Chapter 18

Countability and Separation

In Chapter 15, we introduced a few simple separation definitions (T_0, T_1, and T_2/Hausdorff); in this chapter, we extend the discussion to more sophisticated **separation** axioms, and introduce the notions of **first** and **second countable** spaces.

18.1 Countability Axioms

A **basis at** $x \in X$ is a collection $\mathfrak{B}$ of open sets containing x and such that, for each neighbourhood V of x, there exists $B \in \mathfrak{B}$ with $x \in B \subseteq V$.

We say that a space X is **first countable at** $x \in X$ if there is a **countable** basis at x; X is simply **first countable** if it is first countable at **every** $x \in X$. It is **second countable** if its topology has a countable basis.

Examples (First and Second Countability)

1. If X is second countable, then it has a countable basis $\mathfrak{B}$. Let $x \in X$. If U is an (open) neighbourhood of x, then

$$U = \bigcup_{n \in \mathbb{N}} B_n,$$

where $B_n \in \mathfrak{B}$ for all $n \in \mathbb{N}$. As $x \in U$, then $x \in B_m$ for some m. Hence

$$\mathfrak{B}_x = \{B \in \mathfrak{B} \mid x \in B\}$$

is a countable basis at x since $\mathfrak{B}_x \subseteq \mathfrak{B}$, and so X is first countable at x. But x is arbitrary, so X is first countable.

2. Let $X = \mathbb{R}$ in the usual topology; it is second countable since

$$\mathfrak{B} = \{(a, b) \mid a < b \in \mathbb{Q}\}$$

is a countable basis of X. In light of the previous example, X is also first countable.

3. Let $X = \mathbb{R}$ in the discrete topology. $\mathbb{R}$ is not second countable as every set is open and $\mathbb{R}$ is uncountable. However, it is first countable, since $\{x\}$ is a basis at x for each $x \in X$. □

A space X is **Lindelöf** if every open covering of X contains a **countable** (not necessarily finite) sub-covering. A subset A of X is **Lindelöf** if it is Lindelöf in the subspace topology.

Theorem 248
If X is second countable, then it is Lindelöf.

Proof: let $\mathfrak{F}$ be an open covering and $\mathfrak{B} = \{B_n\}_n$ be a countable basis of X. For each $n \in \mathbb{N}$, whenever it is possible to do so, let $F_n \in \mathfrak{F}$ be such that $B_n \subseteq F_n$. Otherwise let $F_n = \varnothing$. Then

$$X = \bigcup_{n\in\mathbb{N}} B_n \subseteq \bigcup_{n\in\mathbb{N}} F_n$$

and $\{F_n\}_n$ is a countable sub-collection; it is also an open cover. Indeed, let $x \in X$. Then there exists $F \in \mathfrak{F}$ such that $x \in F$. As F is open, there exists a basic set $B_n \in \mathfrak{B}$ such that $x \in B_n \subseteq F$. By construction, $F \in \{F_n\}_n$. Hence X is Lindelöf. ■

Let us take a look at some examples.

Examples (Lindelöf Spaces)

1. The space $\mathbb{R}$ is second countable (hence Lindelöf), since

$$\mathfrak{B} = \{(a, b) \mid a < b \in \mathbb{Q}\}$$

is a countable basis.

2. The space $\mathbb{R}_l$ is Lindelöf but not second countable. Indeed, let $\mathfrak{B}$ be any basis for the lower limit topology on $\mathbb{R}_l$. Then, for any $x \in \mathbb{R}$ and $\varepsilon > 0$, we have $[x, x+\varepsilon) \subseteq_O \mathbb{R}_l$, that is, there is a basic set $B_{x,\varepsilon}$ such that

$$x \in B_{x,\varepsilon} \subseteq [x, x+\varepsilon).$$

If $x < y$, then, for $\varepsilon = y - x$, $y \notin B_{x,\varepsilon}$. So $\mathfrak{B}$ must contain an uncountable sub-collection and $\mathbb{R}_l$ is not second countable.

We show that $\mathbb{R}_l$ is Lindelöf by showing that every open covering by basic sets contains a countable sub-covering. Let $\mathfrak{F} = \{[\alpha_a, \beta_a)\}_{a\in J}$ be an open covering of $\mathbb{R}_l$ and

$$C = \bigcup_{a\in J}(\alpha_a, \beta_a)$$

be a subspace of $\mathbb{R}$. As $\mathbb{R}$ is second countable, so is C; it is thus also Lindelöf, as of Theorem 248. The collection $\{(\alpha_a, \beta_a)\}_{a\in J}$ is an open covering of C, so there exists a sub-covering $\{(\alpha_{a_n}, \beta_{a_n})\}_{n\in\mathbb{N}}$ of C. Then

$$\mathfrak{F}' = \{[\alpha_{a_n}, \beta_{a_n})\}_{n\in\mathbb{N}}$$

also covers C and $\mathfrak{F}' \cup (\mathbb{R} \setminus C)$ is a covering of $\mathbb{R}$.

Let $x \in \mathbb{R} \setminus C$. Then $x = \alpha_a$ for some $a \in J$. Let $q_x \in (\alpha_a, \beta_a) \cap \mathbb{Q}$. Then

$$(x, q_x) \subseteq (\alpha_a, \beta_a) \subseteq C.$$

Now suppose $x < y \in \mathbb{R} \setminus C$. Necessarily, $q_x < q_y$ since, otherwise,

$$y \in (x, q_y) \subseteq (x, q_x) \subseteq C,$$

a contradiction as $y \notin C$. Thus the map $x \mapsto q_x$ is an injection of $\mathbb{R} \setminus C$ into $\mathbb{Q}$, which means that $\mathbb{R} \setminus C$ is countable. Write $\mathbb{R} \setminus C = \{z_n\}_{n\in\mathbb{N}}$, and find $[\alpha_m, \beta_m) \in \mathfrak{F}$ with $z_m \in [\alpha_m, \beta_m)$ for all $m \in \mathbb{N}$ – this can be done as $\mathfrak{F}$ is an open covering of $\mathbb{R}_l$. Then $\mathfrak{F}' \cup \{[\alpha_m, \beta_m)\}_{m\in\mathbb{N}}$ is a countable sub-cover of $\mathbb{R}_l$ extracted from $\mathfrak{F}$.

