $\prod X_n$ is open in the metric topology. As a result, around each point of U, we can fit an open set in the metric topology, i.e. $U \subseteq X$ is open in the metric topology.

Conversely, suppose $U \subseteq X$ is open in the metric topology. Then, if $x \in U$, there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq U$. Choose N such that $\frac{1}{N} < \varepsilon$. Put $V_n = B(x_n, n\varepsilon)$ for all n, so that $V_n = X_n$ whenever $n > N$ (remember, the metrics d_n are standard bounded metrics). If $y = (y_n) \in \prod V_n$, then $d_n(x_n, y_n) < n\varepsilon$ for all $n \in \mathbb{N}$. In particular,

$$\frac{d_n(x_n, y_n)}{n} < \varepsilon$$

whenever $1 \leq n \leq N$ and

$$\frac{d_n(x_n, y_n)}{n} \leq \frac{1}{n} < \frac{1}{N} < \varepsilon$$

for all $n > N$. By construction,

$$d(x, y) = \text{l.u.b.} \left\{ \frac{d_n(x_n, y_n)}{n} \right\}_{n \in \mathbb{N}} < \varepsilon.$$

Then $y \in B(x, \varepsilon)$ and $\prod V_n \subseteq B(x, \varepsilon)$. As a result, around each point of U, we can fit an open set in the product topology, i.e. $U \subseteq X$ is open in the product topology. ■

Let (X_α, d_α) be a collection (not necessarily countable) of metric spaces, where d_α is a standard bounded metric on X_α. Define a metric d on $\prod_\alpha X_\alpha$ by

$$d(x, y) = \text{l.u.b.}\{d_\alpha(x_\alpha, y_\alpha)\}$$

for all $x, y \in X$.[6] This metric is called the **uniform metric**, and the topology it generates on $\prod_\alpha X_\alpha$ is called the **uniform topology** on X. We will in the solved problems that the uniform topology is finer than the product topology and coarser than the box topology, and that for infinite products, the inclusions are strict.

We now introduce another concept that allows us to tell if a space is metrizable. A sequence $\{x_n\}_n \in \mathbb{N}$ in a space X (not necessarily metric) **converges to** $x \in X$ (denoted $x_n \to x$) if for every neighbourhood V of x, there exists $N \in \mathbb{N}$ such that $x_n \in V$ for every $n > N$.

In a general topological space, the **limit** of a sequence is not necessarily unique!

[6] The only non-trivial component here is again the triangle inequality.

Examples (LIMITS)

1. Let $X = [0,1]$, where the basic open sets in X are of the form (a,b) and $[0,a) \cup (b,1]$ for $0 < a < b < 1$. In the topology generated by this basis, every neighbourhood of 0 is a neighbourhood of 1, and vice-versa. Thus $\frac{1}{n} \to 0$ as usual, but $\frac{1}{n} \to 1$ as well.

2. Let X be a space with the indiscrete topology. Then every sequence in X converges to every element of X. □

Suppose $\{a_n\}_{n\in\mathbb{N}}$ is a sequence in a set $A \subseteq X$, and let $a_n \to a \notin A$. Then a is a **limit point** of A. Indeed, for any neighbourhood V of a, there is some index N for which $a_n \in V$ when $n > N$. Consequently, $a_n \in V \cap A$ and $a \neq a_n$ for all $n > N$ (as $a \notin A$), so $a \in \overline{A}$.

In general, if a sequence in A converges to a point not in A, the limit is a limit point. However, the converse statement is false: if $a \in \overline{A}$, there might not be a sequence in A converging to a, as can be seen in the next example.[7]

Example: let Ω be the first uncountable ordinal; let X be the set $\Omega^+ = \Omega \cup \{\Omega\}$, with the order topology. Consider $A = \Omega = [0,\Omega)$. Suppose the sequence $\{\alpha_n\}_{n\in\mathbb{N}}$, where $\alpha_n \in A$, has the limit α. As

$$\alpha_n \leq \bigcup_{m\in\mathbb{N}} \alpha_m = \beta,$$

then $\alpha \leq \beta$. But β is a countable union of countable sets, hence it is countable. Therefore, $\beta < \Omega$, so $\alpha < \Omega$ and $\alpha_n \notin (\beta,\Omega)$ for all $n \in \mathbb{N}$. Now $\overline{A} = [0,\Omega]$, and so $\Omega \in \overline{A}$, but no sequence in A converges to Ω. □

This example may seem a bit far-fetched, but that is the nature of the discipline – in general topology, exotic counter-examples are entirely legitimate. In metric spaces, however, things tend to be substantially better behaved.

Lemma 240 (SEQUENCE LEMMA)
Let X be a metrizable space. For any subset A of X, if $a \in \overline{A}$, then there is a sequence $\{a_n\}_{n\in\mathbb{N}} \subseteq A$ with $a_n \to a$.

