Chapter 19

Advanced Topics

In Chapter 17, we showed that the **finite** product of compact spaces is compact in the **box**, **uniform**, and **product** topologies. **Arbitrary** products of compact spaces, on the other hand, are surprisingly more complicated to handle.

19.1 Tychonoff's Theorem

Our formulation of **compactness** in terms of closed sets uses the **finite intersection property** (f.i.p.).[1] In this section, we will use the following notation:

- a is an element of X;
- A is a subset of X;
- $\mathfrak{A}$ is a collection of subsets of X;
- $\mathbb{A}$ is a family of collections of subsets of X;

as well as a slightly altered re-formulation of that statement (see Theorem 234 (Reprise) in Chapter 17):

Theorem 234 (Reprise, Reprise)
*X is **compact** if and only if for every family $\mathfrak{F}$ of subsets of X satisfying the f.i.p., we have*

$$\bigcap_{F\in\mathfrak{F}} \overline{F} \neq \varnothing.$$

Proof: left as an exercise. ■

[1] We note that the projection mappings are not closed in general.

Our goal is to show that arbitrary products of compact spaces are compact; the following lemmas will bring us to the promise land.

Lemma 262
For any set X and any collection $\mathfrak{F}$ of subsets of X satisfying the f.i.p., there exists a maximal collection $\mathfrak{G}$ with respect to the f.i.p., that is, $\mathfrak{F} \subseteq \mathfrak{G}$, and $\mathfrak{G} \subsetneq \mathfrak{G}' \implies \mathfrak{G}'$ does not satisfy the f.i.p.

Proof: consider all possible collections of subsets of X satisfying the f.i.p., and define a partial order on them by strict inclusion. Then $\{\mathfrak{F}\}$ is a totally ordered family, so by the **maximum principle** of set theory, there is a maximal totally ordered family $\mathbb{A}$ containing it. Define

$$\mathfrak{G} = \bigcup_{\mathfrak{F}' \in A} \mathfrak{F}'.$$

Then $\mathfrak{G}$ satisfies the f.i.p. Indeed, if $G_1, \ldots, G_n \in \mathfrak{G}$, then, for each i, there exists $\mathfrak{F}_i \in \mathbb{A}$ such that $G_i \in \mathfrak{F}_i$. But $\mathbb{A}$ is totally ordered, so one of the $\mathfrak{F}_i$, say $\mathfrak{F}_k$, contains all the others. Then $G_1, \ldots, G_n \in \mathfrak{F}_k$. But $\mathfrak{F}_k$ satisfies the f.i.p., so

$$\bigcap_{i=1}^{n} G_i \neq \varnothing.$$

As $\mathfrak{F} \in \mathbb{A}$, we have $\mathfrak{F} \subseteq \mathfrak{G}$.

Now, suppose $\mathfrak{G} \subseteq \mathfrak{G}'$, where $\mathfrak{G}'$ also satisfies the f.i.p. Then $\mathfrak{F} \subseteq G'$. Furthermore, if $\mathfrak{F}' \in \mathbb{A}$, then $\mathfrak{F}' \subseteq \mathfrak{G}'$. So $\mathfrak{G}'$ is comparable with every collection in $\mathbb{A}$. Thus $\mathbb{A} \cup \{\mathfrak{G}'\}$ is totally ordered and each of its constituent collection satisfies the f.i.p. But $\mathbb{A}$ was maximal with respect to the f.i.p., so $\mathfrak{G}' \in \mathbb{A}$, hence $\mathfrak{G}' \subseteq \mathfrak{G}$ and $\mathfrak{G}' \subseteq \mathfrak{G}$. ■

The **Haussdorf maximum principle** states that in any partially ordered set, every totally ordered subset is contained in a maximal totally ordered subset. This benign looking statement is in fact equivalent to the infamous **axiom of choice**; as it is a a fundamental part of the proof of Lemma 262, it is also a fundamental constituent of its descendents.

Lemma 263
If $\mathfrak{F}$ is maximal with respect to the f.i.p. and $F_1, \ldots, F_n \in \mathfrak{F}$, then $\bigcap_{i=1}^{n} F_i \in \mathfrak{F}$.