3. The space $\mathbb{R}_l^2$ is not Lindelöf. To show this, let $L = \{(x, -x)\}_{x\in\mathbb{R}}$. Then $\mathbb{R}_l^2 \setminus L$ is open in $\mathbb{R}_l^2$. Indeed, let $(x, y) \in \mathbb{R}_l^2 \setminus L$ and put $\varepsilon = \frac{x+y}{2}$. Then

$$(x, y) \in [x, x + \varepsilon) \times [y, y + \varepsilon)$$

and $([x, x + \varepsilon) \times [y, y + \varepsilon)) \cap L = \varnothing$. Now $\mathfrak{F} = \{\mathbb{R}_l^2 \setminus L\} \cup \{F_a\}_{a\in\mathbb{R}}$, where

$$F_a = [a, a + 1) \times [-a, -a + 1),$$

is an open covering of $\mathbb{R}_l^2$. But F_a is the only set in $\mathfrak{F}$ containing $(a, -a)$, so any sub-covering will contain F_a for all $a \in \mathbb{R}$. As $\mathbb{R}$ is uncountable, $\mathfrak{F}$ does not contain a countable sub-covering. Hence $\mathbb{R}_l^2$ is not Lindelöf. This demonstrates that the product of two Lindelöf spaces need not be Lindelöf.

4. Let Ω be the first uncountable ordinal. Then $\Omega = [0, \Omega)$ is first countable but not Lindelöf, so it is not second countable. Indeed, suppose $a \in \Omega$. Then

$$\mathfrak{B}_a = \{(c, a+1)\}_{c<a}$$

is countable as $a < \Omega$. Let U be a neighbourhood of a. Then $a \in (c, a+1) \subseteq U$ for $c < a$. This makes $\mathfrak{B}_a$ a countable basis at $a \in \Omega$. Then Ω is first countable.

To show that Ω is not second countable, consider the open covering $\mathfrak{F} = \{[0, b)\}_{b\in\Omega}$ of Ω, and let $\mathfrak{F}'$ be any countable sub-collection from $\mathfrak{F}$. Let

$$\beta = \bigcup_{[0,b)\in\mathfrak{F}'} b.$$

As it is a countable union of countable sets, β is countable, that is $\beta \in \Omega$. But $\beta \notin [0, b)$ for all $[0, b)$ in $\mathfrak{F}'$, and so $\mathfrak{F}'$ cannot be a sub-covering from $\mathfrak{F}$. Hence Ω is not Lindelöf, nor is it second countable. □

We can show fairly easily that countability behaves as expected for subspaces and products.

Theorem 249
If X is first (resp. second) countable, then any subspace of X is first (resp. second) countable. If X_n is first (resp. second) countable for all $n \in \mathbb{N}$, then

$$\prod_{n\in\mathbb{N}} X_n$$

is first (resp. second) countable.

Proof: the statement about subspaces is clearly true. We show that the countable product of second countable spaces is second countable. The proof for first countable spaces is similar, and is left as an exercise.

Let $X = \prod X_n$, $\mathfrak{B}_n$ be a countable basis for X_n, and define

$$\mathfrak{C}_m = \left\{ \prod_{n\in\mathbb{N}} V_n \,\middle|\, V_n \in \mathfrak{B}_n \text{ for } 0 \leq n \leq m, V_n = X_n \text{ for } m < n \right\}$$

for all $m \in \mathbb{N}$. Then $\mathfrak{C} = \bigcup_{m\in\mathbb{N}} \mathfrak{C}_m$ is countable. Furthermore, it is a basis for the product topology on X. So X is second countable. ■

We shall see in the next section that there is a link between countability and separation.

18.2 Separation Axioms

Let X be a space. In Chapter 15, we introduced a number of separation axioms:

0. X is T_0 if for every pair $x \neq y \in X$, there exist a neighbourhood U of either x or y that misses the other;
1. X is T_1 if for every pair $x \neq y \in X$, there exist neighbourhoods U_x of x and U_y of y such that $y \notin U_x$ and $x \notin U_y$;
2. X is T_2 or **Hausdorff** if for every pair $x \neq y \in X$, there exist disjoint neighbourhoods U_x of x and U_y of y.

We have also seen that if a space X is T_1, then every singleton is closed in X. Note that the condition T_2 is strictly stronger than the condition T_1: there are T_1 spaces that fail to be T_2.

We introduce two new **separation axioms.**[1] We say that a space X is:

3. T_3 or **regular** if X is T_1 and if for every pair consisting of a $x \in X$ and a closed set B disjoint from x, there exist disjoint neighbourhoods U_x of x and U_B containing B;
4. T_4 or **normal** if X is T_1 and if for every pair consisting of disjoint closed sets A and B, there exist disjoint neighbourhoods U_A containing A and U_B containing B.

Some of the conditions imply some of the others: a regular space is Hausdorff, for instance, since singletons are closed. Indeed let $x \neq y$. Then x and the closed set $\{y\}$ are disjoint and there exist U_x and $U_{\{y\}}$ such that $x \in U_x$, $\{y\} \subseteq U_{\{y\}}$ and $U_x \cap U_{\{y\}} = \varnothing$. For the same reasons, a normal space is regular. The following examples (without proof) show that none of the implications

$$T_4 \Longrightarrow T_3 \Longrightarrow T_2 \Longrightarrow T_1 \Longrightarrow T_0$$

can be reversed and that normal spaces are not as well behaved as we might expect.

Example: (REGULARITY AND NORMALITY)

1. Let $K = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ be a subset of $\mathbb{R}$, with basic open sets of the form (a, b) and $(a, b) \setminus K$ for all $a, b \in \mathbb{R}$. With this topology $\mathbb{R}$ is Hausdorff. But it is not regular, since it is possible to separate the point 0 and the closed set K. Hence a Hausdorff space need not be regular.
2. Let Ω be the least uncountable ordinal. The spaces Ω and Ω^+ are normal in the order topology. But their product is not normal. The product $\Omega^+ \times \Omega^+$ is normal however, so a subspace of a normal space need not be normal. And, as we will see later, $\Omega \times \Omega^+$ is regular, being the product of two regular spaces, so a regular space need not be normal.
3. If A is uncountable, the product space $\mathbb{R}^A$ is not normal.

[1]There are other such axioms; see `https://en.wikipedia.org/wiki/Separation_axiom` for more.

We can also formulate the conditions of regularity and normality differently, as that could be more useful in different contexts (as in the next section).

Lemma 250
Let X be T_1. Then

1. *X is regular if and only if given a point of $x \in X$ and a neighbourhood U of x, there is a neighbourhood V of x such that $\overline{V} \subseteq U$.*
2. *X is normal if and only if given a closed set $A \subseteq X$ and an open set U containing A, there is an open set V containing A such that $\overline{V} \subseteq U$.*

Proof:

1. Suppose X is regular and let $x \in X$ have a neighbourhood U. Then

$$x \notin X \setminus U \subseteq_C X.$$

By regularity of X, there exist open subsets V and W such that $x \in V$, $X \setminus U \subseteq W$ and $V \cap W = \varnothing$. Suppose $y \in W$. Then W is a neighbourhood of y that does not meet V, and so $\overline{V} \cap W = \varnothing$. Hence $\overline{V} \subseteq X \setminus W = U$.