Proof: let d be the metric generating the topology on X. For each $n \in \mathbb{N}$, construct the neighbourhood $B(a, \frac{1}{n})$. As $a \in \overline{A}$, we have $A \cap B(a, \frac{1}{n}) \neq \varnothing$ for all $n \in \mathbb{N}$. Let $a_n \in A \cap B(a, \frac{1}{n})$ for all $n \in \mathbb{N}$. Then $a_n \to a$. Indeed, let V be a neighbourhood of a. Then there is a basic neighbourhood $B(a,\varepsilon) \subseteq V$. Let $N \geq \frac{1}{\varepsilon}$. Then, whenever $n > N$, we get $d(a,a_n) < \frac{1}{n} < \frac{1}{N} \leq \varepsilon$, hence $a_n \in V$ and $a_n \to a$. ■

[7] The example requires some familiarity with the **first uncountable ordinal**, see `https://en.wikipedia.org/wiki/First_uncountable_ordinal` for details.

Next, we see that one of the sacred cows of analysis see Proposition 106 in Chapter 8) may not necessarily hold in general topological spaces.

Theorem 241
The function $f : X \to Y$ is continuous if, whenever $a_n \to a$ in X, then $f(a_n) \to f(a)$ in Y. If X is metrizable, the converse holds.

Proof: suppose that f is continuous and $a_n \to a$ in X. Let V be a neighbourhood of $f(a)$. Then $f^{-1}(V)$ is a neighbourhood of a, and so there exists N such that $a_n \in f^{-1}(V)$ whenever $n > N$. Then $f(a_n) \in V$ whenever $n > N$ and $f(a_n) \to f(a)$ in Y.

Conversely, suppose X is metrizable and that the sequence condition holds. Let $A \subseteq X$. By the sequence lemma, if $a \in \overline{A}$, there is a sequence $\{a_n\}_{n\in\mathbb{N}} \subseteq A$ such that $a_n \to a$. By hypothesis, $f(a_n) \to f(a)$, so $f(a) \in \overline{f(A)}$, as $f(a_n) \in f(A)$ for all $n \in \mathbb{N}$. Hence $f(\overline{A}) \subseteq \overline{f(A)}$ which is equivalent to f being continuous. ■

17.2 Limit Point and Sequential Compactness

Throughout the history of topology, many definitions of compactness have been formulated. At the time, each were thought to have isolated the crucial property of a set like $[0, 1]$ that made the maximum/minimum theorem possible, amongst others.

As our understanding of topology increased, these different notions were discarded, to be replaced by the modern concept. But the failed candidates are interesting in their own rights, as they coincide with compactness in the case of metric spaces, as we shall see.

A subset A in a space X is said to be **sequentially compact** if every sequence $\{a_n\}_{n\in\mathbb{N}} \subseteq A$ contains a convergent subsequence. A subset A in a space X is said to be **limit point compact** if every infinite subset of A has a limit point.The next few results show how the various compactness notions are related.

Proposition 242
If X is compact, then X is limit point compact.

Proof: suppose X is compact and let A be a subset of X with no limit point. Then $A = \overline{A}$, and A is closed, so compact. Also, for any $a \in A$, there is a neighbourhood V_a such that $V_a \cap A = \{a\}$. Thus $\{V_a\}_{a\in A}$ is an open covering of A. Since A is compact, there is a finite sub-covering $\{V_{a_i}\}_{i=1}^n$. But

$$A = A \cap \left(\bigcup_{i=1}^{n} V_{a_i}\right) = \bigcup_{i=1}^{n}(A \cap V_{a_i}) = \bigcup_{i=1}^{n}\{a_i\}.$$

Hence A is finite. By contraposition, X is limit point compact. ■

Proposition 243
If X is a limit point compact metric space, then X is sequentially compact.

Proof: let $\{a_n\}_{n\in\mathbb{N}} \subseteq X$, and write $A = \{a_1, a_2, \ldots\}$. If A is finite, there has to exist a constant (hence convergent) subsequence $\{a_{n_m}\}_{m\in\mathbb{N}}$. Otherwise, A is infinite. As X is limit point compact, A has a limit point, say a and every neighbourhood of a contains a point in A different from a. In particular, since X is a metric space, for each $m \in \mathbb{N}$, $B(a, \frac{1}{m})$ is a neighbourhood of a and there exists $a_{n_m} \in B(a, \frac{1}{m}) \cap A$ such that $a_{n_m} \neq a$. By construction, $a_{n_m} \to a$, so X is sequentially compact. ■

Proposition 244
If X is sequentially compact, then X is limit point compact.

Proof: let A be an infinite subset of the X. Then A contains a countable subset $\{a_1, a_2, \ldots\}$. As X is sequentially compact, there is a convergent subsequence $a_{n_m} \to a$. By construction, a is a limit point of A and X is limit point compact. ■

The following result to show that the notions of compactness are equivalent for metric spaces.

Theorem 245
Let X be a compact metric space. For any open covering $\mathfrak{F}$ of X, there is a number $\delta > 0$ satisfying the following property: if $A \subseteq X$ is such that diam$(A) < \delta$, then there exists $F \in \mathfrak{F}$ such that $A \subseteq F$.

Proof: we prove the theorem by contradiction. Let $\mathfrak{F}$ be an open covering of X, and suppose that no δ satisfying the property exists. Then, for each $n \in \mathbb{N}$, we can find a set A_n such that diam$(A_n) < \frac{1}{n}$ where $A_n \nsubseteq F$ for all $F \in \mathfrak{F}$. As $A_n \neq \varnothing$ for all $n \in \mathbb{N}$, we can select $a_n \in A_n$ for all $n \in \mathbb{N}$, and get the sequence $\{a_n\}_{n\in\mathbb{N}}$.

