Chapter 2

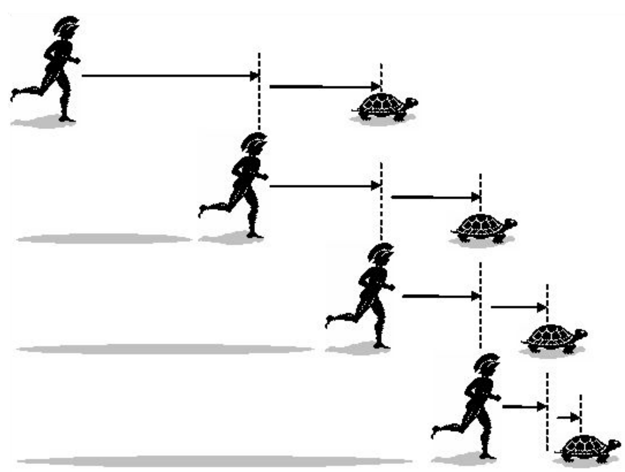
Sequences of Real Numbers

A large chunk of analysis concerns itself with problems of **convergence**. In this chapter, we introduce **sequences** and **limits**, provide results that help to **compute** such limits (when they exist), and identify situations when the limit can be shown to exist without first having to compute it.

2.1 Infinity vs. Intuition

When dealing with infinity, our intuition sometimes falters, as we shall see presently.

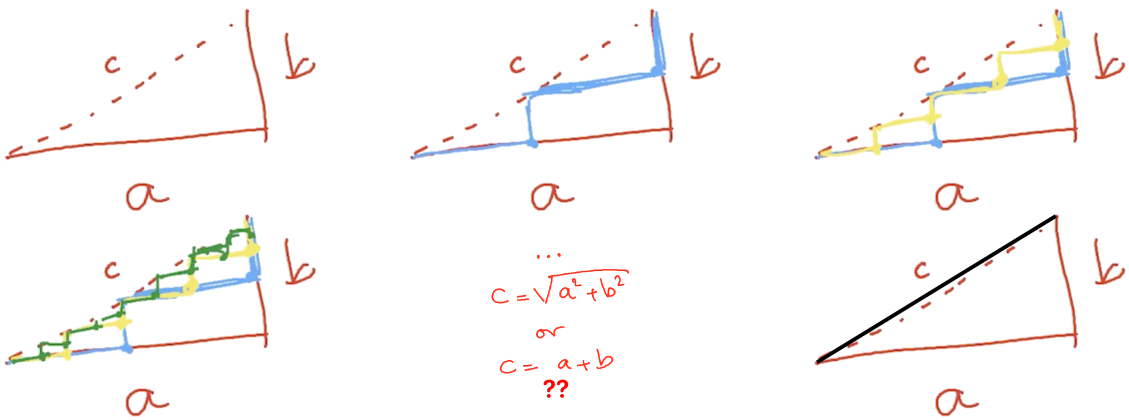
Example (ZENO'S PARADOX)
Achilles pursues a turtle. When he reaches her starting point, she has moved a certain distance. When he crosses that distance, she has moved yet another distance, and so forth. Achilles is always trailing the turtle, so he cannot catch her.



What would happen in reality? □

The next example puts one of the great classical results of planar geometry in doubt.

Example (ANTI-PYTHAGOREAN THEOREM)
Consider a right-angle triangle with base a, height b, and hypotenuse c. We can build staircase structures that each have the same constant length as $a+b$, while increasing the number of stairs (see image below).



This seems to tell us that $c = a + b$. But we know that $c = \sqrt{a^2+b^2}$ according to Pythagoras' Theorem. Thus, we would expect to have $(a+b)^2 = a^2+b^2$ for all right-angle triangles, which is to say, that $2ab = 0$, or, equivalently, that each right-angle triangle has at least one side with length 0. But we know this cannot be true, as the $(3,4,5)$ right-angle triangle demonstrates. What is going on? □

Finally, we present two baffling "results" about infinite sums.

Examples (INFINITE SUMS)

1. Let $S = 1 + (-1) + 1 + (-1) + \cdots$. Then
   $S = (1+(-1)) + (1+(-1)) + \cdots = 0 + 0 + \cdots = 0$
   $S = 1 - (1 + (-1) + 1 + (-1) + \cdots) = 1 + S \implies S = 1/2$
   $S = 1 + ((-1)+1) + ((-1)+1) + \cdots = 1 + 0 + 0 + \cdots = 1$
   Therefore $0 = \frac{1}{2} = 1$. Does this make sense?

2. Let $S = 1 + 2 + 4 + 8 + \cdots$. Then
   $$S = 1 + 2(1 + 2 + 4 = 8 + \cdots) = 1 + 2S \implies S = -1.$$
   Can a sum of positive terms yield a negative result? □

2.2 Limit of a Sequence

In each of the examples provided in Section 2.1, the problem arises with a "..." (implicit in Zeno's paradox, explicit in the others): seen individually, each of the steps makes sense. But when we stitch them all together – letting the number of steps increase without bounds – all hell breaks loose.

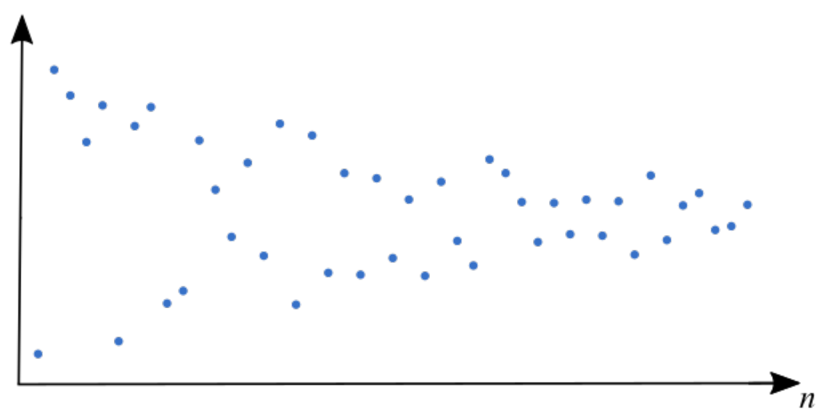
There are instances where letting $n \to \infty$ leads to **convergent behaviour**, others (as in the preceding examples), where it doesn't.[1] We start by formalizing these notions.

A **sequence** of real numbers is a function $X : \mathbb{N} \to \mathbb{R}$ defined by $X(n) = a_n$, where $a_n \in \mathbb{R}$. We denote the sequence X by $(a_n)_{n\in\mathbb{N}}$ or simply by (a_n).

Examples

1. $X : \mathbb{N} \to \mathbb{R}$, $n \mapsto 2n$ is the sequence with $X(1) = 2$, $X(2) = 4$, etc.; we may also write $X = (x_n) = (2, 4, 6, \ldots)$.[2]
2. $X : \mathbb{N} \to \mathbb{R}$, $n \mapsto \frac{1}{n}$ is the sequence with $X(1) = \frac{1}{2}$, $X(2) = \frac{1}{2}$, etc.; we may also write $X = (x_n) = (1, 1/2, 1/3, \ldots)$. □

In general, we let $\mathbb{N}$ stand for whatever countable subset of $\mathbb{N}$ is required for the definition of the sequence to make sense. Graphically, we can display sequences as a "scatterplot", with the horizontal coordinate being the index n and the vertical axis the value $X(n) = x_n$ of the sequence at n. An example is provided below.



We can also see a sequence as an **ordered set** of **terms** a_n, that is, a set of **indexed values**. The set of all values taken by the sequence (a_n) is called the **range** of (a_n) and we denote it by $\{a_n\}$. Sequences and their ranges are different objects.

[1] It isn't much of a stretch to state that mathematical analysis is about coming to terms with infinity – thankfully, this endeavour has proven to have extremely rich consequences, as we shall see throughout these notes.

Examples

1. The terms of the sequence $(\frac{1}{n^2})$ are $(1, \frac{1}{4}, \frac{1}{9}, \ldots)$, while its range is $\{1, \frac{1}{4}, \frac{1}{9}, \ldots\}$.
2. The terms of the sequence $(\frac{1+(-1)^n}{n})$ are $(0, 1, 0, \frac{1}{2}, 0, \frac{1}{3}, \ldots)$, while its range is $\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$. □

Certain sequences are defined with the help of a **recurrence relation**: the first few terms are given, and the subsequent terms are computed using the preceding terms and the relation.

Example (FIBONACCI SEQUENCE)
The classic sequence $(1, 1, 2, 3, 5, 8, 13, \ldots)$ is a recurrence relation, defined by by $x_1 = 1$, $x_2 = 1$, and $x_n = x_{n-1} + x_{n-2}$ for $n \geq 3$. □

We will now examine in detail a specific sequence,

$$(x_n) = \left(\frac{1}{2n}\right) = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots\right).$$

As n increases, the values of x_n seem to **approach** 0. What does this mean, mathematically? Let $\varepsilon > 0$.[3] Then the real number $\frac{1}{2\varepsilon}$ is positive, *i.e.*,

$$\frac{1}{2\varepsilon} > 0.$$

According to the Archimedean property, there exists a **threshold** $N_\varepsilon \in \mathbb{N}$ such that

$$N_\varepsilon > \frac{1}{2\varepsilon}.$$

Different values of ε lead to different thresholds: for instance, if $\varepsilon = \frac{1}{100}$, then any

$$N_\varepsilon > \frac{1}{2(1/100)} = 50$$

would work; if $\varepsilon = \frac{1}{1000}$, then any $N_\varepsilon > 500$ would work, and so on.

No matter what value $\varepsilon > 0$ takes, however, if we look at indices past the threshold (i.e. when $n > N_\varepsilon$), we have

$$n > N_\varepsilon > \frac{1}{2\varepsilon} \Longrightarrow n > \frac{1}{2\varepsilon} \Longleftrightarrow \varepsilon > \frac{1}{2n}.$$

For all indices n **after** the threshold N_ε (i.e. $\forall n > N_\varepsilon$), we have:

$$|x_n - 0| = |x_n| = \left|\frac{1}{2n}\right| = \frac{1}{2n} < \varepsilon \Longrightarrow 0 - \varepsilon < x_n < 0 + \varepsilon.$$

[3]In theory, ε could take on any positive value, but in practice we are interested in small values $\varepsilon \ll 1$.

The interval $(-\varepsilon, \varepsilon)$ thus contains **all** the terms of the sequence x_n **after** the N_εth term, which is to say $x_n \in (-\varepsilon, \varepsilon)$ for all $n > N_\varepsilon$.

Another way of saying this is that the interval $(-\varepsilon, \varepsilon)$ contains all the terms of the sequence (x_n), except maybe for a finite number of terms included in $x_1, \ldots, x_{N_\varepsilon}$.

If $\varepsilon = 1/100$, for instance, $\exists N_{1/100} > \frac{1}{2(1/100)} = 50$ ($N_{1/100} = 51$ works) such that

$$n > 51 \implies |x_n - 0| = |x_n| = \left|\frac{1}{2n}\right| = \frac{1}{2n} < \frac{1}{2(51)} = \frac{1}{102} < \frac{1}{100} = \varepsilon.$$

In other words, the interval $(-1/100, 1/100)$ contains all the terms of the sequence from $n = 52$ onward.

But the threshold $N_{1/100} = 51$ does not may not necessarily work for ε values smaller than $1/100$, however. If $\varepsilon = 1/1000$, say, then we need $N_{1/1000} > \frac{1}{2(1/1000)} = 500$ to guarantee that all the terms after the threshold fall in the interval $(-1/1000, 1/1000)$.

Obviously, we could find an appropriate threshold N_ε in the same manner using any $\varepsilon > 0$. This leads us to the following definition.