Conversely, suppose $B \subseteq_C X$ and $x \notin B$. then

$$x \in X \setminus B \subseteq_O X.$$

By hypothesis, there exists a neighbourhood V of x such that

$$x \in \overline{V} \subseteq X \setminus B.$$

Then by construction, $B \subseteq X \setminus \overline{V} \subseteq_O X$, $x \in V$ and $X \setminus \overline{V} \cap V = \varnothing$. In other words, X is regular.

2. The proof of the second statement uses sensibly the same argument. ■

T_2 and T_3 spaces behave particularly well with respect to subspaces and products.

Theorem 251 Let W, $\{W_\alpha\}$ be Hausdorff, X, $\{X_\beta\}$ be regular.

1. Each subspace Y of W is Hausdorff, and the product $\prod W_\alpha$ is Hausdorff.
2. Each subspace Y of X is regular, and the product $\prod X_\beta$ is regular.

Proof:

1. Let Y be a subspace of W. If $x \neq y \in Y$, then there exist disjoint $U, V \subseteq_O X$ such that $x \in U$ and $y \in V$. But $U \cap Y, V \cap Y \subseteq_O Y$ are disjoint and $x \in U \cap Y$, $y \in V \cap Y$, so Y is Hausdorff.

   Let $W = \prod W_\alpha$. If $x = (x_\alpha) \neq y = (y_\alpha)$, then there is a coordinate γ such that $x_\gamma \neq y_\gamma$. As W_γ is Hausdorff, there exist disjoint $U, V \subseteq_O X_\gamma$ such that $x_\gamma \in U$, $y_\gamma \in V$. Then $\pi_\gamma^{-1}(U), \pi_\gamma^{-1}(V) \subseteq_O W$ are disjoint and $x \in \pi_\gamma^{-1}(U)$, $y \in \pi_\gamma^{-1}(V)$, so W is Hausdorff.

2. Let Y be a subspace of X. Since Y is Hausdorff, one point sets are closed in Y. If $x \in Y$, and B is a closed subset of Y disjoint from x, then

   $$\overline{B} \cap Y \subseteq \overline{B \cap Y} = B \cap Y = B.$$

   So $x \notin \overline{B}$ (in X). By regularity of X, there exist disjoint $U, V \subseteq_O X$ such that $x \in U$ and $\overline{B} \subseteq V$. Then $U \cap Y, V \cap Y \subseteq_O Y$ are disjoint, $x \in U \cap Y$ and $B \subseteq V \cap Y$. Hence Y is regular.

   Let $X = \prod X_\beta$. Since X is Hausdorff, one point sets are closed in X. Let $x = (x_\beta) \in X$ and suppose U is a neighbourhood of x. Choose a basis $\prod U_\beta$ such that

   $$x \in \prod U_\beta \subseteq U.$$

   For each β, X_β is regular. Then there exists a neighbourhood V_β such that $x_\beta \in \overline{V_\beta} \subseteq U_\beta$.[2] Then, $V = \prod V_\beta$ is a neighbourhood of $x \in X$. But $\overline{V} = \prod \overline{V_\beta}$, so

   $$\overline{V} \subseteq \prod U_\beta \subseteq U$$

   and so X is regular according to Lemma 250. ■

The following three theorems give sets of hypotheses under which **normality** is assured.

Theorem 252
Let X be metrizable. Then X is normal.

Proof: let d be the metric on X, and A and B be disjoint closed subsets of X. For each $a \in A$, choose ε_a such that $B(a, \varepsilon_a) \cap B = \varnothing$ – this can always be done as B is closed in X so $X \setminus B \subseteq_O X$.

Similarly, for each $b \in B$, choose ε_b such that $B(b, \varepsilon_b) \cap A = \varnothing$. Then

$$U = \bigcup_{a \in A} B(a, \varepsilon_a/2) \quad \text{and} \quad V = \bigcup_{b \in B} B(b, \varepsilon_b/2)$$

are open subsets of X containing A and B respectively. They are also disjoint. Otherwise, $B(a, \varepsilon_a/2) \cap B(b, \varepsilon_b/2) \neq \varnothing$ for some $a \in A$, $b \in B$. Suppose z lies in that intersection. Then

$$d(a, b) \leq d(a, z) + d(z, b) < \frac{\varepsilon_a + \varepsilon_b}{2}.$$

If $\varepsilon_a \leq \varepsilon_b$, then $d(a, b) < \varepsilon_b$ and $a \in B(b, \varepsilon_b)$. If $\varepsilon_a \geq \varepsilon_b$, then $d(a, b) < \varepsilon_a$ and $b \in B(a, \varepsilon_a)$. But both these statements are false, so $U \cap V = \varnothing$ and X is normal. ■

As usual, compact Hausdorff space behave nicely.

Theorem 253
Let X be a compact Hausdorff space. Then X is normal.

Proof: see the solved problems. ■

We establish a link with second countability below.

Theorem 254
Let X be a second countable regular space. Then X is normal.

Proof: let $\mathfrak{B}$ be a countable basis for X. Suppose A and B are disjoint closed subsets of X. As B is closed, each $x \in A$ has a neighbourhood U_x not meeting B. By regularity, there is a neighbourhood V_x of x such that

$$x \in \overline{V_x} \subseteq U.$$

As $V_x \subseteq_O X$, there exists $W_x \in \mathfrak{B}$ such that $x \in W_x \subseteq V_x$, and

$$\overline{W_x} \subseteq U_x \subseteq X \setminus B,$$

so $\overline{W_x} \cap B = \varnothing$. Then $\{W_x\}_{x \in A}$ is a countable open covering of A since it is contained in $\mathfrak{B}$. Let us re-index it, and write $\{W_n\}_{n \in \mathbb{N}}$. Similarly, it is possible to construct a countable open covering $\{Z_n\}_{n \in \mathbb{N}}$ of B such that $\overline{Z_n} \cap A = \varnothing$ for all $n \in \mathbb{N}$. Given $n \in \mathbb{N}$, define

$$W_n' = W_n \setminus \bigcup_{i=1}^{n} \overline{Z_i} \quad \text{and} \quad Z_n' = Z_n \setminus \bigcup_{i=1}^{n} \overline{W_i}.$$

Then $W_n', Z_n' \subseteq_O X$ as $W_n, Z_n \subseteq_O X$ and $\bigcup_{i=1}^n \overline{W_i}, \bigcup_{i=1}^n \overline{Z_i} \subseteq_C X$. Let

$$W' = \bigcup_{n\in\mathbb{N}} W_n' \quad \text{and} \quad Z' = \bigcup_{n\in\mathbb{N}} Z_n'.$$

Then $W', Z' \subseteq_O X$ and $A \subseteq W'$ and $B \subseteq Z'$. Indeed if $x \in A$, then $x \in W_n$ for some n. But, by construction, $x \notin \overline{Z_i}$ for all $i \in \mathbb{N}$. Then $x \in W_n'$. Similarly, if $y \in B$, then $y \in Z_n'$ for some $n \in \mathbb{N}$. It remains only to show that $W' \cap Z' = \varnothing$.