In a metric space, compactness implies sequential compactness, so there is a convergent subsequence $\{a_{n_m}\}_{m\in\mathbb{N}}$, with $a_{n_m} \to a \in X$. Pick $F \in \mathfrak{F}$ such that $a \in F$. As F is open, there exists $r > 0$ such that $B(a, 2r) \subseteq F$.

Since the subsequence is convergent, there is a number $N \in \mathbb{N}$ such that $a_{n_m} \in B(a, r)$ for all $n_m > N$. Pick $n_m >$ such that $\frac{1}{n_m} < r$. If $x \in A_{n_m}$, then

$$d(x, a) \leq d(x, a_{n_m}) + d(a_{n_m}, a) < \frac{1}{n_m} + r < 2r$$

since diam$(A_{n_m}) < \frac{1}{n_m}$ and $A_{n_k} \subseteq B(a, 2r) \subseteq F$, a contradiction. So there must be a number $\delta > 0$ satisfying the property. ■

The number δ in the proof of Theorem 245 is called a **Lebesgue number** of the covering $\mathfrak{F}$.

We need one more definition before we are ready to prove our big result. A metric space X is **totally bounded** if, for every $\varepsilon > 0$, X can be covered by a finite number of ε-balls.

Theorem 246
In a metric space, compactness, sequential compactness, and limit point compactness are equivalent.

Proof: according to Propositions 242, 243, and 244, it only remains to show that a sequentially compact set X is compact. Let $\mathfrak{F}$ be an open covering of X, and suppose X is not totally bounded. Then there exists $\varepsilon > 0$ such that there is no finite covering of X by ε-balls.

Let $x_1 \in X$. As $B(x_1, \varepsilon) \neq X$, select $x_2 \in X \setminus B(x_1, \varepsilon)$. It is possible to select

$$x_{n+1} \in X \setminus \bigcup_{i=1}^{n} B(x_i, \varepsilon)$$

since $\bigcup_{i=1}^{n} B(x_i, \varepsilon)$ does not cover X. By recursion, $\{x_n\}_{n\in\mathbb{N}}$ is a sequence, and it contains a converging subsequence $\{x_{n_m}\}_{m\in\mathbb{N}}$, where $x_{n_m} \to x$, since X is sequentially compact. Then, there exists M such that $x_{n_m} \in B(x, \frac{\varepsilon}{2})$ and

$$d(x_{n_{m+1}}, x_{n_m}) \leq d(x, x_{n_{m+1}}) + d(x, x_{n_m}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever $m > M$. But this yields $x_{n_{m+1}} \in B(x_{n_m}, \varepsilon)$, which is a contradiction by construction of the sequence $\{x_n\}_{n\in\mathbb{N}}$. Hence X must be totally bounded.

Let 3ε be a Lebesgue number of $\mathfrak{F}$. Then there exists a finite collection $\mathfrak{B} = \{B(y_i, \varepsilon)\}_{i=1}^{n}$ covering X. As diam$(B(y_i, \varepsilon)) \leq 2\varepsilon < 3\varepsilon$, $\exists F_i \in F$ such that $B(y_i, \varepsilon) \subseteq F_i$ for all $1 \leq i \leq n$. If $x \in X$, then $x \in B(y_i, \varepsilon) \subseteq F_i$ for some i. Since $\mathfrak{B}$ is a finite covering of X, $\{F_i\}_{i=1}^{n}$ is a finite sub-covering of X, and so X is compact. ■

The converse of Proposition 242 is not in general true, as can be seen in the following example (which again uses the smallest uncountable ordinal).

Example: let Ω be the first uncountable ordinal, and let X be the set $\Omega^+ = \Omega \cup \{\Omega\}$, with the order topology. Now, Ω is limit point compact. Indeed, suppose C is an infinite (countable) subset of Ω. Then C is bounded above by $\bigcup_{\gamma\in C} \gamma = \beta$, and so $C \subseteq [0, \beta]$. It is clear that Ω has the l.u.b. property, so, by Theorem 236, $[0, \beta]$ is compact. By Proposition 242, $[0, \beta]$ is limit point compact, and so C contains a limit point. Thus Ω is limit point compact. But Ω isn't closed in the Hausdorff space X, so Ω is not compact. □

17.3 Local Compactness and One-Point Compactification

By analogy with local connectedness, we can also define a notion of local compactness: a space X is **locally compact at** $x \in X$ if there exists a compact set C which contains a neighbourhood V of x. We say that X is **locally compact** if it is locally compact at each $x \in X$.

There is an equivalent definition if X is Hausdorff space. For each $x \in X$, if there exists a neighbourhood V and a compact set C such that $x \in V \subseteq C$, then, as X is Hausdorff, C is closed, so $\overline{V} \subseteq C$, and x has a neighbourhood with compact closure.