A sequence (x_n) of real numbers **converges** to a **limit** $L \in \mathbb{R}$, which we denote by

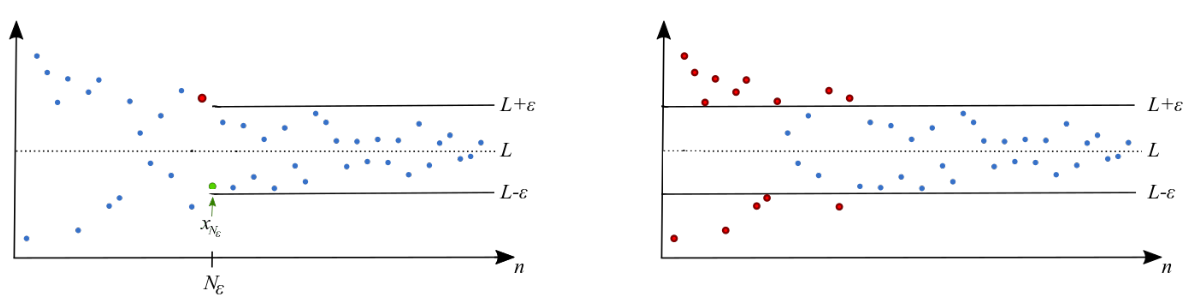
$$x_n \to L \quad \text{or} \quad \lim_{n\to\infty} x_n = L,$$

if

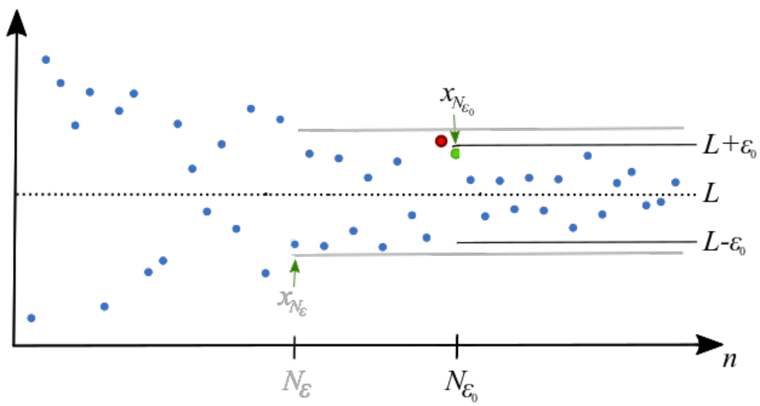
$$\forall \varepsilon > 0,\ \exists N_\varepsilon \in \mathbb{N} \text{ such that } n > N_\varepsilon \implies |x_n - L| < \varepsilon.$$

This may look complicated, but it is just the formalized statement of the example above, where $L = 0$: we look for a **systematic** threshold N_ε after which all terms of the sequence x_n lie in $(L - \varepsilon, L + \varepsilon)$.

In the illustration below where $x_n \to L$, we find an acceptable threshold N_ε for ε on the left, and display the finite number of sequence terms falling outside of the interval $(L - \varepsilon, L + \varepsilon)$ on the right.



We also identify a threshold N_{ε_0} for $\varepsilon_0 \leq \varepsilon$ in the illustration below.



A sequence (x_n) which does not converge to a limit is said to be **divergent**:

$$\forall L \in \mathbb{R},\ \exists \varepsilon_L > 0,\ \forall N \in \mathbb{N},\ \exists n_N > N \text{ such that } |x_{n_N} - L| \geq \varepsilon_L;$$

in other words, no real number L can be the limit of (x_n).

There is only one way for a sequence to converge – its values must eventually get closer and closer to the limit; but there is more than one way for a sequence to diverge.

Examples

1. Show that $\frac{1}{n} \to 0$.

   Proof: let $\varepsilon > 0$. By the Archimedean property, $\exists N_\varepsilon > \frac{1}{\varepsilon}$, so $\varepsilon > \frac{1}{N_\varepsilon}$. If $n > N_\varepsilon$, then $\frac{1}{n} < \frac{1}{N_\varepsilon}$ and

   $$\left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right| = \frac{1}{n} < \frac{1}{N_\varepsilon} < \varepsilon.$$

   This completes the proof. ■

2. Show that $\frac{n+1}{n^2+1} \to 0$.

   Proof: let $\varepsilon > 0$. By the Archimedean property, $\exists N_\varepsilon > \frac{2}{\varepsilon}$, so $\varepsilon > \frac{2}{N_\varepsilon}$. If $n > N_\varepsilon$, then $\frac{1}{n} < \frac{1}{N_\varepsilon}$ and

   $$\left|\frac{n+1}{n^2+1} - 0\right| = \frac{n+1}{n^2+1} \leq \frac{2n}{n^2+1} < \frac{2n}{n^2} = \frac{2}{n} < \frac{2}{N_\varepsilon} < \varepsilon.$$

   This completes the proof. ■

3. Show that $\frac{4-2n-3n^2}{2n^2+n} \to -\frac{3}{2}$.

   Proof: let $\varepsilon > 0$. By the Archimedean property, $\exists N_\varepsilon > \frac{2}{\varepsilon}$, so $\varepsilon > \frac{2}{N_\varepsilon}$. If $n > N_\varepsilon$, then $\frac{1}{n} < \frac{1}{N_\varepsilon}$ and

   $$\left| \frac{4-2n-3n^2}{2n^2+n} - \left(-\frac{3}{2}\right) \right| = \left| \frac{2(4-2n-3n^2)+3(2n^2+n)}{2(2n^2+n)} \right| = \frac{|8-n|}{4n^2+2n}.$$

   Note that $8-n \leq 8n$ if $1 \leq n \leq 8$, and that $n-8 \leq 8n$ if $n \geq 8$, so that $|8-n| \leq 8n$ for all $n \geq 1$. Thus

   $$\frac{|8-n|}{4n^2+2n} \leq \frac{8n}{4n^2+2n} < \frac{8n}{4n^2} = \frac{2}{n} < \frac{2}{N_\varepsilon} < \varepsilon$$

   when $n > N_\varepsilon$, which completes the proof. ■

4. Show that $(x_n) = (n)$ is divergent.

   Proof: suppose instead that (x_n) converges to $a \in \mathbb{R}$. Let $\varepsilon > 0$. By definition, $\exists N_\varepsilon \in \mathbb{N}$ such that $|x_n - a| = |n - a| < \varepsilon$ whenever $n > N_\varepsilon \implies$ that $n < a + \varepsilon$ for all $n > N_\varepsilon$, $\implies a + \varepsilon$ is a an upper bound for $\mathbb{N}$. This contradicts the Archimedean property, so the sequence (n) must diverge. ■

The main benefit of the formal definition of the limit of a sequence is that it **does not call on infinity**: we write $n \to \infty$, but that is a merely a notation of convenience. On the flip side, the formal definition has 2 major inconveniences:

1. it cannot be used to **determine the limit** of a convergent sequence – it can only be used to verify that a given candidate is (or is not) a limit of a sequence;

2. it can seem artificial to some extent, especially upon a first encounter.

In practice, using the definition is in fact rather simple: in order to **determine a threshold** N_ε that does the trick, we often **backtrack** from the end of the string of inequalities rather than to proceed directly from "Let $\varepsilon > 0$".

We have been careful to refer to "a" limit when the sequence converges, but we should really be talking about "the" limit in such cases.

Theorem 12 (UNIQUE LIMIT)
A convergent sequence (x_n) of real numbers has exactly one limit.

Proof: suppose that $x_n \to x'$ and $x_n \to x''$. Let $\varepsilon > 0$. Then there exist 2 integers $N'_\varepsilon, N''_\varepsilon \in \mathbb{N}$ such that

$$|x_n - x'| < \varepsilon \text{ whenever } n > N'_\varepsilon \quad \text{and} \quad |x_n - x''| < \varepsilon \text{ whenever } n > N''_\varepsilon.$$

Set $N_\varepsilon = \max\{N'_\varepsilon, N''_\varepsilon\}$. Then whenever $n > N_\varepsilon$, we have

$$0 \leq |x' - x''| = |x' - x_n + x_n - x''| \leq |x_n - x'| + |x_n - x''| < \varepsilon + \varepsilon = 2\varepsilon.$$

Thus $0 \leq \frac{|x'-x''|}{2} < \varepsilon$. As $\varepsilon > 0$ was arbitrary, $\frac{|x'-x''|}{2} = 0$ and $x' = x''$. ■

Sequences have other properties, which we can sometimes use to show that they converge (or diverge). A sequence $(x_n) \subseteq \mathbb{R}$ is **bounded** by $M > 0$ if $|x_n| \leq M$ for all $n \in \mathbb{N}$.

Theorem 13
Any convergent sequence (x_n) of real numbers is bounded.

Proof: let $(x_n) \subseteq \mathbb{R}$ converge to $x \in \mathbb{R}$. Then for $\varepsilon = 1$, say, $\exists N \in \mathbb{N}$ s.t.

$$|x_n - x| < 1 \quad \text{when } n > N.$$

Thanks to the "reverse" triangle inequality (Theorem 6.6), we also have

$$|x_n| - |x| \leq |x_n - x| < 1 \quad \text{when } n > N,$$

so that $|x_n| < |x| + 1$ when $n > N$.

Finally, we set $M = \max\{|x_1|, \ldots, |x_N|, |x| + 1\}$. Then $|x_n| \leq M$ for all n, which means that (x_n) is bounded. ■

About Proofs In general, we may prove results:

- **directly**, as in Theorem 13;
- by **induction**, as in Bernouilli's inequality (Theorem 3), or
- by **contradiction**, as in the Archimedean property (Theorem 1), and so on.

The **contrapositive** of $P \implies Q$ is $\neg Q \implies \neg P$. They are **logically equivalent**, but one may prove easier to demonstrate than the other. On the other hand, the **converse** of $P \implies Q$ is $Q \implies P$. There is no general link between a statement and its converse: sometimes they are both true, sometimes they are both false, sometimes only of them is true.

Example: the contrapositive of Theorem 13 is "*Any unbounded sequence is divergent*", which is valid since Theorem 13 is true. Its converse is "*Any bounded sequence is convergent*" – if we think that the converse is true, then we try to prove it; if we think that it is false, we look for a **counter-example**. Which one is it? □

2.3 Operations on Sequences and Basic Theorems

The following result removes the need to use the formal definition... as long as we have some "ground-level" building blocks to start with.

Theorem 14 (Operations on Convergent Sequences)
Let $(x_n), (y_n)$ be convergent, with $x_n \to x$ and $y_n \to y$. Let $c \in \mathbb{R}$. Then

1. $|x_n| \to |x|$;
2. $(x_n + y_n) \to (x + y)$;
3. $x_n y_n \to xy$ *and* $cx_n \to cx$;
4. $\frac{x_n}{y_n} \to \frac{x}{y}$, *if* $y_n, y \neq 0$ *for all* n.

Proof: we show each part using the definition of the limit of a sequence.

1. Let $\varepsilon > 0$. As $x_n \to x$, $\exists N'_\varepsilon$ such that $|x_n - x| < \varepsilon$ whenever $n > N'_\varepsilon$. But $||x_n| - |x|| \leq |x_n - x|$, according to Theorem 6. Hence, for $\varepsilon > 0$, $\exists N_\varepsilon = N'_\varepsilon$ such that
$$||x_n| - |x|| \leq |x_n - x| < \varepsilon$$
whenever $n > N_\varepsilon$, *i.e.*, $|x_n| \to |x|$.