Proof: let $G = \bigcap_{i=1}^{n} F_i$ and $\mathfrak{G} = \mathfrak{F} \cup \{G\}$. Suppose $G_1, \ldots, G_m \in \mathfrak{G}$ are all distinct.

1. If G is not one of the G_j's, then $G_j \in \mathfrak{F}$ for $1 \leq j \leq m$. Then
$$\bigcap_{j=1}^{m} G_j \neq \varnothing,$$
since $\mathfrak{F}$ satisfies the f.i.p.

2. If $G = G_m$, then then $G_j \in \mathfrak{F}$ for $1 \leq j \leq m-1$. Then
$$\bigcap_{j=1}^{m} G_j = \left(\bigcap_{j=1}^{m-1} G_j\right) \cap \left(\bigcap_{i=1}^{n} F_i\right) \neq \varnothing,$$
since $\mathfrak{F}$ satisfies the f.i.p.

Thus $\mathfrak{G}$ satisfies the f.i.p. By maximality of $\mathfrak{F}$, $\mathfrak{G} = \mathfrak{F}$, hence $G \in \mathfrak{F}$. ■

We will need one more lemma.

Lemma 264
If $\mathfrak{F}$ is a maximal collection with respect to the f.i.p. and $A \subseteq X$ is such that $A \cap F \neq \varnothing$ for any $F \in \mathfrak{F}$, then $A \in \mathfrak{F}$.

Proof: let $\mathfrak{G} = \{A\} \cup \mathfrak{F}$. We show $\mathfrak{G}$ satisfies the f.i.p. Let $G_1, \ldots, G_n \in \mathfrak{G}$.

1. If $G_i \neq A$ for $1 \leq i \leq n$, then $\bigcap_{i=1}^{n} G_i \neq \varnothing$, since $\mathfrak{F}$ satisfies the f.i.p.
2. If $G_n = A$, let $F = \bigcap_{i=1}^{n-1} G_i$, where $G_i \in F$ for $1 \leq i \leq n-1$. By Lemma 263, $F \in \mathfrak{F}$. But by hypothesis,
$$\bigcap_{i=1}^{n} G_i = A \cap F \neq \varnothing.$$

Hence $\mathfrak{G} = \mathfrak{F}$ and $A \in \mathfrak{F}$. ■

We are now ready to state and prove this section's main result.

Theorem 265 (Tychonoff Theorem)
Let $\{X_\alpha\}_\alpha$ be a family of compact sets. Then $\prod_\alpha X_\alpha$ is compact.

Proof: we show that any collection of subsets of $X = \prod_\alpha X_\alpha$ satisfying the f.i.p. has a non-trivial intersection. If $\mathfrak{A}$ is such a collection, then let $\mathfrak{F}$ be the corresponding maximal collection with respect to the f.i.p., given by Lemma 262.

Then

$$\bigcap_{F\in\mathfrak{F}} \overline{F} \subseteq \bigcap_{F\in\mathfrak{A}} \overline{F},$$

and it will be sufficient to show

$$\bigcap_{F\in\mathfrak{F}} \overline{F} \neq \varnothing.$$

For each α, let $\mathfrak{F}_\alpha = \{\pi_\alpha(F)\}_{F\in\mathfrak{F}}$. Then, since $\mathfrak{F}$ satisfies the f.i.p.,

$$\pi_\alpha \underbrace{\left(\bigcap_{i=1}^{n} F_i\right)}_{\neq\varnothing} \subseteq \bigcup_{i=1}^{n} \pi_\alpha(F_i)$$

for any $F_1, \ldots, F_n \in \mathfrak{F}$. Hence $\mathfrak{F}_\alpha$ satisfies the f.i.p. But X_α is compact, so

$$P(\alpha) = \bigcap_{F\in\mathfrak{F}} \overline{\pi_\alpha(F)} \neq \varnothing.$$

Let $x_\alpha \in P(\alpha) \subseteq X_\alpha$ and set $x = (x_\alpha)_\alpha$. Then $x \in X$. If U_β is a neighbourhood of x_β in X_β, then $\pi_\beta^{-1}(U_\beta)$ is a sub-basic open set in X, and $U_\beta \cap \pi_\beta(F) \neq \varnothing$ for every $F \in \mathfrak{F}$, since $x_\beta \in \overline{\pi_\beta(F)}$ for all $F \in \mathfrak{F}$.