Examples (Local Compactness)

1. Every compact space is locally compact.
2. $\mathbb{R}$ is locally compact, since, for any basic open set $]a, b[$, the closure $[a, b]$ is compact. Similarly, $\mathbb{R}^n$ is locally compact for all $n \in \mathbb{N}$. However $\mathbb{R}^\omega$ is not locally compact in the product topology. Indeed, let

$$V = (a_1, b_1) \times \cdots (a_n, b_n) \times \mathbb{R} \times \cdots$$

be a basic neighbourhood in the product topology. Then

$$\overline{V} = [a_1, b_1] \times \cdots [a_n, b_n] \times \mathbb{R} \times \cdots ,$$

which is not compact in the product topology. □

Let X be a locally compact Hausdorff space, and suppose that ∞ is a point not in X. Construct a new set $Y = X \cup \{\infty\}$, with the following topology: $V \subseteq_O Y$ if either

- $V = U \subseteq_O X$ whenever $\infty \notin V$, or;
- $V = Y \setminus C$, where C is a compact subset of X whenever $\infty \in V$.

This is indeed a topology on Y, as we see presently.

1. $\varnothing$ is an open set of type 1, Y is an open set of type 2.
2. Let $V_1, V_2 \subseteq_O Y$. Then
   a) $V_1, V_2 \subseteq_O X$, so $V_1 \cap V_2 \subseteq_O X$, hence $V_1 \cap V_2 \subseteq_O Y$; or
   b) $V_1 \subseteq_O X$ and $V_2 = Y \setminus C$, where $C \subseteq_K X$. Then
   $$V_1 \cap V_2 = V_1 \cap (Y \setminus C) = V_1 \cap (X \setminus C) \subseteq_O Y,$$
   as C is closed in X, since X is Hausdorff; or
   c) $V_1 = Y \setminus C_1$, $V_2 = Y \setminus C_2$ where $C_1, C_2 \subseteq_K X$. Then
   $$V_1 \cap V_2 = (Y \setminus C_1) \cap (Y \setminus C_2) = Y \setminus (C_1 \cup C_2) \subseteq_O Y,$$
   since $C_1 \cup C_2 \subseteq_K X$ whenever $C_1, C_2 \subseteq_K X$.

3. a) $V_\beta \subseteq_O X$, so $\bigcup_\beta V_\beta \subseteq_O X$, hence $\bigcup_\beta V_\beta \subseteq_O Y$; or
   b) $V_\alpha \subseteq_O X$ (i.e $\bigcup_\alpha V_\alpha \subseteq_O X$) and $V_\beta = Y \setminus C_\beta$, where $C_\beta \subseteq_K X$. Then

$$\left(\bigcup_\alpha V_\alpha\right) \cup \left(\bigcup_\beta V_\beta\right) = \left(\bigcup_\alpha V_\alpha\right) \cup \left(\bigcup_\beta (Y \setminus C_\beta)\right) = \left(\bigcup_\alpha V_\alpha\right) \cup \left(Y \setminus \bigcap_\beta C_\beta\right)$$
$$= Y \setminus \left(\bigcap_\beta C_\beta - \bigcup_\alpha V_\alpha\right) \subseteq_O Y,$$

   as $\bigcap_\beta C_\beta - \bigcup_\alpha V_\alpha$ is compact since it is a closed subset of a compact set; or
   c) $V_\beta = Y \setminus C_\beta$, where $C_\beta \subseteq_K X$. Then

$$\bigcup_\beta V_\beta = \bigcup_\beta (Y \setminus C_\beta) = Y \setminus (\bigcap_\beta C_\beta) \subseteq_O Y,$$

   since $\bigcap_\beta C_\beta \subseteq_K X$ whenever $C_\beta \subseteq_K X$.

The subspace topology on X agrees with the original topology on X. Indeed, in the subspace topology, open sets look like $V \cap X$, where $V \subseteq_O Y$. If $\infty \notin V$, then $V \cap X = V \subseteq_O X$ in the original topology.

On the other hand, if $\infty \in V$, $V = Y \setminus C$ for some compact C, and $V = Y \setminus C = X \setminus C$. But X is Hausdorff, so C is closed, and $V \subseteq_O X$ in the original topology. Conversely, every open set in the original topology is an open set of type 1 in the subspace topology.

Theorem 247

Let X be a non-compact locally compact Hausdorff space and $\infty \notin X$. Then $Y = X \cup \{\infty\}$ is compact Hausdorff with the topology defined above and $\overline{X} = Y$.

Proof: let $\mathfrak{F}$ be an open covering of Y. Then, there exists $F_0 \in \mathfrak{F}$ with $\infty \in F_0$. By definition, $C = Y \setminus F_0$ is a compact subset of X and $\mathfrak{F}' = \{F \cap X\}_{F \in \mathfrak{F}'}$ is an open covering of C in X. As C is compact, there is a finite sub-covering

$$\{F_1 \cap X, \ldots, F_n \cap X\}$$

of C. Hence $\{F_0, F_1, \ldots, F_n\}$ is a finite sub-covering of Y and Y is compact.

As X is not compact, $\{\infty\} = Y \setminus X$ is not open in Y. So every neighbourhood of ∞ looks like $Y \setminus C$, where $C \subsetneq_K X$, and so meets X. By definition, ∞ is a limit point of X in Y, so $\overline{X} = Y$.