2. Let $\varepsilon > 0$; then $\frac{\varepsilon}{2} > 0$. As $x_n \to x$ and $y_n \to y$, $\exists N^x_{\frac{\varepsilon}{2}}, N^y_{\frac{\varepsilon}{2}}$ such that
$$|x_n - x| < \frac{\varepsilon}{2} \quad \text{and} \quad |y_n - y| < \frac{\varepsilon}{2} \tag{2.1}$$
whenever $n > N^x_{\frac{\varepsilon}{2}}$ and $n > N^y_{\frac{\varepsilon}{2}}$, respectively. Set $N_\varepsilon = \max\{N^x_{\frac{\varepsilon}{2}}, N^y_{\frac{\varepsilon}{2}}\}$.

   Then, whenever $n > N_\varepsilon$, which is to say, whenever n is strictly larger than both $N^x_{\varepsilon/2}$ and $N^y_{\varepsilon/2}$ simultaneously, we have:
$$\begin{aligned}|(x_n + y_n) - (x + y)| &= |(x_n - x) + (y_n - y)| \leq |x_n - x| + |y_n - y| \\ &\boxed{\text{by (2.1).}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\end{aligned}$$
i.e., $(x_n + y_n) \to (x + y)$.

3. According to Theorem 13, (x_n) and (y_n) are bounded since they are convergent sequences. Thus $\exists M_x, M_y \in \mathbb{N}$ such that for all n, we have
$$|x_n| < M_x \quad \text{and} \quad |y_n| < M_y.$$

Let $\varepsilon > 0$; then $\frac{\varepsilon}{2M_x}, \frac{\varepsilon}{2M_y} > 0$. As $x_n \to x$, $y_n \to y$, $\exists N^x_{\frac{\varepsilon}{2M_y}}, N^y_{\frac{\varepsilon}{2M_x}} \in \mathbb{N}$ such that

$$|x_n - x| < \frac{\varepsilon}{2M_y} \quad \text{and} \quad |y_n - y| < \frac{\varepsilon}{2M_x} \tag{2.2}$$

whenever $n > N^x_{\frac{\varepsilon}{2M_y}}$ and $n > N^y_{\frac{\varepsilon}{2M_x}}$ respectively. Moreover, $|y| \leq M_y$ (see Theorem 15).

Set $N_\varepsilon = \max\{N^x_{\frac{\varepsilon}{2M_x}}, N^y_{\frac{\varepsilon}{2M_y}}\}$. Then, whenever $n > N_\varepsilon$, we have:

$$\begin{aligned} |x_n y_n - xy| = |x_n y_n - x_n y + x_n y - xy| &= |x_n(y_n - y) + y(x_n - x)| \\ \leq |x_n||y_n - y| + |y||x_n - x| &< M_x|y_n - y| + M_y|x_n - x| \\ \boxed{\text{by (2.2)}} &< M_x \cdot \frac{\varepsilon}{2M_x} + M_y \cdot \frac{\varepsilon}{2M_y} \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

i.e., $x_n y_n \to xy$. Furthermore, if the sequence (y_n) is defined by $y_n = c$ for all n, then the preceding result yields $cx_n \to cx$, since $y_n = c \to c$.[4]

4. It is enough to show $1/y_n \to 1/y$ under the Theorem's assumptions; then the result will hold by part 3. Since $y \neq 0$, $\frac{|y|}{2} > 0$. Hence, as $y_n \to y$, $\exists N_{|y|/2} \in \mathbb{N}$ such that $|y_n - y| < |y|/2$, whenever $n > N_{|y|/2}$. According to Theorem 6, we then have

$$|y| - |y_n| < |y - y_n| < \frac{|y|}{2}, \quad \text{and so} \quad \frac{|y|}{2} < |y_n| \quad \text{or} \quad \frac{1}{|y_n|} < \frac{2}{|y|} \tag{2.3}$$

whenever $n > N_{|y|/2}$ – everything is well-defined as neither y_n nor y is 0 for all n.

Let $\varepsilon > 0$. Then $|y|^2\varepsilon/2 > 0$. As $y_n \to y$, $\exists N_{|y|^2\varepsilon/2} \in \mathbb{N}$ such that

$$|y_n - y| < |y|^2 \frac{\varepsilon}{2} \tag{2.4}$$

whenever $n > N_{|y|^2 \cdot \frac{\varepsilon}{2}}$. Set $N_\varepsilon = \max\{N_{\frac{|y|}{2}}, N_{|y|^2\frac{\varepsilon}{2}}\}$. Then, whenever $n > N_\varepsilon$,

$$\begin{aligned} \left|\frac{1}{y_n} - \frac{1}{y}\right| = \left|\frac{y - y_n}{y_n y}\right| &= \frac{|y - y_n|}{|y_n y|} \\ \boxed{\text{by (2.3)}} &< \frac{2|y - y_n|}{|y|^2} \\ \boxed{\text{by (2.4)}} &< \frac{2}{|y|^2} \cdot |y|^2 \frac{\varepsilon}{2} = \varepsilon, \quad \text{i.e.,} \quad \frac{1}{y_n} \to \frac{1}{y}, \end{aligned}$$

which completes the proof. ■

Now that we have some basic tools to work with, we present two results that allow us to compute limits without operating directly on a sequence.

Theorem 15 (COMPARISON THEOREM FOR SEQUENCES)
Let $(x_n), (y_n)$ be convergent sequences of real numbers with $x_n \to x$, $y_n \to y$, and $x_n \leq y_n \; \forall n \in \mathbb{N}$. Then $x \leq y$.

Proof: suppose that it is not the case, namely, that $x > y$. Then $x - y > 0$. Set $\varepsilon = \frac{x-y}{2} > 0$. Since $x_n \to x$ and $y_n \to y$, $\exists N_\varepsilon^x, N_\varepsilon^y \in \mathbb{N}$ s.t.

$$|x_n - x| < \varepsilon \quad \text{whenever } n > N_\varepsilon^x \quad \text{and} \quad |y_n - y| < \varepsilon \quad \text{whenever } n > N_\varepsilon^y.$$

Let $N_\varepsilon = \max\{N_\varepsilon^x, N_\varepsilon^y\}$. Then, if $n > N_\varepsilon$, we have

$$y_n < y + \varepsilon = y + \frac{x-y}{2} = \frac{x+y}{2} = x - \frac{x-y}{2} = x - \varepsilon < x_n.$$

But this contradicts the assumption that $x_n \leq y_n$ for all n, and so $x \leq y$. ■

Warning: the "$\leq$"s in the statement of Theorem 15 **cannot** be replaced by "$<$"s throughout. For instance, if $(x_n) = (\frac{1}{n+1})$ and $(y_n) = (\frac{1}{n})$, then $x_n < y_n$ for all $n \in \mathbb{N}$, but $x_n \to x = 0$, $y_n \to y = 0$, and $0 = x \not< y = 0$.

Theorem 16 (SQUEEZE THEOREM FOR SEQUENCES)
Let $(x_n), (y_n), (z_n) \subseteq \mathbb{R}$ be such that $x_n, z_n \to \alpha$ and $x_n \leq y_n \leq z_n$, $\forall n \in \mathbb{N}$. Then $y_n \to \alpha$.

Proof: let $\varepsilon > 0$. By convergence of $(x_n), (z_n)$ to α, $\exists N_\varepsilon^x, N_\varepsilon^z \in \mathbb{N}$ s.t.

$$|x_n - \alpha| < \varepsilon \text{ whenever } n > N_\varepsilon^x \quad \text{and} \quad |z_n - \alpha| < \varepsilon \text{ whenever } n > N_\varepsilon^z.$$

Let $N_\varepsilon = \max\{N_\varepsilon^x, N_\varepsilon^z\}$. When $n > N_\varepsilon$, $\alpha - \varepsilon < x_n \leq y_n \leq z_n < \alpha + \varepsilon$, which is to say, that $|y_n - \alpha| < \varepsilon$. Consequently, $y_n \to \alpha$. ■

We can use these various results to compute a fair collection of limits.

Examples

1. Compute $\displaystyle\lim_{n\to\infty} \frac{3n+1}{n}$, if the limit exists.

   Solution: note that $\frac{3n+1}{n} = 3 + \frac{1}{n}$. According to Theorem 14, **if the limit exists** we must have

   $$\lim_{n\to\infty} \frac{3n+1}{n} = \lim_{n\to\infty} \left(3 + \frac{1}{n}\right) = \lim_{n\to\infty} 3 + \lim_{n\to\infty} \frac{1}{n} = 3 + 0 + 3.$$

Reading the string of equations backwards, we see that the original limit must exist and be equal to 3. □

2. Compute $\lim_{n\to\infty} \frac{\sin(n^2+212)}{n}$, if the limit exists.

   Solution: we cannot use Theorem 14 since neither the numerator nor the denominator limit exists. This does not necessarily mean that the limit of the quotient does not exist. In order to determine if it does, we need to use another approach.

   By definition of the sin function (which we take for granted for now), we have $-1 \leq \sin x \leq 1,\ \forall x \in \mathbb{R}$. Thus

$$-1 \leq \sin(n^2+212) \leq 1,\ \forall n \Longrightarrow -\frac{1}{n} \leq \frac{\sin(n^2+212)}{n} \leq \frac{1}{n},\ \forall n.$$

   As $\pm\frac{1}{n} \to 0$, we can use the squeeze theorem to conclude that

$$\lim_{n\to\infty} \frac{\sin(n^2+212)}{n} = 0. \qquad \square$$

3. Compute $\lim_{n\to\infty} \frac{2n-1}{n+7}$, if the limit exists.

   Solution: we cannot apply Theorem 14 directly since neither the numerator nor the denominator limits exist. However,

$$\frac{2n-1}{n+7} = \frac{1/n\cdot(2n-1)}{1/n\cdot(n+7)} = \frac{2-1/n}{1+7/n} \quad \text{when } n \neq 0.$$

   Because each of the constituent parts converge (and because the denominator is never equal to 0, either in the limit or in the sequence), repeated applications of Theorem 14 yield

$$\lim_{n\to\infty} \frac{2n-1}{n+7} = \frac{\lim_{n\to\infty}(2-1/n)}{\lim_{n\to\infty}(1+7/n)} = \frac{2-\lim_{n\to\infty} 1/n}{1+7\cdot\lim_{n\to\infty} 1/n} = \frac{2-0}{1+7\cdot 0} = 2.$$

   This is basically a calculus argument. □

4. Let (x_n) be such that $|x_n| \to 0$. Show that $x_n \to 0$.

   Proof: since $-|x_n| \leq x_n \leq |x_n|$ for all $n \in \mathbb{N}$ according to Theorem 6, and since $-|x_n|, |x_n| \to 0$ by assumption, then $x_n \to 0$ according to the squeeze theorem (note, however that if $|x_n| \to \alpha \neq 0$, we cannot necessarily conclude that $x_n \to \alpha$. Consider, for instance, the sequence $(x_n) = (-1)^n$). ■

5. Let $|q| < 1$. Compute $\lim_{n\to\infty} q^n$, if the limit exists.

   Solution: if $q = 0$, then $q^n = 0 \to 0$. If $q \neq 0$, then $\frac{1}{|q|} > 1$. Thus, $\exists t > 0$ such that $\frac{1}{|q|} = 1 + t$.

   From Bernoulli's inequality, we have

$$\left(\frac{1}{|q|}\right)^n = (1+t)^n \geq 1 + nt, \ \forall n \in \mathbb{N},$$

   so that $0 \leq |q^n| \leq |q|^n \leq \frac{1}{1+nt}$. But $\frac{1}{1+nt} = 0$ when $n \to \infty$ (does this need to be proven?); thus $|q^n| \to 0$ according to the squeeze theorem, and so $q^n \to 0$ by the previous example. □

6. Let $|q| < 1$. Compute $\lim_{n\to\infty} nq^n$, if the limit exists.

   Solution: the proof that $nq^n \to 0$ is left as an exercise; it is similar to the proof of part of the previous example, but uses an extension of Bernoulli's inequality:

$$(1+t)^n \geq 1 + nt + \frac{n(n-1)}{2}t^2, \text{ for } t > 0, n \geq 1,$$

   which can be proven by induction. □

7. Show that $\sqrt[n]{n} \to 1$.

   Solution: let $\varepsilon > 0$. Then $1 + \varepsilon > 1$ and $0 < \frac{1}{1+\varepsilon} < 1$.