Consequently, $\pi_\beta^{-1}(U_\beta) \cap F \neq \varnothing$ for all $F \in \mathfrak{F}$. Then, by Lemma 264, $\pi_\beta^{-1}(U_\beta) \in \mathfrak{F}$. If V is a neighbourhood of x in X, then V contains a basic neighbourhood $U = \prod_\alpha U_\alpha$ around x, where $U_\alpha = X_\alpha$ for all but finitely many α.

But

$$U = \bigcap_{i=1}^{n} \pi_{\beta_i}^{-1}(U_{\beta_i}).$$

By Lemma 19.1, $U \in \mathfrak{F}$. Then $U \cap F \neq \varnothing$ for all $F \in \mathfrak{F}$ since $\mathfrak{F}$ satisfies the f.i.p., so $V \cap F \neq \varnothing$ for all $F \in \mathfrak{F}$. But V was arbitrary, so $x \in \overline{F}$ for all $F \in \mathfrak{F}$ and $x \in \bigcap_{F\in\mathfrak{F}} \overline{F}$. Hence X is compact. ■

Note that as $[0,1]$ is compact, $[0,1]^A$ is compact in the product topology. As a result, any completely regular space can be embedded in $[0,1]^A$ for some index set A, according to the embedding theorem (Theorem 259). Hence, any completely regular space is homeomorphic to a subspace of a compact Hausdorff space, which is to say, a **normal space**. This opens the door for us to continue the discussion on **compactification**.

19.2 Stone-Čech Compactification

A **compactification** of a space X is a compact Hausdorff space Y which contains X as a subspace and such that $\overline{X} = Y$. For X to have a compactification, it must be **completely regular**.

As Y is compact Hausdorff, it is necessarily normal, and so completely regular, and its subspaces are also completely regular. We now show that this condition is sufficient.

Theorem 266
If X is completetly regular, then X has a compactification Y.

Proof: since X is completely regular, it is possible to embed X into a space $Z = [0,1]^A$. If $f : X \to Z$ is the embedding, let $X_0 = f(X)$ and take $Y = \overline{X}$. Then Y_0 is compact, since it is closed in the compact space Z. Let X_1 be a set disjoint from X, in one-to-one correspondence with $Y_0 \setminus X_0$. Then, put $Y = X \cup X_1$. If $g : X_1 \to Y_0 \setminus X_0$ is the bijection, then define $h : Y \to Y_0$ by

$$h(x) = \begin{cases} f(x) & \text{if } x \in X, \\ g(x) & \text{if } x \in X_1. \end{cases}$$

Then h is a bijection. Topologize Y by setting

$$V \subseteq_O Y \iff h(V) \subseteq_O Y_0.$$

This clearly makes $h : Y \to Y_0$ a homeomorphism, and so Y is compact, Hausdorff. But the restriction of h on X is a homeomorphism of X onto X_0, so X is a subspace of Y and $\overline{X_0} = Y_0$ implies $\overline{X} = Y$. ■

The compactification clearly depends on the embedding $f : X \to Z$.

Examples: let $X = (0,1)$ in the usual topology and $f : X \to Z$.

1. If $Z = [0,1]^2$ in the usual topology and $f(x) = e^{2\pi i x}$, then the resulting compactification is the one-point compactification.

2. If $Z = [0,1]$ in the usual topology and $f(x) = x$, then the resulting compactification is a two-point compactification.

3. If $Z = [0,1]^2$ in the usual topology and $f(x) = (x, \sin(1/x))$, then the resulting compactification is given by adding the sets $\{0\} \times [-1,1]$ and $\{(1, \sin 1)\}$ to the topologist's sine curve.