We show now that Y is Hausdorff. If $x \neq y \in X$, there are open neighbourhoods in X satisfying the T_2 condition as X is Hausdorff. So suppose $x \in X$ and $y = \infty$. As X is locally compact, there is a compact set C and a neighbourhood V of x such that $x \in V \subseteq C$. Then $U = Y \setminus C$ is a neighbourhood of ∞ and $U \cap V = \varnothing$, which proves that Y is Hausdorff. ■

The space Y is the **one-point compactification of** X.

Examples (One-Point Compactification)

1. Let $X = \mathbb{R}$. Then X is a non-compact locally compact Hausdorff space. By Theorem 247, there is a one-point compactification $Y = X \cup \{\infty\}$ of X. Y is in fact homeomorphic to

$$S^1 = \{(x, y) \mid x^2 + y^2 = 1\},$$

through the homeomorphism $f : S^1 \to Y$ defined by $f(x, y) = \frac{x}{1-y}$ whenever $y \neq 1$ and $f(0, 1) = \infty$.

2. Let $X = \mathbb{R}^2$. Then X is a non-compact locally compact Hausdorff space. By Theorem 247, there exists a one-point compactification Y, or $X \cup \{\infty\}$ of X. Y is in fact homeomorphic to

$$S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\},$$

through the homeomorphism $f : S^2 \to Y$ defined by

$$f(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$

whenever $z \neq 1$ and $f(0, 0, 1) = \infty$.

17.4 Solved Problems

1. Let $\mathfrak{B}$ be a basis for a topology on a space X. Show that a subset A of X is compact if and only if every covering of A by sets from $\mathfrak{B}$ has a finite subcovering.

Proof: if $A \subseteq_K X$, then every open covering of A contains a finite subcovering of A. But every covering of A by sets from $\mathfrak{B}$ is an open covering of A as all sets in $\mathfrak{B}$ are open, and so contains a finite subcovering of A.

Conversely, suppose that every covering of A by sets from $\mathfrak{B}$ contains a finite subcovering of A, and let $\mathfrak{U} = \{U_\gamma\}_{\gamma \in \Gamma}$ be an open covering of A. Since $U_\gamma \subseteq_O X$, and since $\mathfrak{B}$ is a basis for the topology on X, there exists, for each $\gamma \in \Gamma$, a subset $\mathfrak{B}_\gamma \subseteq \mathfrak{B}$ such that

$$U_\gamma = \bigcup_{B \in \mathfrak{B}_\gamma} B.$$

Thus, the collection $\{B | B \in \mathfrak{B}_\gamma \text{ for some } \gamma \in \Gamma\}$ is a covering of A by sets from $\mathfrak{B}$, and by hypothesis, it contains a finite subcovering of A, say $\{B_1, \ldots, B_n\}$. Now, for $1 \leq i \leq n$, choose $U_i \in \mathfrak{U}$ such that $B_i \subseteq U_i$. Then $\{U_1, \ldots, U_n\}$ is a finite subcovering of A, and $A \subseteq_K X$. ■

2. Let A and B be disjoint compact subsets of the Hausdorff space X. Show that there exist disjoint open sets U and V containing A and B, respectively.

Proof: assume that $A, B \neq \varnothing$, otherwise the statement is vacuously true. Let $b \in B$. Since X is Hausdorff and $A \cap B = \varnothing$, for every $a \in A$, there exists $U_{b,a}, V_{b,a} \subseteq_O X$, such that $a \in U_{b,a}$, $b \in V_{b,a}$ and $U_{b,a} \cap V_{b,a} = \varnothing$. The collection $\{U_{b,a}\}_{a \in A}$ is an open covering of $A \subseteq_K X$, and so we can extract from it a finite subcovering, say $\{U_{b,a_1}, \ldots, U_{b,a_n}\}$. Now, put

$$U(b) = \bigcup_{i=1}^{n} U_{b,a_i} \quad \text{and} \quad V(b) = \bigcap_{i=1}^{n} V_{b,a_i}.$$

Then $A \subseteq U(b), U(b), V(b) \subseteq_O X$ and $U(b) \cap V(b) \neq \varnothing$. This process can be repeated for every $b \in B$ so that $\{V(b)\}_{b \in B}$ covers B. Since $B \subseteq_K X$, we can extract a finite subcovering of B, say $\{V(b_1), \ldots, V(b_m)\}$. Let

$$V = \bigcup_{i=1}^{m} V(b_i) \quad \text{and} \quad U = \bigcap_{i=1}^{m} U(b_i).$$

Then $U, V \subseteq_O X$, $A \subseteq U$, $B \subseteq V$ and $U \cap V = \varnothing$. Indeed

$$\begin{aligned} U \cap V &= U \cap (\textstyle\bigcup_{i=1}^{m} V(b_i)) = \bigcup_{i=1}^{m} (U \cap V(b_i)) \\ &\subseteq \bigcup_{i=1}^{n} (V(b_i) \cap U(b_i)) = \bigcup_{i=1}^{m} \varnothing = \varnothing, \end{aligned}$$

and the statement is proven. ■

3. Show that $[0,1]$ is not compact in $\mathbb{R}_l$. Is it compact in the countable complement topology on $\mathbb{R}$?