Suppose $\xi \in W' \cap Z'$. Then $\xi \in W_n' \cap Z_m'$ for some $m, n \in \mathbb{N}$. If $m \geq n$, then

$$\xi \in W_n' \Longrightarrow \xi \in W_n, \quad \text{and} \quad \xi \in Z_m' \Longrightarrow \xi \notin \overline{W_n},$$

which is a contradiction. If $m \leq n$, then

$$\xi \in W_n' \Longrightarrow \xi \notin \overline{Z_m}, \quad \text{and} \quad \xi \in Z_m' \Longrightarrow \xi \in Z_m,$$

another contradiction. Then $W' \cap Z' = \varnothing$ and X is normal. ■

18.3 Results of Urysohn and Tietze

Let X be a normal space, and A, B be disjoint closed subsets of X. Put $U_1 = X \setminus B \subseteq_O X$, so $A \subseteq U_1$. As X is normal, there exists $U_0 \subseteq_O X$ such that $A \subseteq U_0$ and $\overline{U_0} \subseteq U_1$. For each dyadic rational $r = \frac{m}{2^n}$ in $[0,1]$, we can associate an open set U_r such that

$$r < s \Longrightarrow \overline{U_r} \subseteq U_s. \tag{18.1}$$

To do so, we start with any $U_{\frac{1}{2}} \subseteq_O X$ such that

$$\overline{U_0} \subseteq U_{\frac{1}{2}} \subseteq \overline{U_{\frac{1}{2}}} \subseteq U_1$$

(this can be done as X is normal, $U_1 \subseteq_O X$, $\overline{U_0} \subseteq_C X$, and $\overline{U_0} \subseteq U_1$). Then, by the same process, it is possible to obtain $U_{\frac{1}{4}}, U_{\frac{3}{4}} \subseteq_O X$ satisfying

$$\overline{U_0} \subseteq U_{\frac{1}{4}} \subseteq \overline{U_{\frac{1}{4}}} \subseteq U_{\frac{1}{2}} \quad \text{and} \quad \overline{U_{\frac{1}{2}}} \subseteq U_{\frac{3}{4}} \subseteq \overline{U_{\frac{3}{4}}} \subseteq U_1.$$

Recursively, suppose we have sets $U_{\frac{m}{2^n}}$ satisfying (18.1), for $m = 0, 1, \ldots, 2^n$.

Then $\overline{U_{\frac{m}{2^n}}} \subseteq U_{\frac{m+1}{2^n}}$ for all $m = 0, 1, \ldots, 2^n - 1$. By normality of X, for $m = 0, 1, \ldots, 2^n - 1$, there is an set $U_{\frac{2m+1}{2^{n+1}}} \subseteq_O X$ such that

$$\overline{U_{\frac{m}{2^n}}} \subseteq U_{\frac{2m+1}{2^{n+1}}} \subseteq \overline{U_{\frac{2m+1}{2^{n+1}}}} \subseteq U_{\frac{m+1}{2^n}}.$$

Let r be a **dyadic rational** not in $[0,1]$.[3] If $r > 1$, take $U_r = X$. If $r < 0$, take $U_r = \varnothing$. Then (18.1) holds for all dyadic rational.

[3] A dyadic rational is a real number that can be written as a fraction with denominator 2^q for some non-negative integer q.

Now, let $x \in X$, and define

$$Q(x) = \{p \mid x \in U_p\}.$$

For all $x \in X$, $p \notin Q(x)$ whenever $p < 0$ since $x \notin U_p = \varnothing$, and $q \in Q(x)$ whenever $q > 1$ since $x \in U_p = X$. Hence $Q(x)$ is bounded below and its **greatest lower bound** lies in $[0, 1]$. Define $f : X \to [0, 1]$ by

$$f(x) = \text{g.l.b.}\{Q(x)\}.$$

Then, $f(a) = 0$ for $a \in A$ since $Q(a)$ is the set of dyadic rational in $[0, \infty)$, and $f(b) = 1$ for $b \in B$ since $Q(b)$ is the set of dyadic rational in $(1, \infty)$. By construction,

1. $x \in \overline{U_p} \implies f(x) \leq p$;
2. $x \notin U_p \implies f(x) \geq p$.

Indeed, if $x \in \overline{U_p}$, then $x \in U_q$ for all $q > p$. Then $q \in Q(x)$ for all $q > p$, and so that $f(x) \leq p$. If $x \notin U_p$, then p is a lower bound for $Q(x)$ so that $f(x) \geq p$.

Theorem 255 (Urysohn Lemma)
The function f defined above is continuous.

Proof: suppose $x_0 \in X$ and (a, b) is a neighbourhood of $f(x_0)$. We find a set $U \subseteq_O X$ such that $x_0 \in U \subseteq f^{-1}((a, b))$. Choose two dyadic rationals $p < q$ such that $a < p < f(x_0) < q < b$. Let $U = U_q \setminus \overline{U_p}$. Then $U \subseteq_O X$ as U_q is open and $\overline{U_p}$ is closed. Since $q > f(x_0) > p$, we have $x_0 \notin \overline{U_p}$ and $x_0 \in U_q$, so $x_0 \in U$.

If $x \in U$, then $x \in U_q \subseteq \overline{U_q}$ and $f(x) \leq q < b$; but $x \notin \overline{U_p}$ so $x \notin U_p$ and $a < p \leq f(x)$. Thus $f(U) \subseteq (a, b)$ and $U \subseteq f^{-1}((a, b))$, so $f^{-1}((a, b)) \subseteq_O X$ and f is continuous. ■

We have shown that in any normal space X, it is possible to separate any two **disjoint closed sets** A and B by a continuous function $f : X \to [0, 1]$, where $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$. Note that this does not necessarily mean that $f^{-1}(\{0\}) = A$ and $f^{-1}(\{1\}) = B$. This prompts the following definition.

A T_1-space is $T_{3\frac{1}{2}}$ or **completely regular** if, given a point x_0 and a closed subset A with $x_0 \notin A$, there is a continuous function $f : X \to [0, 1]$ such that $f(x_0) = 0$ and $f(a) = 1$ for all $a \in A$. Suppose X is completely regular. Let x_0 and A be a closed subset of X such that $x_0 \notin A$. Then there is a continuous function $f : X \to [0, 1]$ with $f(x_0) = 0$ and $f(a) = 1$ for all $a \in A$. Define

$$U = f^{-1}([0, 1/3)) \quad \text{and} \quad V = f^{-1}((2/3, 1]).$$

Then $U, V \subseteq_O X$, $x_0 \in U$, $A \subseteq V$, and $U \cap V = \varnothing$, and so X is regular.

One of the most important corollaries of the Urysohn lemma is the Tietze extension theorem. Before stating and proving it, we first prove the following useful lemma.