Now, suppose X is a completely regular space. Let $\{f_\alpha\}_{\alpha\in A}$ be the set of all continuous bounded real-valued functions on X. For each $\alpha \in A$, let

$$I_\alpha = \left[\inf_{x\in X}\{f_\alpha(x)\}, \sup_{x\in X}\{f_\alpha(x)\}\right].$$

Then I_α is a closed bounded interval in $\mathbb{R}$, so I_α is compact and $\prod_\alpha I_\alpha$ is compact by Tychonoff's theorem. Define $\hat{F} : X \to \prod_\alpha I_\alpha$ by

$$\hat{F}(x) = (f_\alpha(x))_\alpha,$$

and so $\hat{F}$ is continuous as $f_\alpha(x)$ is continuous for all α. Since X is completely regular, the set $\{f_\alpha\}_{\alpha\in A}$ satisfies the conditions of the embedding theorem.

Consequently, X is homeomorphic to a subspace of $Z = \prod_\alpha I_\alpha$, and we obtain a compactification of X that is homeomorphic to the closure of $\hat{F}(X)$ in Z. This compactification is called the **Stone-Čech compactification** of X, and is denoted $\beta(X)$.[2]

If Y and Z are compactifications of X for which there exists an homeomorphism $f : Y \to Z$, we say that Y and Z are **equivalent** if $f(x) = x$ for all $x \in X$.

> **Theorem 267**
> *If X is completely regular, then every continuous bounded real-valued function on X can be uniquely extended to a continuous function on $\beta(X)$.*
>
> **Proof:** let f_γ be a continuous bounded real-valued function on X. Then
>
> $$f_\gamma = \pi_\gamma \circ F|_X,$$
>
> where $F : \beta(X) \to \prod I_\alpha$ is the embedding given in footnote 2. Define g on $\beta(X)$ by
>
> $$g(x) = \pi_\gamma F(x).$$
>
> Then $g|_X = f_\gamma$; according to a previous solved problem, the extension is unique as $\beta(X) = \overline{X}$. ■

This leads to the following useful result.

> **Theorem 268**
> *Suppose that $g : X \to Z$ is continuous, where Z is compact Hausdorff. Suppose Y is a compactification of X such that every continuous real-valued function on X can be extended to Y. Then g can be extended to Y.*

[2] Note that we have just uniquely extended the continuous function $\hat{F}$ on X to a continuous function F on $\beta(X) = \overline{X}$ using one of the solved problems from a previous section.

Proof: since Z is a compact Hausdorff it is normal, and so completely regular. Then Z can be embedded into $[0,1]^A$ for some A. Without loss of generality, we may take Z as a subspace of $[0,1]^A$. Note that Z is closed in $[0,1]^A$, since it is a compact subset of $[0,1]^A$. Then $g : X \to [0,1]^A$ is continuous and $g_\alpha = \pi_\alpha \circ g : X \to [0,1]$ is continuous for all $\alpha \in A$. By hypothesis, g_α can be extended to a continuous function $f_\alpha : Y \to \mathbb{R}$. Define $f : Y \to \mathbb{R}^A$ by

$$f(y) = (f_\alpha(y))_{\alpha \in A}.$$

As each coordinate function is continuous, f is continuous. Furthermore, $f|_X = g$. It remains only to show that f maps Y into Z. But

$$f(Y) = f(\overline{X}) \subseteq \overline{f(X)} = \overline{g(X)}.$$

But $g(X) \subseteq Z$ and Z is closed, so $\overline{g(X)} \subseteq Z$. Consequently, $f(Y) \subseteq Z$. Thus $f : Y \to [0,1]^A$ is the desired extension. ■

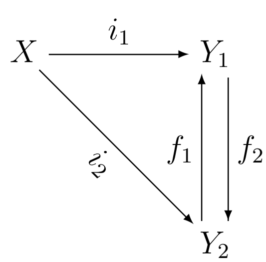
In a certain sense, the Stone-Čech compactification is unique.

Theorem 269
Suppose Y_1 and Y_2 are compactifications of X satisfying the conditions of Theorem 268. If every continuous function $g : X \to Z$ can be extended, Y_1 and Y_2 are equivalent.