Proof: we prove the statement by exhibiting an open covering of $[0,1]$ from which it is impossible to extract a finite subcovering. Let

$$\mathfrak{U} = \{[1,2)\} \cup \{[0, 1 - 1/n)\}_{n \geq 2}.$$

Then $\mathfrak{U}$ is an open covering of $[0,1]$ since

$$[0,1) \subseteq \bigcup_{U \in \mathfrak{U}} U = [0,2),$$

and since $[a,b)$ is open in $\mathbb{R}_l$ for all $a < b$ in $\mathbb{R}$. Any subcovering of $[0,1)$ must contain $[1,2)$ as $1 \notin [0, 1-1/n)$ for all $n \geq 2$. Any finite subcovering must then look like

$$\mathfrak{V} = \{[1,2), [0, 1 - 1/n_1), \ldots, [0, 1 - 1/n_m)\},$$

where the n_i's are ordered such that $n_1 > n_2 > \ldots > n_m \geq 2$. With this ordering,

$$\bigcup_{i=1}^{m-1} [0, 1 - 1/n_i) \subseteq [0, 1 - 1/n_m).$$

However, $1 - 1/(n_m - 1) \notin [0, 1 - 1/n_m)$ and $1 - 1/(n_m - 1) \notin [1, 2)$. Any finite subcollection $\mathfrak{V}$ taken from $\mathfrak{U}$ cannot cover all of $[0, 1]$, so $[0, 1]$ is not compact in $\mathbb{R}_l$.

We show now that $[0, 1]$ is not compact in $\mathbb{R}$ with the finite complement topology. First, recall that a space is compact if and only if every family $\mathfrak{C} = \{C_\alpha\}$ of closed subsets having the finite intersection property, that is $\bigcap_{i=1}^n C_{\alpha_i} \neq \varnothing$ for all $C_{\alpha_i} \in \mathfrak{C}$, $1 \le i \le n$, has a non-empty intersection:

$$\bigcap_\alpha C_\alpha \neq \varnothing.$$

We construct a family of closed subsets having the finite intersection property, while their full intersection is empty. The closed subsets of $[0, 1]$ in this topology are the countable subsets of $[0, 1]$, as well as $[0, 1]$ itself. Now let

$$A_n = \left\{ \frac{1}{m} \right\}_{m \ge n} \subseteq [0, 1]$$

for all $n \in \mathbb{N}$. Each of the $A_n \neq \varnothing$ is countable and so closed in $[0, 1]$. Now, take $A_{n_1}, \ldots, A_{n_k}$, where $n_k > n_{k-1} > \ldots > n_1$. By construction,

$$\bigcap_{i=1}^k A_{n_i} = A_{n_k} \neq \varnothing.$$

But $\bigcap_{n \in \mathbb{N}} A_n = \varnothing$, since, otherwise, there would exist $m \in \mathbb{N}$ such that $\frac{1}{m} \le \frac{1}{n}$ for all $n \in \mathbb{N}$, a contradiction. We can thus conclude that $[0, 1]$ is not a compact subspace in the finite complement topology. ■

4. Let X be a locally compact space. If $f : X \to Y$ is continuous, is the space $f(X)$ necessarily locally compact? What if f is both continuous and open?

Proof: let $X = \{-1\} \cup (0, 1)$ be a subspace of $\mathbb{R}$ and

$$T = \{(x, \sin(1/x)) \mid 0 < x < 1\} \cup \{(0, 0)\}$$

be a subspace of $\mathbb{R}^2$. This is the *topologist's sine curve*. Let $f : X \to T$ be the map sending -1 to $(0, 0)$ and x to $(x, \sin(1/x))$ for $0 < x < 1$. This map is continuous, since the pre-image of open subsets of T in the subspace topology are unions of open intervals in X, possibly with $\{-1\}$. Furthermore, $f(X) = T$. X is clearly locally compact at x for $0 < x < 1$. And $\{-1\}$ is a compact neighbourhood of $\{-1\}$, so that X is locally compact at -1. But T is not locally compact at $(0, 0)$. Indeed any candidate for a compact subset around $(0, 0)$ must contain an infinity of points who are as close as desired from the slice $\{0\} \times [-1, 1]$. Hence, no such sets are closed in $\mathbb{R}^2$, and so they can not be compact.

Suppose f is continuous and open. Then for any $y \in f(X)$ there exists $x \in X$ such that $f(x) = y$. The space X is locally compact so there is a compact set C_x and an open set U_x such that $x \in U_x \subseteq C_x$. Now, applying f yields

$$y = f(x) \in f(U_x) \subseteq f(C_x).$$

Since f is continuous and open $f(C_x)$ is compact and $f(U_x)$ is open. So $f(X)$ is locally compact at $f(x)$ for all $f(x) \in f(X)$, and $f(X)$ is locally compact. ■

5. Show that $[0,1]^\omega$ is not locally compact in the uniform topology.

Proof: Throughout, we assume that $[0,1]^\omega$ has the uniform topology. Let $\delta_{m,n}$ be the Kronecker δ, and let $d_m = (\delta_{m,n})_{n\in\mathbb{N}}$. Hence $d_m \in [0,1]^\omega$ for all $m \in \mathbb{N}$ and $d_U(d_m, d_k) = 1$ when $m \neq k$. Consequently, the sequence $d_1, d_2, \ldots$ has no convergent subsequence. Now, consider the open ball $B(x,r)$ in $[0,1]^\omega$. It contains a sequence $(x_n \pm \frac{rd_n}{2})_{n\in\mathbb{N}}$,[8] with no convergent subsequence. Hence $B(x,r)$ can not be contained in a compact set as compact set are sequentially compact in $[0,1]^\omega$. Thus $[0,1]^\omega$ is not locally compact in the uniform topology. ■