   Claim: $n\left(\frac{1}{1+\varepsilon}\right)^n \to 0$ when $n \to \infty$ (use previous example with $q = \frac{1}{1+\varepsilon}$).

   Hence, $\exists M_1 \in \mathbb{N}$ such that

$$\left|\frac{n}{(1+\varepsilon)^n} - 0\right| < 1 \text{ when } n > M_1 \Longrightarrow 1 \leq n < (1+\varepsilon)^n \text{ when } n > M_1.$$

   Set $N_\varepsilon = M_1$. Then $1 - \varepsilon < 1 \leq n^{1/n} < 1 + \varepsilon$ when $n > N_\varepsilon$. But this is precisely the same as $|n^{1/n} - 1| < \varepsilon$ when $n > N_\varepsilon$; thus $n^{1/n} \to 1$. □

8. Compute $\lim_{n\to\infty} \frac{n!}{n^n}$, if the limit exists.

   Solution: since

$$0 \leq \frac{n!}{n^n} = \frac{n \cdot (n-1) \cdot \cdots \cdot 2 \cdot 1}{n \cdot n \cdot \cdots \cdot n \cdot n} \leq \frac{1}{n}, \ \forall n \in \mathbb{N},$$

   and $\frac{1}{n} \to 0$, the squeeze theorem implies $\frac{n!}{n^n} \to 0$. □

9. Let $a > 0$. Compute $\lim_{n\to\infty} a^{1/n}$, if the limit exists.

   Solution: since $a > 0$, we have $\frac{1}{a} > 0$. According to the Archimedean property, $\exists N_a \geq \max\{a, \frac{1}{a}\}$. For every $n \geq N_a$, we then have $\frac{1}{n} \leq a \leq n$. Thus $\frac{1}{\sqrt[n]{n}} \leq \sqrt[n]{a} \leq \sqrt[n]{n}$ for all $n \geq N_a$. But $\sqrt[n]{n} \to 1$ by a previous example, so $\sqrt[n]{a} \to 1$ by the squeeze theorem. □

10. Compute $\lim_{n\to\infty} \sqrt[n]{3^n + 5^n}$, if the limit exists.

    Solution: since

$$5^n \leq 3^n + 5^n \leq 5^n + 5^n = 2 \cdot 5^n \leq n \cdot 5^n,\ \forall n \geq 2,$$

    then

$$5 \leq \sqrt[n]{3^n + 5^n} \leq \sqrt[n]{n} \cdot 5,\ \forall n \geq 2.$$

    But we have seen previously that $\sqrt[n]{n} \to 1$.

    The squeeze theorem can then be applied to the above chain of inequalities to conclude $\sqrt[n]{3^n + 5^n} \to 5$. □

We can also use the definition and theorems to demonstrate general results (that is, results about general sequences rather than about specific examples).

Theorem 17
Let $y_n \to y$. If $y_n \geq 0\ \forall n \in \mathbb{N}$, then $\sqrt{y_n} \to \sqrt{y}$.

Proof: according to Theorem 15, we must have $y \geq 0$. There are 2 cases:

- If $y = 0$, let $\varepsilon > 0$. Then $\varepsilon^2 > 0$. Since $y_n \to 0$, $\exists M_{\varepsilon^2} \in \mathbb{N}$ s.t. whenever $n > M_{\varepsilon^2}$, we must have $|y_n - 0| = y_n < \varepsilon^2$. Now, set $N_\varepsilon = M_{\varepsilon^2}$.

  Then whenever $n > N_\varepsilon$, $|\sqrt{y_n} - 0| = \sqrt{y_n} < \sqrt{\varepsilon^2} = \varepsilon$.

- If $y > 0$, let $\varepsilon > 0$. Then $\varepsilon\sqrt{y} > 0$. Since $y_n \to y$, $\exists M_{\varepsilon\sqrt{y}} \in \mathbb{N}$ s.t. whenever $n > M_{\varepsilon\sqrt{y}}$, $|y_n - y| < \varepsilon\sqrt{y}$. Now, set $N_\varepsilon = M_{\varepsilon\sqrt{y}}$.

  Then whenever $n > N_\varepsilon$, $|\sqrt{y_n} - \sqrt{y}| = \frac{|y_n - y|}{\sqrt{y_n} + \sqrt{y}} \leq \frac{|y_n - y|}{\sqrt{y}} < \frac{\varepsilon\sqrt{y}}{\sqrt{y}} = \varepsilon$.

In both cases, we have $\sqrt{y_n} \to \sqrt{y}$. ■

2.4 Bounded Monotone Convergence Theorem

A sequence (x_n) is **increasing** if $x_1 \leq x_2 \leq \cdots x_n \leq x_{n+1} \leq \cdots$, $\forall n \in \mathbb{N}$; it is **decreasing** if $x_1 \geq x_2 \geq \cdots \geq x_n \geq x_{n+1} \cdots$, $\forall n \in \mathbb{N}$. If (x_n) is either increasing or decreasing, we say that it is **monotone**. If it is both increasing and decreasing, it is **constant.**[5]

Monotone sequences play an important role in the theory of convergence, assuming that they satisfy an additional condition.

Theorem 18 (Bounded Monotone Convergence)
Let (x_n) be an increasing sequence bounded above. Then $x_n \to \sup\{x_n \mid n \in \mathbb{N}\}$.

Proof: since the sequence (x_n) is bounded above, so it its range $\{x_n\}$. By completeness of $\mathbb{R}$, $x^* = \sup\{x_n\}$ exists. It remains only to show $x_n \to x^*$.

Let $\varepsilon > 0$. By definition, $x^* - \varepsilon$ is not an upper bound for $\{x_n\}$. Then $\exists N_\varepsilon \in \mathbb{N}$ s.t.

$$x^* - \varepsilon < x_{N_\varepsilon} \leq x^* < x^* + \varepsilon.$$

But (x_n) is increasing; in particular, $x_{N_\varepsilon} \leq x_n$ when $n > N_\varepsilon$. Thus

$$n > N_\varepsilon \Longrightarrow x^* - \varepsilon < x_n < x^* + \varepsilon,$$

so $x_n \to x^*$. ■

A similar result holds for decreasing sequences bounded below.

Examples

- Does the sequence $(x_n) = (1 - \frac{1}{n})$ converge? If so, what is its limit?

  Solution: as $\frac{1}{n} \geq \frac{1}{n+1}$ for all $n \in \mathbb{N}$,

  $$x_n - 1 - \frac{1}{n} \leq 1 - \frac{1}{n+1} \leq x_{n+1},$$

  and so (x_n) is increasing. Furthermore, $x_n \leq 1$ for all $n \in \mathbb{N}$. Then (x_n) converges according to the bounded monotone convergence theorem, and

  $$\lim_{n\to\infty} x_n = \sup_{n\in\mathbb{N}}\{x_n\} = \sup_{n\in\mathbb{N}}\{1 - 1/n\} = 1 + \sup_{n\in\mathbb{N}}\{-1/n\} = 1 - \inf_{n\in\mathbb{N}}\{1/n\} = 1,$$

  which agrees with our intuition. □

[5] When the inequalities are strict, then the sequence is **strictly increasing** or **strictly decreasing**, depending on the specific situation, and is thus **strictly monotone**.

- Let (x_n) be defined by $x_n = \sqrt{2x_{n-1}}$ when $n \geq 2$, with $x_1 = 1$. Does (x_n) converge? If so, to what limit?

  Solution: we first show, by induction, that (x_n) is increasing.

  - **Base Case:** $x_2 = \sqrt{2} \geq 1 = x_1$.
  - **Induction Step:** Suppose $x_k \geq x_{k-1}$. Then

  $$2x_k \geq 2x_{k-1} \implies \sqrt{2x_k} \geq \sqrt{2x_{k-1}} \implies x_{k+1} \geq x_k.$$

  Thus $x_{n+1} \geq x_n$ for all $n \in \mathbb{N}$.

  Next we show, again by induction, that (x_n) is bounded above by 2.

  - **Base Case:** $1 \leq x_1 = 1 \leq 2$.
  - **Induction Step:** Suppose $1 \leq x_k \leq 2$. Then

  $$2 \leq 2x_k \leq 2 \cdot 2 = 4 \implies 1 \leq \sqrt{2} \leq \sqrt{2x_k} \leq \sqrt{4} = 2 \implies 1 \leq x_{k+1} \leq 2.$$

  Thus $x_n \leq 2$ for all $n \in \mathbb{N}$ (why did we include the lower bound 1?).

  We then have, according to the bounded monotone convergence theorem,

  $$x_n \to x = \sup\{x_n \mid n \in \mathbb{N}\}.$$

  But

  $$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{2x_n} = \sqrt{2 \lim_{n\to\infty} x_n} = \sqrt{2x},$$

  whence $x^2 = 2x$. So either $x = 0$ or $x = 2$. But $x_n \geq 1$ for all $n \in \mathbb{N}$, so $x \geq 1$ according to Theorem 15. Thus $x_n \to 2$. □

2.5 Bolazano-Weierstrass Theorem

The main result of this section, concerning bounded sequences and their subsequences, is a corner stone of analysis.

Let $(x_n) \subseteq \mathbb{R}$ be a sequence and $n_1 < n_2 < \cdots$ be an increasing string of positive integers. The sequence

$$(x_{n_k})_k = (x_{n_1}, x_{n_2}, \ldots)$$

is a **subsequence** of (x_n), denoted by $(x_{n_k}) \subseteq (x_n)$. Note that $n_k \geq k$ for all $k \in \mathbb{N}$.

Examples

- Let $(x_n) = (\frac{1}{n})$. Both $(\frac{1}{2k}) = (\frac{1}{2}, \frac{1}{4}, \ldots)$ and $(1, \frac{1}{3}, \frac{1}{6}, \frac{1}{10}, \frac{1}{15}, \frac{1}{21}, \ldots)$ are subsequences of (x_n) as they sample the original sequence while preserving the order in which the terms appear. But $(1, \frac{1}{3}, \frac{1}{2}, \frac{1}{8}, \ldots)$ is not a subsequence of (x_n) as $\frac{1}{3} = x_3$ appears before $\frac{1}{2} = x_2$.
- The sequence $(x_{3n}) = (x_3, x_6, x_9, \ldots)$ is a subsequence of (x_n) for any sequence (x_n).
- Every sequence (x_n) is a **(non-proper)** subsequence of itself.
- If $(y_k) = (x_{n_k})$ is a subsequence of (x_n) and $(z_j) = (y_{k_j})$ is a subsequence of (y_k), then $(z_j) = (x_{n_{k_j}})$ is a subsequence of (x_n). $\square$

Convergent sequences have well-behaved subsequences, as we see below.

Theorem 19 Let $x_n \to x$. If $(x_{n_k}) \subseteq (x_n)$, then $x_{n_k} \to x$ as well.

Proof: Let $\varepsilon > 0$. Since $x_n \to x$, $\exists N_\varepsilon \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ whenever $n > N_\varepsilon$. But (x_{n_k}) is a subsequence of (x_n), so $n_k \geq k$ for all $k \in \mathbb{N}$. Then $|x_{n_k} - x| < \varepsilon$ whenever $n_k \geq k > N_\varepsilon$, so $x_{n_k} \to x$ when $k \to \infty$. $\blacksquare$

Note that the converse of Theorem 19 is false (see Exercises).

The next result is surprising (at first glance) and deep, and will prove quite useful.