Proof: let $i_1 : X \to Y_1$ be the injection of X into the compact normal space Y_1. Then, i_1 can be extended to $f_1 : Y_2 \to Y_1$. Similarly, we can extend $i_2 : X \to Y_2$ to $f_2 : Y_1 \to Y_2$. Then $f_1 f_2 : Y_1 \to Y_1$, and

$$f_1 f_2(x) = f_1 i_2(x) = f_1(x) = i_1(x) = x$$

for $x \in X$. Hence $f_1 f_2$ extends id $: X \to Y_1$ to $Y_1 = \overline{X}$. Since id_{Y_1} is also such a continuous extension, $f_1 f_2 = \text{id}_{Y_1}$ and, similarly, $f_2 f_1 = \text{id}_{Y_2}$. Hence f_1 and f_2 are homeomorphisms and Y_1 and Y_2 are equivalent.



■

19.3 Solved Problems

1. Let $\{X_\alpha\}$ be a family of non-empty topological spaces. Prove that the product space is locally compact if and only if each X_α is locally compact and all but a finite number of the X_α are compact.

   Proof: let $X = \prod X_\alpha$ and assume the axiom of choice holds. Suppose $x = (x_\alpha)_\alpha \in X$. If X is locally compact, then it is locally compact at x and there exist a compact set C and a basic neighbourhood U such that $x \in U \subseteq C \subseteq X$. But U takes the form

   $$U = U_{\alpha_1} \times \cdots \times U_{\alpha_n} \times \prod_{\alpha \neq \alpha_1, \ldots, \alpha_n} X_\alpha,$$

   where U_{α_i} is open in X_{α_i} for all $1 \leq i \leq n$. Since $U \subseteq C$, then

   $$C = C_{\alpha_1} \times \cdots \times C_{\alpha_n} \times \prod_{\alpha \neq \alpha_1, \ldots, \alpha_n} X_\alpha,$$

   where $U_{\alpha_i} \subseteq C_{\alpha_i}$. But C is compact, so X_α is compact for all $\alpha \neq \alpha_i$, and so is C_{α_i}, for $1 \leq i \leq n$. Now, consider X_{α_i} for $1 \leq i \leq n$. By construction, C_{α_i} is compact, U_{α_i} is open and

   $$x_{\alpha_i} \in U_{\alpha_i} \subseteq C_{\alpha_i} \subseteq X_{\alpha_i}$$

   for $1 \leq i \leq n$. But this means that X_{α_i} is locally compact at x_{α_i}, so X_{α_i} is locally compact for $1 \leq i \leq n$.

   Conversely, suppose X_{α_i} is locally compact for $1 \leq i \leq n$ and X_α is compact for $\alpha \neq \alpha_i$, $1 \leq i \leq n$. Write

   $$W = \prod_{\alpha \neq \alpha_1, \ldots, \alpha_n} X_\alpha.$$

   By Tychonoff's theorem, W is compact, and so locally compact. Then

   $$X = X_{\alpha_1} \times \cdots \times X_{\alpha_n} \times W$$

   is a finite product of locally compact spaces, and so is locally compact. ■

2. Show that if X is completely regular and B is a closed set with $a \notin B$, then there is a continuous function $f : X \to [0,1]$ such that $f(x) = 1$ for all $x \in B$ and $f(x) = 0$ in some neighbourhood of a.

   Proof: since X is completely regular, it is homeomorphic to a subspace of a normal space Y (we identify X with its homeomorphic copy in Y). Since B is closed in X, there exists B_Y closed in Y such that $B = B_Y \cap X$. As $a \in X$, $a \in Y \setminus B_Y$. By normality of Y, there is an open set U_Y in Y such that

   $$a \in U_Y \subseteq \overline{U_Y} \subseteq Y \setminus B_Y.$$

Then $\overline{U_Y} \cap B_Y = \varnothing$, and we can apply the Urysohn lemma to find a continuous function $f : Y \to [0,1]$ such that $f(\overline{U_Y}) = \{0\}$ and $f(B_Y) = \{1\}$. The restriction of a continuous function to a subspace is continuous, so the restriction

$$f|_X : X \to [0,1]$$

is continuous. Put $U = U_Y \cap X$ and $\overline{U} = \overline{U_Y} \cap X$, so that U is open and $\overline{U}$ is closed in X and $U \subseteq \overline{U}$. Then

$$f|_X(B) = f(B_Y \cap X) = \{1\} \quad \text{and} \quad f|_X(\overline{U}) = f(\overline{U_Y} \cap X) = \{0\},$$

so that $f|_X(U) = \{0\}$. But by construction, $a \in U$, so $f|_X$ is the desired function. ■