6. Let $\mathfrak{T}_P$, $\mathfrak{T}_U$, $\mathfrak{T}_B$ denote the product, uniform and box topologies respectively on $\mathbb{R}^\omega$.

a) Show that $\mathfrak{T}_B$ is strictly finer than $\mathfrak{T}_U$.

b) In which of the topologies are the following functions from $\mathbb{R}$ to $\mathbb{R}^\omega$ continuous?

i. $f(t) = (t, 2t+1, 3t+2, 4t+3, \ldots)$

ii. $g(t) = (t/2, t/3, t/4, t/5, \ldots)$

c) In which of the topologies do the following sequences converge?

$$x_1 = (1,1,1,1,\ldots) \qquad y_1 = (1,0,0,0,\ldots)$$
$$x_2 = (0,2^2,2^2,2^2,\ldots) \qquad y_2 = ((1/2)^2,(1/2)^2,0,0,\ldots)$$
$$x_3 = (0,0,3^3,3^3,\ldots) \qquad y_3 = ((1/3)^3,(1/3)^3,(1/3)^3,0,\ldots)$$
$$x_4 = (0,0,0,4^4,\ldots) \qquad y_4 = ((1/4)^4,(1/4)^4,(1/4)^4,(1/4)^4,\ldots)$$
$$\vdots$$
$$z_1 = (1,1,0,0,\ldots)$$
$$z_2 = ((1/2)^2,(1/2)^2,0,0,\ldots)$$
$$z_3 = ((1/3)^2,(1/3)^2,0,0,\ldots)$$
$$z_4 = ((1/4)^4,(1/4)^4,0,0,\ldots)$$
$$\vdots$$

Solution:

a) Let $B(x,\varepsilon_x)$ be an open ball in the uniform topology. The set

$$B_x = \prod_{n\in\mathbb{N}} \left(x_n - \frac{\varepsilon_x}{4}, x_n + \frac{\varepsilon_x}{4}\right)$$

is open in the box topology, and $x \in B_x \subseteq B(x,\varepsilon_x)$. Indeed, let $z \in B_x$. Then $d_n(x_n,z_n) < \frac{\varepsilon_x}{2}$ for all $n \in \mathbb{N}$, so

$$d(x,z) = \text{l.u.b.}\{d_n(x_n,z_n)\} \leq \frac{\varepsilon_x}{2} < \varepsilon_x,$$

[8] Select either one of $+$ or $-$ so that x_n stays in $[0,1]$.

and thus $z \in B(x, \varepsilon_x)$. Now suppose $x \neq y \in B(x, \varepsilon_x)$. As $B(x, \varepsilon_x)$ is open in the uniform (metric) topology, there exists $\varepsilon_y > 0$ such that $B(y, \varepsilon_y) \subseteq B(x, \varepsilon_x)$. Using the same reasoning as above yields

$$y \in B_y \subseteq B(y, \varepsilon_y) \subseteq B(x, \varepsilon_x),$$

where B_y is open in the box topology for all $y \in B(x, \varepsilon_x)$. Hence, around each point of $B(x, \varepsilon_x)$, we can fit an open set in the box topology, i.e. $B(x, \varepsilon_x)$ is open in the box topology and $\mathfrak{T}_U \subseteq \mathfrak{T}_B$.

We show that $\mathfrak{T}_U \subsetneq \mathfrak{T}_B$ by showing that $\mathbb{R}^\omega$ is not metrizable in the box topology. Since $\mathbb{R}^\omega$ has a metric in the uniform topology, $\mathfrak{T}_B$ is strictly finer than $\mathfrak{T}_U$. Let $X = \mathbb{R}^\omega$ and $A = (0,1)^\omega$. Clearly $0 = (0, 0, 0, \ldots) \in \overline{A}$ since, in the box topology every neighbourhood of 0 contains positive sequences. However, there is no sequence $x_n \in A$ such that $x_n \to 0$. Suppose x_n is a sequence in A. Then,

$$\begin{aligned} x_1 &= (x_{1,1}, x_{1,2}, x_{1,3}, \ldots) \\ x_2 &= (x_{2,1}, x_{2,2}, x_{2,3}, \ldots) \\ x_3 &= (x_{3,1}, x_{3,2}, x_{3,3}, \ldots) \\ &\vdots \end{aligned}$$

Let $\varepsilon < 0$, and construct the open set (in the box topology)

$$U_\varepsilon = \prod_{m \in \mathbb{N}} (\varepsilon, y_m),$$

where $0 < y_m < x_{m,m}$ for all $m \in \mathbb{N}$. By construction, U_ε is a neighbourhood of 0 in the box topology, and $x_n \notin U_\varepsilon$ for all $n \in \mathbb{N}$. Hence, x_n can not converge to 0. By the Sequence Lemma, $\mathbb{R}^\omega$ (in the box topology) is not metrizable.