Theorem 20 (BOLZANO-WEIERSTRASS)
If $(x_n) \subseteq \mathbb{R}$ is bounded, it has (at least) one convergent subsequence.

Proof: we build a subsequence as follows: as (x_n) is bounded, there is an interval $I_1 = [a, b]$ s.t. $(x_n) \subseteq I_1$. Let $n_1 = 1$. Then $x_{n_1} = x_1 \in I_1$ and

$$\text{length}(I_1) = b - a = \frac{b-a}{2^0}.$$

Set $I_1' = [a, \frac{a+b}{2}]$ and $I_1'' = [\frac{a+b}{2}, b]$,

$$A_1 = \{n \in \mathbb{N} \mid n > n_1 \text{ and } x_n \in I_1'\}, \quad B_1 = \{n \in \mathbb{N} \mid n > n_1 \text{ and } x_n \in I_1''\}.$$

At least one of A_1, B_1 must be infinite as $A_1 \cup B_1 = \{n \in \mathbb{N} \mid n > n_1\}$:

- If A_1 is infinite, set $I_2 = I_1'$. Since A_1 is an infinite set of integers, it is not empty. By the well-ordering axiom, A_1 contains a smallest element, say n_2.

- If A_1 is finite, set $I_2 = I_1''$. Since B_1 is an infinite set of integers, it is not empty. By the well-ordering axiom, B_1 contains a smallest element, say n_2.

Either way, there is an integer $n_2 > n_1$ such that $x_{n_2} \in I_2$, $I_1 \supseteq I_2$ and

$$\text{length}(I_2) = \frac{b-a}{2^1}.$$

Now, suppose that $I_{k-1} \supseteq I_k$ are intervals with

$$\text{length}(I_{k-1}) = \frac{b-a}{2^{k-2}} \quad \text{and} \quad \text{length}(I_k) = \frac{b-a}{2^{k-1}},$$

that $\exists n_{k-1}, n_k \in \mathbb{N}$ such that $n_{k-1} < n_k$, $x_{n_{j-1}} \in I_{k-1}$, $x_{n_k} \in I_k$, and that at least one of the corresponding sets A_{k-1}, B_{k-1} is infinite.

Write $I_k = [\alpha, \beta]$. Set $I_k' = [\alpha, \frac{\alpha+\beta}{2}]$ and $I_k'' = [\frac{\alpha+\beta}{2}, \beta]$,

$$A_k = \{n \in \mathbb{N} \mid n > n_k \text{ and } x_n \in I_k'\}, \quad B_k = \{n \in \mathbb{N} \mid n > n_k \text{ and } x_n \in I_k''\}.$$

One of A_k, B_k must be infinite as $A_k \cup B_k = \{n \in \mathbb{N} \mid n > n_k \text{ and } x_n \in I_k\}$ is infinite.

- If A_k is infinite, set $I_{k+1} = I_k'$. Since A_k is an infinite set of integers, it is not empty. By the well-ordering axiom, A_k contains a smallest element, say n_{k+1}.
- If A_k is finite, set $I_{k+1} = I_k''$. Since B_k is an infinite set of integers, it is not empty. By the well-ordering axiom, B_k contains a smallest element, say n_{k+1}.

Either way, there is an integer $n_{k+1} > n_k$ s.t. $x_{n_{k+1}} \in I_{k+1}$, $I_k \supseteq I_{k+1}$ and

$$\text{length}(I_{k+1}) = \frac{b-a}{2^k}.$$

By induction, we have

1. $I_1 \supseteq I_2 \supseteq \cdots I_k \supseteq I_{k+1} \supseteq \cdots$;
2. for each $k \in \mathbb{N}$, $\text{length}(I_k) = \frac{b-a}{2^{k-1}}$;
3. for each $k \in \mathbb{N}$, $x_{n_k} \in I_k$, and
4. $n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$.

Furthermore, $\frac{b-a}{2^k} \to 0$ (since it is a subsequence of $\frac{b-a}{n} \to 0$). According to the nested intervals theorem, then, $\exists \xi \in [a, b]$ such that

$$\bigcap_{k \geq 1} I_k = \{\xi\}.$$

It remains only to show that $x_{n_k} \to \xi$.

Let $\varepsilon > 0$. By the Archimedean property, $\exists K_\varepsilon \in \mathbb{N}$ such that $2^{K_\varepsilon - 1} > \frac{b-a}{\varepsilon}$, and so

$$k > K_\varepsilon \Longrightarrow 2^{K_\varepsilon - 1} < 2^{k-1} \Longrightarrow 0 \leq \frac{b-a}{2^{k-1}} < \frac{b-a}{2^{K_\varepsilon - 1}} < \varepsilon.$$

Since $\xi \in I_k$ for all $k \in \mathbb{N}$, then

$$k > K_\varepsilon \Longrightarrow |x_{n_k} - \xi| \leq \frac{b-a}{2^{k-1}} < \frac{b-a}{2^{K_\varepsilon - 1}} < \varepsilon,$$

which is to say $x_{n_k} \to x$. ■

We have mentioned that a sequence (x_n) which diverges is one for which

$$\forall L \in \mathbb{R},\ \exists \varepsilon_L > 0,\ \forall N \in \mathbb{N},\ \exists n_N > N \text{ such that } |x_{n_N} - L| \geq \varepsilon_L.$$

If (x_n) **does not converge to** L, it is easy to construct a subsequence (x_{n_k}) which also fails to converge to L:

- let $n_1 \in \mathbb{N}$ be such that $n_1 \geq 1$ and $|x_{n_1} - L| \geq \varepsilon_L$;
- let $n_2 \in \mathbb{N}$ be such that $n_2 \geq n_1$ and $|x_{n_2} - L| \geq \varepsilon_L$;
- etc.

Note that if $x_n \not\to L$, some subsequences of (x_n) **might still converge** to L: for instance, $x_n = (-1)^n \not\to 1$, but $x_{2n} = (-1)^{2n} = 1 \to 1$.

Theorem 21
Let $(x_n) \subseteq \mathbb{R}$ be a bounded sequence such that every one of its proper converging subsequence converges to the same $x \in \mathbb{R}$. Then $x_n \to x$.

Proof: Let $M > 0$ be a bound for (x_n). Then $|x_n| \leq M$ for all $n \in \mathbb{N}$. If (x_n) does not converge to x, then $\exists (x_{n_k}) \subseteq (x_n)$ and an $\varepsilon_0 > 0$ such that

$$|x_{n_k} - x| \geq \varepsilon_0 \quad \text{for all } k \in \mathbb{N}.$$

But (x_{n_k}) is also a bounded sequence, and so, by the Bolzano-Weierstrass theorem, there is convergent subsequence $(x_{n_{k_j}}) \subseteq (x_{n_k}) \subseteq (x_n)$.

But all subsequences of (x_n) converge to x, by assumption, so $x_{n_{k_j}} \to x$. That is to say, for $\varepsilon_0 > 0$, $\exists N_{\varepsilon_0} \in \mathbb{N}$ such that $|x_{n_{k_j}} - x| < \varepsilon_0$ whenever $k_j > j > N_{\varepsilon_0}$, which contradicts the above property. Hence $x_n \to x$. ■

2.6 Cauchy Sequences

One of the main challenge with the definition of a limit is that we need to know what L is **before** we can show what it is. Thankfully, we can bypass the circularity of the situation.We say that a sequence (x_n) is a **Cauchy sequence** if

$$\forall \varepsilon > 0,\ \exists N_\varepsilon \in \mathbb{N} \text{ such that } m,n > N_\varepsilon \Longrightarrow |x_m - x_n| < \varepsilon.$$

Incidentally, (x_n) is not a Cauchy sequence if

$$\exists \varepsilon_0 > 0,\ \forall N \in \mathbb{N},\ \exists m_N, n_N > N \text{ such that } |x_{m_N} - x_{n_N}| \geq \varepsilon_0.$$

Examples:

1. Is $(x_n) = (\frac{1}{n})$ a Cauchy sequence?

   Solution: let $\varepsilon > 0$. By the Archimedean property, $\exists N_\varepsilon > \frac{2}{\varepsilon}$. Thus

   $$m,n > N_\varepsilon \Longrightarrow \left|\frac{1}{m} - \frac{1}{n}\right| \leq \frac{1}{m} + \frac{1}{n} < \frac{1}{N_\varepsilon} + \frac{1}{N_\varepsilon} = \frac{2}{N_\varepsilon} < \varepsilon.$$

   Thus (x_n) is Cauchy. □

2. Is $(x_n) = (1 + \frac{1}{2} + \cdots + \frac{1}{n})$ a Cauchy sequence?
   Solution: let $m > n$. Then $\frac{1}{n} \geq \frac{1}{n+1} \geq \cdots \geq \frac{1}{m}$ and

   $$|x_m - x_n| = \frac{1}{n+1} + \cdots + \frac{1}{m} \geq \underbrace{\frac{1}{m} + \cdots + \frac{1}{m}}_{m-n \text{ terms}} = \frac{(m-n)}{m} = 1 - \frac{n}{m}.$$

   In particular, if $m = 2n$, then $|x_m - x_n| \geq \frac{1}{2}$ for every $n \in \mathbb{N}$, and so (x_n) is not a Cauchy sequence. □

In essence, a Cauchy sequence is a sequence for which the terms can get as close to one another as one wishes, after a certain index threshold.

The next result shows that Cauchy sequences have at least one of the traits of convergent sequences in $\mathbb{R}$ – we will soon see that the similarity is not pure happenstance.

Theorem 22
If (x_n) is a Cauchy sequence, then it is bounded.

Proof: let $1 > \varepsilon > 0$. If (x_n) is Cauchy, $\exists N_\varepsilon \in \mathbb{N}$ such that $|x_m - x_n| < \varepsilon$ whenever $m,n > N_\varepsilon$. Set $m^* = N_\varepsilon + 1$. If $n > N_\varepsilon$, then

$$|x_n| = |x_{m^*} + (x_n - x_{m^*})| \leq |x_{m^*}| + |x_n - x_{m^*}| < |x_{m^*}| + \varepsilon.$$

Set $M = \max\{|x_1| + 1, \ldots, |x_{N_\varepsilon}| + 1, |x_{m^*}| + 1\}$. Then $|x_n| \leq M$ for all $n \in \mathbb{N}$. ■

We could also show that the sum of two Cauchy sequences is a Cauchy sequence, that every bounded Cauchy sequence admits at least one convergent subsequence, and so on. In fact, any result that applies to convergent sequences in $\mathbb{R}$ also applies to Cauchy sequences in $\mathbb{R}$ (and vice-versa) because of the following result.

Theorem 23
A sequence of real numbers is convergent if and only if it is a Cauchy sequence.

Proof: let (x_n) be the sequence under consideration. Suppose that $x_n \to x$, say. Let $\varepsilon > 0$. Then $\frac{\varepsilon}{2} > 0$ and $\exists M_{\varepsilon/2}$ such that

$$n > M_{\varepsilon/2} \Longrightarrow |x_n - x| < \frac{\varepsilon}{2}.$$

Set $N_\varepsilon = M_{\varepsilon/2}$. When $n, m > N_\varepsilon$, we have

$$|x_m - x_n| \leq |x_m - x + x - x_n| \leq |x_m - x| + |x - x_n| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which is to say that (x_n) is Cauchy.

Now suppose that (x_n) is Cauchy. According to Theorem 22, it is a bounded sequence, and so must admit a convergent subsequence $(x_{n_k}) \subseteq (x_n)$ by the Bolzano-Weierstrass theorem, with $x_{n_k} \to x$, say.