Lemma 256
Let X be a normal space and $A \subseteq_C X$. If $h : A \to [-r, r]$ is continuous, then there is a continuous function $g : X \to \left[-\frac{r}{3}, \frac{r}{3}\right]$ such that $|h(x) - g(x)| \leq \frac{2}{3}r$ for all $x \in A$.

Proof: let $B = h^{-1}([r/3, r])$ and $C = h^{-1}([-r, -r/3])$. Then $B, C \subseteq_C A$ as h is continuous, so $B, C \subseteq_C X$ as $A \subseteq_C X$, and $B \cap C = \varnothing$. Since X is normal, we can use the Urysohn lemma to construct a continuous function $g : X \to \left[-\frac{r}{3}, \frac{r}{3}\right]$ such that $g(b) = \frac{r}{3}$ for all $b \in B$ and $g(c) = -\frac{r}{3}$ for all $c \in C$. Now, let $x \in A$.

Then there are three cases:

1. If $x \in B$, then $r \geq h(x) \geq \frac{r}{3} = g(x)$, so $\frac{2}{3}r \geq h(x) - g(x) \geq 0$.
2. If $x \in C$, then $-r \leq h(x) \leq -\frac{r}{3} = g(x)$, so $\frac{2}{3}r \geq g(x) - h(x) \geq 0$.
3. If $x \in A \setminus (B \cup C)$ then $|h(x)| < r$ and $|g(x)| \leq \frac{r}{3}$, so

$$|h(x) - g(x)| \leq |h(x)| + |g(x)| \leq 2r/3.$$

Hence $|h(x) - g(x)| \leq 2r/3$ whenever $x \in A$. ■

We are now ready to prove the extension result.

Theorem 257 (Tietze Extension Theorem)
Let X be a normal space and A a closed subset of X.

1. *If $f : A \to [a, b]$ is continuous, there is a continuous function $g : X \to [a, b]$ such that $g|_A = f$.*
2. *If $f : A \to \mathbb{R}$ is continuous, there is a continuous function $g : X \to \mathbb{R}$ such that $g|_A = f$.*

Proof:

1. It is sufficient to prove the theorem for $a = -1$, $b = +1$, as $[-1, 1]$ is homeomorphic to $[a, b]$ for all $a < b \in \mathbb{R}$. Let $r = 1$, $h = f$ and apply Lemma 256 to get a a continuous function g_1 on X such that

$$|g_1(x)| \leq 1/3 \quad \text{and} \quad |f(a) - g_1(a)| \leq 2/3$$

for all $x \in X$, $a \in A$.

Then, for $r = \frac{2}{3}$, $h = f - g_1$, we repeat the process to get a continuous function g_2 on X such that, for all $x \in X$, $a \in A$, we have:

$$|g_2(x)| \leq 1/3 \cdot (2/3) \quad \text{and} \quad |f(a) - g_1(a) - g_2(a)| \leq (2/3)^2 .$$

By recursion, suppose that $s_n = \sum_{k=1}^n g_k$, where $|f(a) - s_n(a)| \leq \left(\frac{2}{3}\right)^n$ for all $a \in A$. Take $r = \left(\frac{2}{3}\right)^n$, and $h = f - s_n$. Then by Lemma 256, there is a continuous function g_{n+1} on X such that

$$|g_{n+1}(x)| \leq 1/3 \cdot (2/3)^n \quad \text{and} \quad |f(a) - s_n(a) - g_{n+1}(a)| \leq (2/3)^{n+1}$$

for all $x \in X$, $a \in A$. By induction, the continuous functions g_n are defined for all $n \in \mathbb{N}$, and $|g_n(x)| \leq 1/3 \cdot (2/3)^{n-1} = M_n$ for all $x \in X$. By the Weierstrass M-test (Theorem 79),

$$g = \sum_{n \in \mathbb{N}} g_n$$

is uniformly convergent hence continuous. By construction

$$|g(x)| \leq \sum_{n \in \mathbb{N}} |g_n(x)| \leq \sum_{n \in \mathbb{N}} \frac{1}{3} \left(\frac{2}{3}\right)^{n-1} = \frac{1/3}{1 - 2/3} = 1$$

for all $x \in X$. For $a \in A$, $|f(a) - s_n(a)| \leq \left(\frac{2}{3}\right)^n$. Then, as $n \to \infty$, $s_n(a) \to f(a)$ and $s_n(a) \to g(a)$. As X is Hausdorff, limits are unique, so $g|_A = f$.

2. It is sufficient to prove the theorem for continuous $f : A \to (-1, 1)$, as $(-1, 1)$ is homeomorphic to $\mathbb{R}$. If $f : A \to (-1, 1) \subseteq [-1, 1]$ is a continuous function, using part 1 of the theorem, there is a continuous extension $h : X \to [-1, 1]$. Define

$$D = h^{-1}(\{-1\}) \cup h^{-1}(\{1\}) \subseteq X.$$

As h is continuous and X is Hausdorff, $D \subseteq_C X$. Since

$$h(A) = f(A) \subseteq (-1, 1),$$

then $A \cap D = \varnothing$. Using the Urysohn lemma, there is a continuous function $\phi : X \to [0, 1]$ such that $\phi(D) = \{0\}$ and $\phi(A) = \{1\}$. Let $g(x) = \phi(x)h(x)$ for all $x \in X$.

Then g is continuous and $g|_A = f$ since $g(a) = \phi(a)h(a) = 1 \cdot h(a) = f(a)$ for all $a \in A$. Finally, $g : X \to (-1, 1)$. Indeed, if $x \in D$, then $g(x) = \phi(x)h(x) = 0 \cdot h(x) = 0 \in (-1, 1)$. If $x \notin D$, then $|h(x)| < 1$, so $|g(x)| \leq |\phi(x)||h(x)| < 1$. ■

In the remaining part of this chapter, we prove a result that provides conditions under which a topological space is **metrizable**.

Theorem 258 (Urysohn Metrization Theorem)
Every regular second countable space X is metrizable.

Proof: we show X is metrizable by showing it is homeomorphic to a subspace of $\mathbb{R}^\omega$ in the product topology. Let $\mathfrak{B} = \{B_n\}_{n\in\mathbb{N}}$ be a basis of X. Then, using appendix A, section 4, question 3, there is a countable collection of continuous function $f_n : X \to [0,1]$, where $f_n(x) > 0$ for $x \in B_n$ and $f(X \setminus B_n) = \{0\}$ for all $n \in \mathbb{N}$. Given $x_0 \in X$, and a neighbourhood U of x_0, there is an index $n \in \mathbb{N}$ such that $f_n(x_0) > 0$ and $f(X \setminus U) = \{0\}$. Indeed, choose a basis element B_n such that $x_0 \in B_n \subseteq U$. Then the index n satisfies the property. Now, define a function $F : X \to \mathbb{R}^\omega$ (in the product topology) by

$$F(x) = (f_1(x), f_2(x), \ldots).$$

We show that F is an embedding. Clearly, F is continuous, since f_n is continuous for all $n \in \mathbb{N}$. Furthermore, it is injective. Indeed, let $x \neq y$. As X is Hausdorff, there is a neighbourhood U of x disjoint from y. Using the property above, there is an index $n \in \mathbb{N}$ such that $f_n(x) > 0$ and $f_n(y) = 0$. Hence $F(x) \neq F(y)$. It remains only to show that F is an homeomorphism from X to $F(X)$. As F is already continuous, it will be sufficient to show that F is open.