b) Both of the functions are continuous in the product topology as each of the components are continuous. In the uniform topology, f is not continuous. Indeed, let $\varepsilon = 1/2$. Then, for every $\delta > 0$,

$$d_U(f(x), f(x+\delta)) = \text{l.u.b.}\{\min\{n\delta, 1\}\} = 1 > \varepsilon.$$

In the box topology f is not continuous. Indeed, let

$$U = \prod_{n \in \mathbb{N}} \left(\frac{n^2 - n - 1}{n}, \frac{n^2 - n + 1}{n} \right).$$

U is open, but $f^{-1}(U) = \{0\}$[9] which is closed in $\mathbb{R}$. Similarly, g isn't continuous in the box topology. Let

$$V = \prod_{n \in \mathbb{N}} \left(-\frac{1}{(n+1)^2}, \frac{1}{(n+1)^2} \right).$$

[9] $t \in f^{-1}(U) \iff f(t) \in U \iff \frac{n^2-n-1}{n} < nt + (n-1) < \frac{n^2-n+1}{n}\ \forall n \iff -\frac{1}{n} < nt < \frac{1}{n}\ \forall n \iff t = 0.$

V is open, but $g^{-1}(V) = \{0\}$[10] which is closed in $\mathbb{R}$. But it is continuous in the uniform topology. Indeed, let $\varepsilon > 0$ and put $\delta = 2\varepsilon$. If $|x-y| < \delta$, then

$$d_Y(g(x), g(y)) < \text{l.u.b.}\{\min\{\delta/(n+1), 1\}\} = \frac{1}{2}\delta = \varepsilon.$$

c) All three sequences have to converge to $0 = (0,0,0\ldots)$ if they converge at all. They all converge in the product topology. Indeed, suppose U is a basic neighbourhood of 0 in the product topology. Then

$$U = U_1 \times U_2 \times \cdots \times U_m \times \mathbb{R} \times \mathbb{R} \times \cdots$$

for some $m \in \mathbb{N}$, and where each of the $U_i =]a_i, b_i[$ are basic neighbourhood of 0 in $\mathbb{R}$. The sequence x_n lies in U for $n > m$, y_n lies in U for all n such that $\left(\frac{1}{n}\right)^n < \min_{1\leq i\leq m}\{b_i\}$,[11] and z_n lies in U for all

$$n > \frac{1}{(\min_{1\leq i\leq m}\{b_i\})^2}.$$

Let's look at what happens in the uniform topology. The sequence x_n does not converge to 0. Indeed, let $B(0,\varepsilon)$ be a ε-neighbourhood of 0, so

$$B(0,\varepsilon) \quad = \{\xi \mid \text{l.u.b.}|\xi_i| < \varepsilon\}.$$

For the sequence x_n,

$$\text{l.u.b.}\{\min\{|x_{nj}|, 1\}\} = 1,$$

which is bigger than every $\varepsilon < 1$. Hence, there does not exist a N for which $x_n \in B(0,\varepsilon)$ when $n > N$, and $\varepsilon < 1$. At the same time, (x_n) does not converge in the box topology. For y_n, z_n, all elements of the sequence are less than 1 for large enough n, so we can forget about the metric being bounded, and

$$\begin{aligned} \text{l.u.b.}\{|y_{nj}|\} \quad &= (1/n)^n \\ \text{l.u.b.}\{|z_{nj}|\} \quad &= (1/n)^2. \end{aligned}$$

For these least upper bounds, there exists N such that $y_n, z_n \in B(0,\varepsilon)$ whenever $n > N$, so the sequences converge to 0 in the uniform topology.[12] In the box topology, (y_n) doesn't converge, but (z_n) does. $\square$

[10] $t \in g^{-1}(V) \iff g(t) \in V \iff -\frac{1}{(n+1)^2} < \frac{t}{n+1} < \frac{1}{(n+1)^2}\ \forall n \iff -\frac{1}{n+1} < t < \frac{1}{n+1}\ \forall n \iff t = 0.$

[11] As $f(x) = x^{-x}$ is eventually decreasing, y_n is eventually in U for all $n > N$.

[12] Wait, you say. For this sequence to converge to 0, every neighbourhhod of 0 must contain all y_n when $n > N$ for some N. Ah, but every neighbourhood of 0 contains $B(0,\varepsilon)$ for some $\varepsilon > 0$, and this ball contains all y_n when $n > N$, so the original neighbourhood did as well...

17.5 Exercises

1. Prepare a 2-page summary of this chapter, with important definitions and results.

2. Let X be the subspace

$$X = \left\{ \frac{t}{1+t} e^{it} \,\middle|\, t \geq 0 \right\} \cup \{e^{i\pi}\}.$$

   Give detailed answers to the following:

   a) Is X compact?

   b) Is X locally compact?

3. Prove that if Y is compact and N is an open set in $X \times Y$ containing $\{x_0\} \times Y$, then there is a neighbourhood W of x_0 such that $W \times Y \subseteq N$.

4. If Y is compact, show that the projection $\pi_1 : X \times Y \to X$ is closed.

5. Prove Theorem 234 (Reprise) and Corollary 235.

6. Show that the standard bounded metric $\overline{d}$ and the uniform metric are indeed metrics on (X, d).