Let $\varepsilon > 0$. Since (x_n) is Cauchy, $\exists M_{\varepsilon/2} \in \mathbb{N}$ such that

$$n, m > M_{\varepsilon/2} \Longrightarrow |x_m - x_n| < \frac{\varepsilon}{2}.$$

Since (x_{n_k}) converges to x, $\exists N > M_{\varepsilon/2}$ such that $|x_N - x| < \frac{\varepsilon}{2}$. Set $N_\varepsilon = M_{\varepsilon/2}$. Then

$$n > N_\varepsilon \Longrightarrow |x_n - x| = |x_n - x_N + x_N - x| \leq |x_n - x_N| + |x_N - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

and so $(x_n)$is convergent. ∎

This result can help simplify proofs and computations to a considerable extent.

Examples

1. As the sequence $(x_n) = (1 + \frac{1}{2} + \cdots + \frac{1}{n})$ is not a Cauchy sequence, it does not converge.

2. Compute the limit of the sequence defined by $x_n = \frac{1}{2}(x_{n-2} + x_{n-1})$, $n > 2$, with $x_1 = 1$ and $x_2 = 2$.

Solution: we cannot use the bounded monotone convergence theorem as (x_n) is not monotone. However, (x_n) is a Cauchy sequence. Indeed,

$$|x_{n+1}-x_n| = \left|\tfrac{1}{2}(x_{n-1}+x_n)-x_n\right| = \tfrac{1}{2}|x_n-x_{n-1}| = \tfrac{1}{2^2}|x_{n-1}-x_{n-2}| \\ = \tfrac{1}{2^3}|x_{n-2}-x_{n-3}| = \cdots = \tfrac{1}{2^{n-1}}|x_2-x_1| = \tfrac{1}{2^{n-1}}.$$

Let $\varepsilon > 0$. By the Archimedean property, $\exists N_\varepsilon \in \mathbb{N}$ such that $\frac{1}{2^{N_\varepsilon-2}} < \varepsilon$. Then, whenever $m \geq n > N_\varepsilon$,

$$|x_m - x_n| \leq |x_m - x_{m-1}| + \cdots + |x_{n+1}-x_n| \\ = \frac{1}{2^{m-2}} + \cdots + \frac{1}{2^{n-1}} < \frac{1}{2^{n-2}} < \frac{1}{2^{N_\varepsilon-2}} < \varepsilon.$$

Being a Cauchy sequence, (x_n) is convergent according to Theorem 23. Let $x_n \to x$. From Theorem 19, we must have $x_{2n+1} \to x$ as well.

It is left as an induction exercise to show that

$$x_{2n+1} = 1 + \frac{1}{2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{2n-1}} = 1 + \frac{3}{4}\left(1 - \frac{1}{4^n}\right).$$

Then $x_{2n+1} \to 1 + \frac{2}{3} = \frac{5}{3} = x$. □

Cauchy sequences illustrate the fundamental difference between $\mathbb{R}$ and $\mathbb{Q}$. A sequence is Cauchy if the points of the sequence "**accumulate**" on top of one another. We have seen that in $\mathbb{R}$, every Cauchy sequence is convergent, and *vice-versa*.

In $\mathbb{Q}$, the converging sequences are Cauchy, but there are Cauchy sequences that do not converge: it is possible that the points of such a sequence "accumulate" around one of the (uncountably infinitely) many holes of $\mathbb{Q}$. For instance, the sequence $(1, 1.4, 1.41, 1.414, \ldots)$ is Cauchy in $\mathbb{Q}$, but does not converge in $\mathbb{Q}$.

This remark leads to one of the ways of building $\mathbb{R}$ from $\mathbb{Q}$: we take all Cauchy sequences in $\mathbb{Q}$ and add whatever point the sequences "accumulates" around to $\mathbb{R}$ (there is more to it than that, but that is the main idea – We will revisit this idea in much more detail in Chapter 7). In the example above, the Cauchy sequence would lead us to add $\sqrt{2}$ to $\mathbb{Q}$.

2.7 Solved Problems

1. The first few terms of a sequence (x_n) are given below. Assuming that the "natural pattern" indicated by these terms persists, give a formula for the nth term x_n.

   a) $(5, 7, 9, 11, \ldots)$;

b) $\left(\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \ldots\right)$;

c) $\left(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right)$;

d) $(1, 4, 9, 16, \ldots)$.

Solution: there is no general method (this question is a wee bit on the easy side...).

a) Odd integers ≥ 5: $x_n = 2n + 3$ for all $n \geq 1$;

b) Alternating powers of $\frac{1}{2}$: $x_n = (-1)^{n+1}\frac{1}{2^n}$ for all $n \geq 1$;

c) Fractions where the denominator is one more than the numerator: $x_n = \frac{n}{n+1}$ for all $n \geq 1$;

d) Perfect squares ≥ 1: $x_n = n^2$ for all $n \geq 1$. □

2. Use the definition of the limit of a sequence to establish the following limits.

a) $\lim_{n\to\infty} \left(\frac{1}{n^2+1}\right) = 0$;

b) $\lim_{n\to\infty} \left(\frac{2n}{n+1}\right) = 2$;

c) $\lim_{n\to\infty} \left(\frac{3n+1}{2n+5}\right) = \frac{3}{2}$, and

d) $\lim_{n\to\infty} \left(\frac{n^2-1}{2n^2+3}\right) = \frac{1}{2}$.

Proof:

a) Let $\varepsilon > 0$. By the Archimedean property, there is a positive integer $N_\varepsilon > \frac{1}{\varepsilon}$. Then

$$\left|\frac{1}{n^2+1} - 0\right| = \frac{1}{n^2+1} < \frac{1}{n^2} \leq \frac{1}{n} < \frac{1}{N_\varepsilon} < \varepsilon,$$

whenever $n > N_\varepsilon$.

b) Let $\varepsilon > 0$. By the Archimedean property, there is a positive integer $N_\varepsilon > \frac{2}{\varepsilon}$. Then

$$\left|\frac{2n}{n+1} - 2\right| = \left|-\frac{2}{n+1}\right| = \frac{2}{n+1} < \frac{2}{n} < \frac{2}{N_\varepsilon} < \varepsilon,$$

whenever $n > N_\varepsilon$.

c) Let $\varepsilon > 0$. By the Archimedean property, there is a positive integer $N_\varepsilon > \frac{13}{4} \cdot \frac{1}{\varepsilon}$. Then

$$\left|\frac{3n+1}{2n+5} - \frac{3}{2}\right| = \left|-\frac{13}{2(2n+5)}\right| = \frac{13}{2} \cdot \frac{1}{2n+5} < \frac{13}{2} \cdot \frac{1}{2n} = \frac{13}{4} \cdot \frac{1}{n} < \frac{13}{4} \cdot \frac{1}{N_\varepsilon},$$

which is smaller than ε whenever $n > N_\varepsilon$.

d) Let $\varepsilon > 0$. By the Archimedean property, there is a positive integer $N_\varepsilon > \frac{5}{4} \cdot \frac{1}{\varepsilon}$. Then

$$\left|\frac{n^2-1}{2n^2+3} - \frac{1}{2}\right| = \left|-\frac{5}{2(2n^2+3)}\right| = \frac{5}{2} \cdot \frac{1}{2n^2+3} < \frac{5}{2} \cdot \frac{1}{2n^2} \leq \frac{5}{4} \cdot \frac{1}{n} < \frac{5}{4} \cdot \frac{1}{N_\varepsilon},$$

which is smaller than ε whenever $n > N_\varepsilon$. ■

3. Show that

a) $\lim_{n\to\infty} \left(\frac{1}{\sqrt{n+7}}\right) = 0$;

b) $\lim_{n\to\infty} \left(\frac{2n}{n+2}\right) = 2$;

c) $\lim_{n\to\infty} \left(\frac{\sqrt{n}}{n+1}\right) = 0$, and

d) $\lim_{n\to\infty} \left(\frac{(-1)^n n}{n^2+1}\right) = 0$.

Proof:

a) Let $\varepsilon > 0$. By the Archimedean property, there is a positive integer $N_\varepsilon > \frac{1}{\varepsilon^2}$. Then

$$\left|\frac{1}{\sqrt{n+7}} - 0\right| = \frac{1}{\sqrt{n+7}} < \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N_\varepsilon}} < \varepsilon,$$

whenever $n > N_\varepsilon$.

b) Let $\varepsilon > 0$. By the Archimedean property, there is a positive integer $N_\varepsilon > \frac{4}{\varepsilon}$. Then

$$\left|\frac{2n}{n+2} - 2\right| = \left|-\frac{4}{n+2}\right| = \frac{4}{n+2} < \frac{4}{n} < \frac{4}{N_\varepsilon} < \varepsilon,$$

whenever $n > N_\varepsilon$.

c) Let $\varepsilon > 0$. By the Archimedean property, there is a positive integer $N_\varepsilon > \frac{1}{\varepsilon^2}$. Then

$$\left|\frac{\sqrt{n}}{n+1} - 0\right| = \frac{\sqrt{n}}{n+1} < \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N_\varepsilon}} < \varepsilon,$$

whenever $n > N_\varepsilon$.

d) Let $\varepsilon > 0$. By the Archimedean property, there is a positive integer $N_\varepsilon > \frac{1}{\varepsilon}$. Then

$$\left|\frac{(-1)^n n}{n^2+1} - 0\right| = \frac{n}{n^2+1} < \frac{n}{n^2} = \frac{1}{n} < \frac{1}{N_\varepsilon} < \varepsilon,$$

whenever $n > N_\varepsilon$. ■

4. Show that $\lim_{n\to\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)=0.$

 Proof: let $\varepsilon>0$. By the Archimedean property, there is a positive integer $N_\varepsilon>\frac{1}{\sqrt{\varepsilon}}$. Then

$$\left|\frac{1}{n}-\frac{1}{n+1}-0\right|=\frac{1}{n(1+n)}<\frac{1}{n^2}<\frac{1}{N_\varepsilon^2}<\varepsilon,$$

 whenever $n>N_\varepsilon$. ■

5. Find the limit of the following sequences:

 a) $\lim_{n\to\infty}\left(\left(2+\frac{1}{n}\right)^2\right)$;

 b) $\lim_{n\to\infty}\left(\frac{(-1)^n}{n+2}\right)$;

 c) $\lim_{n\to\infty}\left(\frac{\sqrt{n}-1}{\sqrt{n}+1}\right)$, and

 d) $\lim_{n\to\infty}\left(\frac{n+1}{n\sqrt{n}}\right)$.