3. Let X be completely regular. Show that X is connected if and only if $\beta(X)$ is connected.

Proof: if X is connected, $\beta(X) \simeq \overline{X}$ is connected. Now suppose X is not connected, and let A, B be a separation of X. Note that

$$\beta(X) \simeq \overline{X} = \overline{A \cup B} = \overline{A} \cup \overline{B}.$$

Hence $\beta(X)$ is disconnected if $\overline{A}$, $\overline{B}$ is a separation of $\beta(X)$. It will be sufficient to show that $\overline{A} \cap \overline{B} = \varnothing$. Define $f : X \to [0,1]$ by $f(A) = \{0\}$ and $f(B) = \{1\}$. Then, f is continuous. Indeed,

$$\begin{aligned} f^{-1}([0,1]) &= X \\ f^{-1}((a,b)) &= \varnothing \text{ for } 0 \leq a < b \leq 1 \\ f^{-1}([0,b)) &= A \text{ for } 0 < b \leq 1 \\ f^{-1}((a,1]) &= B \text{ for } 0 \leq a < 1, \end{aligned}$$

and X, $\varnothing$, A and B are all open in X. Then f can be extended to a continuous function $\widehat{f} : \beta(X) \to Y$ where $\widehat{f}|_X = f$. As $\widehat{f}$ is continuous,

$$\{0\} \subseteq \widehat{f}(A) \subseteq \widehat{f}(\overline{A}) \subseteq \overline{\widehat{f}(A)} = \overline{f(A)} = \overline{\{0\}} = \{0\}$$

and

$$\{1\} \subseteq \widehat{f}(B) \subseteq \widehat{f}(\overline{B}) \subseteq \overline{\widehat{f}(B)} = \overline{f(B)} = \overline{\{1\}} = \{1\}.$$

Then $\widehat{f}(\overline{A}) = \{0\}$ and $\widehat{f}(\overline{B}) = \{1\}$. Hence $\overline{A} \cap \overline{B} = \varnothing$, since otherwise there would be a $x \in \beta(X)$ such that $\widehat{f}(x) = 0$ and $\widehat{f}(x) = 1$, a contradiction as $\widehat{f}$ is a function. ■

4. Let Y be an arbitrary compactification of X. Show there is a continuous surjective closed map $g : \beta(X) \to Y$ such that $g|_X = \text{id}_X$.

Proof: if Y is a compactification of X, there is an embedding $f : X \to Y$ with $\overline{f(X)} = Y$. Hence, by the properties of the Stone-Čech compactification, and since Y is compact Hausdorff, f can be extended continuously to $g : \beta(X) \to Y$, where $g|_X = f$. As $\beta(X)$ is compact and Y is Hausdorff, the map g is closed. Indeed, let C be a closed subset of $\beta(X)$. As $\beta(X)$ is compact, C is compact, so $g(C)$ is compact in Y. But Y is Hausdorff, so $g(C)$ is closed.

It remains only to show that g is surjective. To do this, we show that $Y \subseteq g(\beta(X))$. As g is an extension of f on X, $f(X) \subseteq g(\beta(X))$. But g is closed, so $g(\beta(X))$ is closed in Y. Thus $Y = \overline{f(X)} \subseteq g(\beta(X))$ and g is surjective. ■

19.4 Exercises

1. Prepare a 2-page summary of this chapter, with important definitions and results.
2. Provide a proof for Theorem 234 (Reprise, Reprise).
3. If $X \neq \beta(X)$, show that $\beta(X)$ is not metrizable.
4. Let X be a discrete.
   a) If $A \subseteq X$, show that $\overline{A}, \overline{X \setminus A} \subseteq_C \beta(X)$ are disjoint.
   b) If $U \subseteq_O \beta(X)$, show that $\overline{U} \subseteq_O \beta(X)$.
   c) Is $\beta(X)$ totally disconnected?