Let $U \subseteq_O X$ and $z_0 \in F(X)$. Then there exists $W \subseteq_O F(X)$ such that $z_0 \in W \subseteq F(U)$. Indeed, let $x_0 \in U$ such that $F(x_0) = z_0$. As above, there is an index $N \in \mathbb{N}$ such that $f_N(x_0) > 0$ and $f_N(X \setminus U) = \{0\}$. Let

$$V = \pi_N^{-1}((0,\infty)) \subseteq_O \mathbb{R}^\omega,$$

and set $W = V \cap F(X)$. Then $W \subseteq_O F(X)$, and

$$\pi_N(z_0) = \pi_N(F(x_0)) = f_N(x_0) > 0,$$

so that $z_0 \in W$. We show now that $W \subseteq F(U)$. If $z \in W$, there exists an $x \in X$ such that $z = F(x)$ and $\pi_N(z) > 0$. But $0 < \pi_N(z) = f_N(x)$, so $x \in U$. Then $F(x) \in F(U)$ and $W \subseteq F(U)$. Hence F is open. ■

In the proof, we have called upon a special countable collection of continuous functions. The following theorem shows how to generalize to arbitrary collections.

Theorem 259 (Embedding Theorem)
Suppose X is Hausdorff and $\{f_\alpha\}$ is a family of real-valued continuous functions (indexed by A) such that if U is a neighbourhood of $x_0 \in X$, there is an $\alpha \in A$ such that $f_\alpha(x_0) > 0$ and $f_\alpha(x) = 0$ if $x \notin U$. Then X is homeomorphic to a subspace of $\mathbb{R}^A$.

Proof: the proof is similar to that of the Urysohn metrization theorem, just replace $n \in \omega$ throughout by $\alpha \in A$. ■

Let's take a look at another embedding result.

Theorem 260
Let X be a completely regular space. Then X can be embedded in $\mathbb{R}^A$ for some A.

Proof: we first define an index set. Let $A = \{(C, x) \mid C \subseteq_C X, x \notin C\}$. For $x_0 \in X$, if U is a neighbourhood of x_0, then $C = X \setminus U \subseteq_C X$ and $x_0 \notin C$, so $\alpha = (C, x_0) \in A$.

Since X is completely regular, there is a continuous function $f_\alpha : X \to [0, 1]$ such that $f(x_0) = 1$ and $f(x) = 0$ for all $x \in C$, so for all $x \notin U$. Hence there is a family of continuous functions $\{f_\alpha\}_\alpha$ satisfying the hypotheses of the embedding theorem. As X is Hausdorff, we apply the embedding theorem to obtain the desired result. ■

The next result shows that $T_{3\frac{1}{2}}$ spaces behave in a nice fashion, not unlike their T_3 cousins.

Theorem 261
Subspaces and product of completely regular spaces are completely regular.

Proof: suppose Y is a subspace of the completely regular space X. If $y \in Y$ and $y \notin A \subseteq_C Y$, then $A = Y \cap \overline{A}$ (closure in X) and $y \notin \overline{A}$. Since X is completely regular, there is a continuous function $f : X \to [0, 1]$ such that $f(\overline{A}) = \{0\}$ and $f(y) = 1$. Then the restriction of f to Y is continuous and $f|_Y(A) = \{0\}$ and $f|_Y(y) = 1$, so that Y is completely regular.

Now suppose that X_α is completely regular for every α. Let $X = \prod_\alpha X_\alpha$. If $C \subseteq_C X$ and $x_0 = (x_\alpha)_\alpha \notin C$, then there is a basic neighbourhood $\prod_\alpha U_\alpha$ of x_0 disjoint from C. By definition, $U_\alpha = X_\alpha$, except when $\alpha = \alpha_i$, $1 \leq i \leq n$ for some n. Let $i \in \{1, \ldots, n\}$. Then $x_{\alpha_i} \in U_{\alpha_i} \subsetneq_O X_{\alpha_i}$, so $X_{\alpha_i} \setminus U_{\alpha_i} \subseteq_C X_{\alpha_i}$. By complete regularity of X_{α_i}, there is a continuous function $f_{\alpha_i} : X_{\alpha_i} \to [0, 1]$ such that $f_{\alpha_i}(x_{\alpha_i}) = 1$ and $f_{\alpha_i}(X_{\alpha_i} \setminus U_{\alpha_i}) = \{0\}$. This can be done for all $1 \leq i \leq n$. Now define a function $f : X \to [0, 1]$ by

$$f(x) = f_{\alpha_1}(\pi_{\alpha_1}(x)) \cdots f_{\alpha_n}(\pi_{\alpha_n}(x)).$$

Then, f is continuous, being the product of continuous functions. Furthermore, $f(x_0) = f_{\alpha_1}(\pi_{\alpha_1}(x_0)) \cdots f_{\alpha_n}(\pi_{\alpha_n}(x_0)) = 1$. Now suppose $y \notin \prod_\alpha U_\alpha$. Then, there exists α_i such that $\pi_{\alpha_i}(y) \notin U_{\alpha_i}$ and $f_{\alpha_i}(\pi_{\alpha_i}(y)) = 0$. Hence $f(y) = 0$, and X is completely regular. ■

18.4 Solved Problems

1. Show that if X is Lindelöf and Y is compact, then $X \times Y$ is Lindelöf.

Proof: the proof is nearly identical to that showing $X \times Y$ is compact whenever X and Y are compact. Let $\mathfrak{F}$ be an open covering for $X \times Y$. For each $x \in X$ we get a finite subcovering of $\{x\} \times Y$ from $\mathfrak{F}$, say $F(x)_1, \ldots F(x)_n$. Let N be the open set

$$N = \bigcup_{i=1}^{n} F(x)_i.$$

By the Tube Lemma, there is a neighbourhood W_x of x in X such that $W_x \times Y \subseteq N$. Repeating this procedure for all $x \in X$, we get that $\{W_x\}_{x \in X}$ is an open covering of X. But X is Lindelöf, so there is a countable subcovering

$$W_{x_1}, W_{x_2}, \ldots$$

For each of these W_{x_i}, there were n_i corresponding sets $F(x_i)_j$ in $\mathfrak{F}$. Define

$$\mathfrak{F}' = \{F(x_i)_j \mid i \in \mathbb{N}, 1 \leq j \leq n_i\}.$$

$\mathfrak{F}'$ is an open countable collection. For any $(x, y) \in X \times Y$, $x \in W_{x_i}$ for some i. Then $(x, y) \in W_{x_i} \times Y$ and $(x, y) \in F(x_i)_j$ for some j, so

$$X \times Y \subseteq \bigcup_{i \in \mathbb{N}} \bigcup_{j=1}^{n_i} F(x_i)_j.$$