 Solution: we can only use the definition if we have a candidate. Throughout, we will assume that it is known that $\frac{1}{n}\to 0$.

 a) Note that $(2+\frac{1}{n})^2=4+\frac{2}{n}+\frac{1}{n^2}$. Then, by Theorem 14 (operations on sequences and limits),

$$\frac{2}{n}=2\cdot\frac{1}{n}\to 2\cdot 0=0\quad\text{and}\frac{1}{n^2}=\frac{1}{n}\cdot\frac{1}{n}\to 0\cdot 0=0,$$

 so that $4+\frac{2}{n}+\frac{1}{n^2}\to 4+0+0=4$.

 b) Clearly,

$$\frac{-1}{n+2}\leq\frac{(-1)^n}{n+2}\leq\frac{1}{n+2},\quad\forall n\in\mathbb{N}.$$

 Note that $n+2\geq n$ for all n so that

$$0\leq\frac{1}{n+2}\leq\frac{1}{n},\quad\forall n\in\mathbb{N};$$

 as a result, $\frac{1}{n+2}\to 0$ by the squeeze theorem. Then $-\frac{1}{n+2}\to -0=0$ by Theorem 14, so that $\frac{(-1)^n}{n+2}\to 0$ by the squeeze theorem.

 c) Re-write $\frac{\sqrt{n}-1}{\sqrt{n}+1}=1-\frac{2}{\sqrt{n}+1}$. Note that

$$0\leq\frac{1}{\sqrt{n}+1}<\frac{1}{\sqrt{n}},\quad\forall n\in\mathbb{N}.$$

 We have seen that $\frac{1}{\sqrt{n}}\to 0$; as a result of the squeeze theorem, $\frac{1}{\sqrt{n}+1}\to 0$. Then $1-\frac{2}{\sqrt{n}+1}\to 1-2\cdot 0=1$, by theorem 14.

d) Note that $n \leq n\sqrt{n} \leq n^2$ for all $n \in \mathbb{N}$ so

$$\frac{1}{n^2} \leq \frac{1}{n\sqrt{n}} \leq \frac{1}{n}, \quad \forall n \in \mathbb{N}.$$

But $\frac{1}{n}, \frac{1}{n^2}, \frac{1}{\sqrt{n}} \to 0$ (see previous problems) so that $\frac{1}{n\sqrt{n}} \to 0$ by the squeeze theorem. Furthermore,

$$\frac{n+1}{n\sqrt{n}} = \frac{1}{\sqrt{n}} + \frac{1}{n\sqrt{n}} \to 0 + 0 = 0,$$

by Theorem 14. □

6. Let $y_n = \sqrt{n+1} - \sqrt{n}$. Show that (y_n) and $(\sqrt{n}y_n)$ converge.

Proof: as

$$0 \leq \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \leq \frac{1}{\sqrt{n}}, \quad \forall n \in \mathbb{N},$$

and $\frac{1}{\sqrt{n}} \to 0$, then $\sqrt{n+1} - \sqrt{n} \to 0$ by the squeeze theorem.
Note that $\sqrt{n}y_n = \sqrt{n(n+1)} - n = \frac{1}{\sqrt{1+\frac{1}{n}}+1}$ for all $n \in \mathbb{N}$. Then, according to theorem 14,

$$\lim_{n\to\infty} \sqrt{n}y_n = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{\lim_{n\to\infty}\left(\sqrt{1+\frac{1}{n}}+1\right)} = \frac{1}{2},$$

since $\sqrt{1+\frac{1}{n}} + 1 > 2$ for all $n \in \mathbb{N}$. ■

7. Let $(x_n) \subseteq \mathbb{R}^+$ be such that $x_n^{1/n} \to L < 1$ for all n. Show $\exists r \in (0,1)$ such that $0 < x_n < r^n$ for all sufficiently large $n \in \mathbb{N}$. Use this result to show that $x_n \to 0$.

Proof: since $L < 1$, $\exists \varepsilon_0 > 0$ such that $L < L + \varepsilon_0 < 1$. Then, $\exists N_0 \in \mathbb{N}$ such that

$$|x_n^{1/n} - L| < \varepsilon_0 \quad \text{whenever } n > N_0.$$

Hence $L - \varepsilon_0 < x_n^{1/n} < L + \varepsilon_0$ for all $n > N_0$. Set $r = L + \varepsilon_0$. Then $r \in (0,1)$ and

$$0 < x_n < r^n, \quad \forall n > N_0.$$

Let $\varepsilon > 0$. $r^n \to 0$ (do you know how to show this?), $\exists N_\varepsilon \geq N_0$ such that $r^n < \varepsilon$ whenever $n > N_\varepsilon$, hence

$$|x_n - 0| = x_n < r^n < \varepsilon$$

whenever $n > N_\varepsilon$. ■

8. Give an example of a convergent (resp. divergent) sequence (x_n) of positive real numbers with $x_n^{1/n} \to 1$.

Solution: the sequences $(x_n) = \frac{1}{n}$ and $(x_n) = (n)$ do the trick, among others. □

9. Let $x_1 = 1$, $x_{n+1} = \sqrt{2 + x_n}$ for $n \in \mathbb{N}$. Show that (x_n) converges; find the limit.

Proof: we show (x_n) is increasing and bounded by induction; according to the bounded monotone convergence theorem, (x_n) must then converge.

A quick computation shows that $x_2 = \sqrt{3}$.

Initial case: Clearly, $1 \leq x_1 \leq x_2 \leq 2$.

Induction hypothesis: Suppose $1 \leq x_k \leq x_{k+1} \leq 2$. Then

$$3 \leq x_k + 2 \leq x_{k+1} + 2 \leq 4$$

and so

$$1 \leq \sqrt{3} \leq \sqrt{x_k + 2} \leq \sqrt{x_{k+1} + 2} \leq \sqrt{4} = 2,$$

i.e. $1 \leq x_{k+1} \leq x_{k+2} = 2$.

Hence (x_n) is increasing and bounded above by 2; as such $x_n \to x$ for some $x \in \mathbb{R}$. But

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{2 + x_n} = \sqrt{2 + \lim_{n\to\infty} x_n} = \sqrt{2 + x},$$

that is, $x^2 = 2 + x$. The only solutions are $x = 2$ or $x = -1$, but $x = -1$ must be rejected since $1 \leq x_n$ for all n.

Thus, $x_n \to 2$.

10. Let $x_n = \sum_{k=1}^{n} \frac{1}{k^2}$ for all $n \in \mathbb{N}$. Show that (x_n) is increasing and bounded above.

Proof: as $\frac{1}{(n+1)^2} > 0$ for all $n \in \mathbb{N}$, we have

$$x_n = \frac{1}{1^2} + \cdots + \frac{1}{n^2} \leq \frac{1}{1^2} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2} = x_{n+1}.$$

Furthermore, for any $k \geq 2 \in \mathbb{N}$, we have $\frac{1}{k^2} < \frac{1}{k-1} - \frac{1}{k}$. Then

$$\begin{aligned} x_n &= \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \\ &\leq 1 + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) \\ &= 1 + 1 + 0 + \cdots + 0 - \frac{1}{n} < 2 \end{aligned}$$

for all $n \in \mathbb{N}$. Hence (x_n) is increasing and bounded above by 2.

11. Show that $c^{1/n} \to 1$ if $0 < c < 1$.

    Proof: let $x_n = c^{1/n}$ for all $n \in \mathbb{N}$. Since $x_{n+1} = c^{1/(n+1)} > c^{1/n} = x_n$ for all $n \in \mathbb{N}$ (as $c < 1$), then (x_n) is increasing. Furthermore, $0 < c^{1/n} < 1^{1/n} = 1$ for all $n \in \mathbb{N}$, so (x_n) is bounded above.

    Hence (x_n) converges, and $x_n \to x$, for some $x \in \mathbb{R}$. As all subsequences of a convergent sequence converge to the same limit as the convergent sequence, $x_{2n} = c^{1/2n} \to x$. As such,

    $$x = \lim_{n\to\infty} c^{1/2n} = \lim_{n\to\infty} \sqrt{c^{1/n}} = \lim_{n\to\infty} \sqrt{x_n} = \sqrt{\lim_{n\to\infty} x_n} = \sqrt{x},$$

    and so either $x = 0$ or $x = 1$. But as x_n increases to 1, there comes a point after which all x_n are "far" from 0 (you should mathematicize this statement...), so $x_n \to 1$. ■

12. Let (x_n) be a bounded sequence and let $s_n = \sup\{x_k : k \geq n\}$. If $S = \inf\{s_n\}$, show that there is a subsequence of (x_n) that converges to S.

    Proof: as (x_n) is bounded, $\exists M > 0$ such that $-M < x_n < M$ for all $n \in \mathbb{N}$. By definition, $s_1 \geq s_2 \geq \cdots$ and $s_n \geq x_k$ for all $n \in \mathbb{N}$, $k \geq n$.

    Hence $s_n > -M$ for all n and (s_n) is bounded below and decreasing, i.e. (s_n) is convergent. Furthermore, for each $n \in \mathbb{N}$, as $s_n = \sup\{x_k : k \geq n\}$, $\exists k_n \in \mathbb{N}$ s.t.

    $$s_n - \frac{1}{n} \leq x_{k_n} < s_n$$

    (otherwise s_n is not the supremum).

    The sequence (x_{k_n}) might not necessarily be a subsequence of (x_n), but by deleting the terms that are out of order, the resulting sequence, which we will also denote by (x_{k_n}) is a subsequence of (x_n).

    Then

    $$0 \leq |x_{k_n} - s_n| \leq \frac{1}{n}, \quad \forall n \in \mathbb{N}.$$

    By the squeeze theorem,

    $$0 \leq \lim_{n\to\infty} |x_{k_n} - s_n| \leq 0, \quad \text{so } \lim_{n\to\infty} |x_{k_n} - s_n| = 0.$$

    But this means that

    $$\lim_{n\to\infty} x_{k_n} = \lim_{n\to\infty} s_n = S, \quad \text{(why?)}$$

    where the last equation comes from the theorem on bounded increasing/decreasing sequences. ■

13. Suppose that $x_n \geq 0$ for all $n \in \mathbb{N}$ and that $((-1)^n x_n)$ converges. Show that (x_n) converges.

Proof: Let $(-1)^n x_n \to \alpha$. Consider its subsequences

$$\left((-1)^{2n} x_{2n}\right) = (x_{2n}) \quad \text{and} \quad \left((-1)^{2n+1} x_{2n+1}\right) = (-x_{2n+1}).$$

Then $x_{2n} \to \alpha$ and $(-x_{2n+1}) \to \alpha$. But $x_{2n} \geq 0 \; \forall n \in \mathbb{N}$ so $\alpha \geq 0$. Similarly, $-x_{2n+1} \leq 0 \; \forall n \in \mathbb{N}$ so $\alpha \leq 0$. Since $0 \leq \alpha \leq 0$, we must then have $\alpha = 0$. By Theorem 14 (operations on limits), we have:

$$\lim_{n\to\infty} |(-1)^n x_n| = |0| = 0.$$

But $|(-1)^n x_n| = x_n \; \forall n$, so $x_n \to 0$. ■

14. Show that if (x_n) is unbounded, there exists a subsequence (x_{n_k}) with $1/x_{n_k} \to 0$.

Proof: as (x_n) is unbounded, $\exists n_1 \in \mathbb{N}$ such that $|x_{n_1}| \geq 1$. Moreover, $\forall k \geq 2$, $\exists n_k \in \mathbb{N}$ such that $|x_{n_k}| \geq k$ and $n_{k+1} > n_k$ (otherwise the sequence would be bounded).

Let $\varepsilon > 0$. According to the Archimedean property, $\exists K_\varepsilon \in \mathbb{N}$ such that $K_\varepsilon > \frac{1}{\varepsilon}$ and

$$\left|\frac{1}{x_{n_k}} - 0\right| = \frac{1}{|x_{n_k}|} \leq \frac{1}{k} < \frac{1}{K_\varepsilon} < \varepsilon$$

whenever $k > K_\varepsilon$. Thus, $1/x_{n_k} \to 0$. ■

15. If $x_n = \frac{(-1)^n}{n}$, find the convergent subsequence in the proof of the Bolzano-Weierstrass theorem, with $I_1 = [-1, 1]$.

Proof: we first note that (x_n) is bounded by -1 and 1, so the question makes sense. Let $n_1 = 1$. Then $x_{n_1} = x_1 = -1$ and length$(I_1) = 2$. Set $I_1' = [-1, 0]$ and $I_1'' = [0, 1]$.

We have

$$A_1 = \{n \in \mathbb{N} \mid n > n_1 \text{ and } x_n \in I_1'\} = \{3, 5, 7, 9, 11, \ldots\}$$

and

$$B_1 = \{n \in \mathbb{N} \mid n > n_1 \text{ and } x_n \in I_1''\} = \{2, 4, 6, 8, 10, \ldots\}.$$

Since A_1 is infinite (why?), set $I_2 = I_1' = [-1, 0]$ and $n_2 = \min A_1 = 3$, so that $x_{n_2} = -1/3$. Note that $n_2 > n_1$, $I_2 \subseteq I_1$, and length$(I_2) = 1$. Set $I_2' = [-1, -1/2]$ and $I_2'' = [-1/2, 0]$.