So $\mathfrak{F}'$ is a countable subcovering of $X \times Y$ extracted from $\mathfrak{F}$, so $X \times Y$ is Lindelöf. ■

2. Let X be a space with the order topology. Show that X is regular.

Proof: let $A \subseteq_C X$, with $b \notin A$. As A is closed, there exists an open interval (c, d) such that

$$b \in (c, d) \subseteq X \setminus A.$$

There are now four possibilities.

a) If there exists $e, f \in X$ such that $c < e < b < f < d$, put $U = (e, f)$ and $V = (-\infty, e) \cup (f, +\infty)$.
b) If there is an f, but no such e, that is $(c, b) = \varnothing$, put $U = (c, f)$ and $V = (-\infty, b) \cup (f, +\infty)$.
c) If there is an e, but no such f, that is $(b, d) = \varnothing$, put $U = (e, d)$ and $V = (-\infty, e) \cup (b, +\infty)$.
d) If there are no such e, f, that is $(c, b) = (b, d) = \varnothing$, put $U = (c, d) = \{b\}$ and $V = (-\infty, b) \cup (b, +\infty)$.

In all cases, $b \in U \subseteq_O X$, $A \subseteq V \subseteq_O X$ and $U \cap V = \varnothing$, so X is regular. ■

3. a) If X is a Lindelöf space, show that every closed subset of X is Lindelöf.
   b) If A, B are disjoint closed subsets of a regular space, show that there are open coverings $\mathfrak{E}$, $\mathfrak{F}$ of A, B respectively such that $\overline{U} \cap B = \varnothing$ and $\overline{V} \cap A = \varnothing$ for all $U \in \mathfrak{E}, V \in \mathfrak{F}$.
   c) If X is a regular Lindelöf space, show that X is normal.

Proof:

a) Let A be a closed subset of X, and suppose that $\mathfrak{F}$ is an open covering of A. Then $X \setminus A$ is open and $\mathfrak{F} \cup \{X \setminus A\}$ is an open covering of X. But X is Lindelöf, so there is a countable sub-covering of X, say

$$\{X \setminus A, F_1, F_2, \ldots\},$$

where $F_n \in \mathfrak{F}$ for all $n \in \mathbb{N}$. Consequently,

$$\{F_1, F_2, \ldots\} \subseteq \mathfrak{F}$$

is a countable sub-covering of A, and A is Lindelöf.

b) Let $a \in A$. Since $X \setminus B$ is open, there exists an open set W_a such that $a \in W_a$ and $W_a \cap B = \varnothing$. By regularity of X, there exists an open set U_a such that

$$a \in U_a \subseteq \overline{U_a} \subseteq W_a.$$

Then $\overline{U_a} \cap B \subseteq W_a \cap B = \varnothing$. The collection $\{U_a\}_{a \in A}$ is an open covering of A satisfying the requisite property. Similarly, we can construct an open covering of B satisfying the property.

c) Let A and B be disjoint closed subsets of the regular Lindelöf space X. Then A and B are Lindelöf, by part (a), and there are open coverings $\mathfrak{E}$ and $\mathfrak{F}$ of A and B respectively such that

$$\overline{U} \cap B = \varnothing \quad \text{and} \quad \overline{V} \cap A = \varnothing$$

for all $U \in \mathfrak{E}, V \in \mathfrak{F}$. Since A and B are Lindelöf, it is possible to extract countable sub-coverings

$$\{U_1, U_2, \ldots\} \subseteq \mathfrak{E} \quad \text{and} \quad \{V_1, V_2, \ldots\} \subseteq \mathfrak{F}$$

of A and B respectively. Now define

$$U'_n = U_n \setminus \bigcup_{i=1}^{n} \overline{V_i} \quad \text{and} \quad V'_n = V_n \setminus \bigcup_{i=1}^{n} \overline{U_i}.$$

Then U'_n, V'_n are open in X as U_n, V_n are open in X and $\bigcup_{i=1}^{n} \overline{U_i}, \bigcup_{i=1}^{n} \overline{V_i}$ are closed in X. Let

$$U' = \bigcup_{n \in \mathbb{N}} U'_n \quad \text{and} \quad V' = \bigcup_{n \in \mathbb{N}} V'_n.$$

Then U', V' are open in X with $A \subseteq U'$ and $B \subseteq V'$. Indeed if $x \in A$, then $x \in U_n$ for some n. But, by construction, $x \notin \overline{V_i}$ for all $i \in \mathbb{N}$. Then $x \in U_n'$. Similarly, if $y \in B$, then $y \in V_n'$ for some $n \in \mathbb{N}$. It remains only to show that $U' \cap V' = \varnothing$. Suppose $\xi \in U' \cap V'$. Then $\xi \in U_n' \cap V_m'$ for some $m, n \in \mathbb{N}$. If $m \geq n$, then

$$\begin{aligned} \xi \in U_n' &\implies \xi \in U_n \\ \xi \in V_m' &\implies \xi \notin \overline{U_n}, \end{aligned}$$

a contradiction. If $m \leq n$, then

$$\begin{aligned} \xi \in U_n' &\implies \xi \notin \overline{V_m} \\ \xi \in V_m' &\implies \xi \in V_m, \end{aligned}$$

another contradiction. Then $U' \cap V' = \varnothing$ and X is normal. ■

4. Let X be a second countable regular space and let U be an open set.

a) Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued functions on a space X. If there exists $M \in \mathbb{R}$ such that $|f_n(x)| \leq M$ for all $x \in X$, $n \in \mathbb{N}$ show that

$$\sum_{n\in\mathbb{N}} \frac{1}{2^n} f_n$$

converges uniformly on X.

b) Show that U is a countable union of closed sets in X.

c) Show that there is a continuous function $f : X \to [0,1]$ such that $f(x) > 0$ for all $x \in U$ and $f(x) = 0$ for all $x \notin U$.

Proof:

a) Let $\varepsilon > 0$, and choose $N_\varepsilon \in \mathbb{N}$ such that

$$N_\varepsilon > \frac{\log M - \log \varepsilon}{\log 2}.$$

Then, for all $x \in X$ and $n > N_\varepsilon$,

$$\left| \sum_{i=n+1}^{\infty} \frac{1}{2^i} f_i(x) \right| \leq \sum_{i=n+1}^{\infty} \frac{1}{2^i} |f_i(x)| \leq M \left(\sum_{i=n+1}^{\infty} \frac{1}{2^i} \right) = \frac{M}{2^n} < \frac{M}{2^{N_\varepsilon}} < \varepsilon,$$

and so $\sum 2^{-n} f_n$ converges uniformly on X.

b) Suppose $\mathfrak{B} = \{B_n\}_{n\in\mathbb{N}}$ is a basis for X and U is open in X. Then $X \setminus U$ is closed in X. Since X is regular, if $x \in U$, there exist $B_x \in \mathfrak{B}$ and an open set V_x such that $x \in B_x$, $X \setminus U \subseteq V_x$ and $B_x \cap V_x = \varnothing$. But

$$\bigcup_{x \in U} B_x = U$$

since $B_x \cap (X \setminus U) \subseteq B_x \cap V_x = \varnothing$ for all $x \in U$. As $\mathfrak{B}$ is countable, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in U such that

$$U = \bigcup_{x\in U} B_x = \bigcup_{n\in\mathbb{N}} B_{x_n}.$$

By construction, $X \setminus U \subseteq V_{x_n}$, so $X \setminus V_{x_n} \subseteq U$ for all $n \in \mathbb{N}$, and

$$\bigcup_{n\in\mathbb{N}} (X \setminus V_{x_n}) \subseteq U.$$

Now, suppose $x \in U$. Then $x \in B_{x_n}$ for some $n \in \mathbb{N}$, so $x \notin V_{x_n}$ and $x \in X \setminus V_{x_n}$ for that n. Hence

$$\bigcup_{n\in\mathbb{N}} (X \setminus V_{x_n}) = U.$$

But $X \setminus V_{x_n}$ is closed in X for all $n \in \mathbb{N}$ so U is a countable union of closed sets.

c) By hypothesis, $U = \bigcup_{n\in\mathbb{N}} C_n$, where C_n is closed in X and $X \setminus U$ is closed. But X is normal, as it is regular and second countable, so, by the Urysohn lemma, there exists a family of continuous functions $\{f_n\}_{n\in\mathbb{N}}$, where $f_n : X \to [0,1]$, such that $f_n(X \setminus U) = \{0\}$ and $f_n(C_n) = \{1\}$. Define the function f on X by

$$f(x) = \sum_{n\in\mathbb{N}} \frac{1}{2^n} f_n(x).$$

Since each f_n is bounded by 1 above and 0 below, we can apply the result obtained in q. 2 to show that f is defined for all $x \in X$ and that f is continuous, since the series is uniformly convergent. Now we show that $f : X \to [0,1]$. Let $x \in X$. Then $f_n(x) \in [0,1]$ for all $n \in \mathbb{N}$, so

$$0 \leq \underbrace{\sum_{n\in\mathbb{N}} 2^{-n} f_n(x)}_{=f(x)} \leq \sum_{n\in\mathbb{N}} 2^{-n} = 1.$$

It remains to show that f satisfies the requisite property. Suppose $x \notin U$. Then $f(x) = \sum 2^{-n} f_n(x) = \sum 2^{-n} \cdot 0 = 0$. Now suppose $x \in U$. Then $x \in C_n$ for some n, and $f(x) \geq 2^{-n} > 0$. ■

5. For disjoint closed sets A, B in a completely regular space, if A is compact show that there is a continuous function $f : X \to [0,1]$ with $f(A) = \{0\}$ and $f(B) = \{1\}$.

Proof: let $a \in A$. Then, by the previous question, there exists a continuous function $f_a : X \to [0,1]$ such that $f_a(B) = \{1\}$ and $f_a(U_a) = \{0\}$ for some neighbourhood U_a of a, disjoint from B. The collection $\mathfrak{F} = \{U_a\}_{a\in A}$ is then an open covering of A, disjoint from B. But A is compact, so there is a finite sub-covering

$$\{U_{a_1}, \ldots, U_{a_n}\} \subseteq \mathfrak{F}$$

of A. Pick the associated functions f_{a_i}, $1 \leq i \leq n$, and construct the function $f : X \to [0,1]$ defined by

$$f(x) = f_{a_1}(x) f_{a_2}(x) \cdots f_{a_n}(x).$$

Then f is continuous, since the finite product of continuous functions is continuous. Furthermore, $f(A) = \{0\}$ and $f(B) = \{1\}$. Indeed, suppose $x \in A$. Then $x \in U_{a_i}$ for some i and $f_{a_i}(x) = 0$, so $f(x) = 0$. If $x \in B$, $f_{a_i}(x) = 1$ for all $1 \leq i \leq n$, so $f(x) = 1$. ■

6. a) Show that a connected normal space X having more than one point is uncountable.

   b) Show that a connected regular space X having more than one point is uncountable.

   Proof:

   a) By hypothesis, there exists $x \neq y \in X$. Since X is normal, singletons are closed in X and X is completely regular. Then there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f(y) = 1$. But X is connected. By the intermediate value theorem, for every $0 = f(x) \leq r \leq f(y) = 1$, there exists $z_r \in X$ such that $f(z_r) = r$. Then f is a surjection of X onto $[0,1]$. Hence X is uncountable.

   b) Suppose X was a countable connected regular space with at least two points. Then X is clearly Lindelöf, so it is normal by a previous solved problem. But this would make X uncountable by this problem's first part, which is a contradiction. Hence X has to be uncountable. ■

7. Show that every locally compact Hausdorff space is completely regular.

   Proof: as X is a locally compact Hausdorff space, it has a one-point compactification $Y = X \cup \{\infty\}$, where $Y = \overline{X}$ and X is a subspace of Y. But Y is compact Hausdorff, so X is homeomorphic to a subspace of a compact Hausdorff space, hence X is completely regular. ■

18.5 Exercises

1. Prepare a 2-page summary of this chapter, with important definitions and results.

2. If A is a subspace of a first countable space X, show that $x \in \overline{A}$ if and only if there is a sequence of points in A converging to x.

3. If X is a first countable space, show that $f : X \to Y$ is continuous if and only if for any convergent sequence $x_n \to x$, the sequence $f(x_n)$ converges to $f(x)$.

4. If X is second countable, show that every collection of disjoint open sets in X is countable.

5. If Y is compact and X is Lindelöf, show that $X \times Y$ is Lindelöf.

6. Let X be a regular, second countable space. Show that every open set U in X is a countable union of closed sets.
7. Use the fact that X is completely regular to show that there is a continuous function $f : X \to [0, 1]$ such that $f(x) > 0$ for all $x \in U$ and $f(x) = 0$ for $x \notin U$.
8. Show that subspaces and products of completely regular spaces are completely regular.
9. If X_n is first countable for all $n \in \mathbb{N}$, show that $\prod_{n \in \mathbb{N}} X_n$ is first countable.
10. Provide proofs for the examples of p. 431.
11. Complete the proof of Lemma 250.
12. Illustrate the separation axioms as in Chapter 15 (see p. 321, and footnote for a list).