We have

$$A_2 = \{n \in \mathbb{N} \mid n > n_2 \text{ and } x_n \in I_2'\} = \varnothing$$

and

$$B_2 = \{n \in \mathbb{N} \mid n > n_2 \text{ and } x_n \in I_2''\} = \{5, 7, 9, 11, 13, \ldots\}.$$

Since A_2 is finite, set $I_3 = I_2'' = [-1/2, 0]$ and $n_3 = \min B_2 = 5$, so that $x_{n_3} = -1/5$. Note that $n_3 > n_2 > n_1$, $I_3 \subseteq I_2 \subseteq I_1$, and length$(I_3) = 1/2$.

For $k \geq 3$, we set $I'_k = [-1/2^{k-2}, -1/2^{k-1}]$ and $I''_k = [-1/2^{k-1}, 0]$. Then

$$A_k = \{n \in \mathbb{N} \mid n > n_k \text{ and } x_n \in I'_k\} = \varnothing$$

and

$$B_k = \{n \in \mathbb{N} \mid n > n_k \text{ and } x_n \in I''_k\} = \{2k+1, 2k+3, 2k+5, \dots\}.$$

A_k is finite, so set $I_{k+1} = I''_k = [-1/2^{k-1}, 0]$. Furthermore, $n_{k+1} = \min B_k = 2k+1$ so that $x_{n_k} = \frac{-1}{2k+1}$.

Note that $n_{k+1} > n_k > \cdots > n_2 > n_1$, $I_{k+1} \subseteq I_k \subseteq \cdots \subseteq I_2 \subseteq I_1$ and $\text{length}(I_{k+1}) = 1/2^{k-2}$. The convergent subsequence is thus $-1, -1/3, -1/5, \ldots \to 0$. ∎

16. Show directly that a bounded increasing sequence is a Cauchy sequence.

Proof: let $\varepsilon > 0$. By completeness of $\mathbb{R}$, $x^* = \sup\{x_n \mid n \in \mathbb{N}\}$ exists as $\{x_n \mid n \in \mathbb{N}\}$ is bounded and non-empty. In particular, $\exists M_{\frac{\varepsilon}{2}} \in \mathbb{N}$ such that

$$x^* - \frac{\varepsilon}{2} < x_{M_{\frac{\varepsilon}{2}}} \leq x^*.$$

But $x^* \geq x_n > x_{M_{\frac{\varepsilon}{2}}}$ whenever $n > M_{\frac{\varepsilon}{2}}$.

Let $N_\varepsilon = M_{\frac{\varepsilon}{2}}$. Then

$$|x_m - x_n| = |x_m - x^* + x^* - x_n| \leq |x^* - x_m| + |x^* - x_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever $m, n > N_\varepsilon$. ∎

17. If $0 < r < 1$ and $|x_{n+1} - x_n| < r^n$ for all $n \in \mathbb{N}$, show that (x_n) is Cauchy.

Proof: let $\varepsilon > 0$. By the Archimedean property, $\exists N_\varepsilon > \log_r(\varepsilon(1-r)) + 1$, i.e. $r^{N_\varepsilon - 1} < \varepsilon$. Then

$$\begin{aligned} |x_m - x_n| &\leq |x_m - x_{m-1}| + \cdots + |x_{n+1} - x_n| \\ &< r^{m-1} + \cdots + r^n < \frac{r^{n-1}}{1-r} < \frac{r^{N_\varepsilon - 1}}{1-r} < \varepsilon \end{aligned}$$

whenever $m > n > N_\varepsilon$.[6] ∎

18. If $x_1 < x_2$ and $x_n = \frac{1}{2}(x_{n-1} + x_{n-2})$ for all $n \in \mathbb{N}$, show that (x_n) is convergent and compute its limit.

Proof: we start by showing that (x_n) is Cauchy. Let $L = x_2 - x_1$; by induction,

$$|x_n - x_{n-1}| \leq \frac{L}{2^{n-2}}.$$

[6] The third last inequality holds since $r^{m-1} + \cdots + r^n$ is a geometric progression.

Let $\varepsilon > 0$. By the Archimedean property, $\exists N_\varepsilon \in \mathbb{N}$ such that $\frac{L}{2^{N_\varepsilon - 2}} < \varepsilon$. Then

$$\begin{aligned}|x_m - x_n| &\leq |x_m - x_{m-1}| + \cdots + |x_{n+1} - x_n| \\ &\leq \frac{L}{2^{m-2}} + \cdots \frac{L}{2^{n-1}} \leq \frac{L}{2^{n-2}} < \frac{L}{2^{N_\varepsilon - 2}} < \varepsilon\end{aligned}$$

whenever $m > n > N_\varepsilon$. Hence (x_n) is a Cauchy sequence, and so it converges, say to $x_n \to x$. We can show by induction (do it!) that

$$x_{2n+1} = x_1 + \frac{L}{2} + \frac{L}{2^3} + \cdots + \frac{L}{2^{2n-1}}$$

for all $n \in \mathbb{N}$. In particular,

$$\begin{aligned}x = \lim_{n\to\infty} x_{2n+1} &= x_1 + \lim_{n\to\infty}\left(\frac{L}{2} + \frac{L}{2^3} + \cdots + \frac{L}{2^{2n-1}}\right) \\ &= x_1 + \frac{L}{2}\lim_{n\to\infty}\left(1 + \frac{1}{2^2} + \cdots + \frac{1}{2^{2n-2}}\right) \\ &= x_1 + \frac{L}{2}\lim_{n\to\infty}\left(\frac{1-(1/2^2)^n}{1-(1/2^2)}\right) = x_1 + \frac{2}{3}L = \frac{1}{3}(x_1 + 2x_2).\end{aligned}$$

For instance, when $x_1 = 1$ and $x_2 = 2$, $x_n \to 5/3$. ■

19. Suppose that (a_n) is a bounded sequence and $b_n \to 0$. Show that $a_n b_n \to 0$.

Proof: since (a_n) is bounded, there exists some $0 \leq M < \infty$ so that $\sup_n |a_n| \leq M$. Next, we will check that $a_n b_n \to 0$.

Fix some $\varepsilon > 0$. Since $b_n \to 0$, there exists some N_ε so that for all $n > N_\varepsilon$, $|b_n| \leq \frac{\varepsilon}{M}$. Thus, for all $n > N_\varepsilon$,

$$|a_n b_n| \leq M|b_n| \leq M\,\frac{\varepsilon}{M} = \varepsilon.$$

Thus, $a_n b_n \to 0$. ■

20. Let (a_n) be a sequence with no convergent subsequences. Show that $|a_n| \to \infty$.

Proof: we prove this by contradiction. Assume that $|a_n|$ does not diverge to infinity. Then there exists some $M < \infty$ such that the set $\{n \in \mathbb{N} \mid |a_n| < M\}$ is infinite. Let

$$1 \leq m_1 \leq m_2 \leq m_3 \leq \ldots$$

be the indices satisfying $|a_{m_n}| < M$. Set $b_n = a_{m_n}$. Then $\{b_n\}$ is a bounded sequence and so has a convergent subsequence $\{b_{k_n}\}_n$ according to the Bolzano-Weierstrass theorem.

But $\{a_{m_{k_n}}\}_n = \{b_{k_n}\}_n$ is in fact a convergent subsequence of (a_n), contradicting the information given in the question. We conclude that our assumption was false, and so that $|a_n|$ diverges to infinity. ■

21. We define the **limit inferior** and the **limit superior** of a sequence as follows:

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf\{a_k \mid k \geq n\}$$
$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup\{a_k \mid k \geq n\}.$$

Let (a_n) be bounded. Show that $\liminf_{n\to\infty} a_n$ and $\limsup_{n\to\infty} a_n$ exist and are in $\mathbb{R}$.

Proof: define the sequence of sets $B_n = \{a_k \mid k \geq n\}$ and the sequence of numbers $b_n = \sup(B_n)$, so that

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} b_n.$$

We note that $B_1 \supset B_2 \supset \ldots$, which implies $\sup(B_1) \leq \sup(B_2) \leq \ldots$, which means that $\{b_n\}$ is monotone decreasing. Furthermore, since (a_n) is bounded, there exists some $-\infty < M < \infty$ so that $a_n \geq M$ for all $n \in \mathbb{N}$.

But this M is a lower bound for (a_n), which means it must be a lower bound for B_n for all $n \in \mathbb{N}$, which means $b_n = \sup(B_n) \geq M$ for all $n \in \mathbb{N}$ as well.

Thus, we have shown that $\{b_n\}$ is a monotone decreasing sequence that is bounded from below. Hence, by the monotone convergence theorem, it has a limit and so

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

exists. The proof for the lim inf statement follows a similar path. ■

22. Let (a_n) be unbounded. Show that $\liminf_{n\to\infty} a_n = -\infty$ or $\limsup_{n\to\infty} a_n = \infty$.

Proof: since (a_n) is unbounded, for all $0 < M < \infty$, there exists $n = n(M)$ satisfying $|a_n| > M$.

Define the subsequence $\{b_k\}$ by setting $b_k = a_{n(k)}$, so that $|b_k| > k$ for all $k \in \mathbb{N}$. Since this is an infinite sequence, we have by the Pigeonhole Principle that at least one of the two sets $I_+ = \{k \in \mathbb{N} \mid b_k \geq 0\}$, $I_- = \{k \in \mathbb{N} \mid b_k \leq 0\}$ is infinite.

In the case that I_+ is infinite, write the elements $i_1 < i_2 < i_3 < \ldots$ in order and define the subsequence $\{c_\ell\}$ of $\{b_n\}$ by the formula $c_\ell = b_{i_\ell} = a_{n(i_\ell)}$. But then for all n, we have

$$\begin{aligned}\sup\{a_k \mid k \geq n\} &\geq \sup\{a_{n(i_\ell)} \mid \ell \geq n\} \\ &= \sup\{c_k \mid k \geq n\} \geq \sup\{k \mid k \geq n\} = \infty.\end{aligned}$$

Thus,

$$\limsup_{n\to\infty} a_n = \infty.$$

The case that I_- is infinite is essentially the same, with the conclusion

$$\liminf_{n\to\infty} a_n = -\infty.$$

This completes the proof.[7] ■

23. Let (a_n), (b_n) be two sequences. Show that

$$\liminf_{n\to\infty} a_n + \limsup_{n\to\infty} b_n \leq \limsup_{n\to\infty}(a_n+b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

Proof: fix $\varepsilon > 0$. Then there exists some $N_\varepsilon \in \mathbb{N}$ such that, for all $m > N_\varepsilon$, the following inequalities all hold:

$$\frac{\varepsilon}{2} + \limsup_{n\to\infty} a_n \geq a_m \geq -\frac{\varepsilon}{2} + \liminf_{n\to\infty} a_n;$$
$$\frac{\varepsilon}{2} + \limsup_{n\to\infty} b_n \geq b_m \geq -\frac{\varepsilon}{2} + \liminf_{n\to\infty} b_n.$$

Adding the left-hand sided inequalities, we get:

$$a_m + b_m \leq \varepsilon + \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

We conclude with our first desired inequality,

$$\limsup_{n\to\infty}(a_n+b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

To obtain the reverse inequality, again fix $\varepsilon > 0$. Then there exists a sequence $\{k_n\}$ so that

$$b_{k_m} \geq -\frac{\varepsilon}{2} + \limsup_{n\to\infty} b_n \quad \text{for all } m.$$

Chopping off the finitely-many terms in the sequence occurring before the threshold N_ε and applying the above inequalities, we have, for all $m \in \mathbb{N}$:

$$a_{k_m} + b_{k_m} \geq -\frac{\varepsilon}{2} + \liminf_{n\to\infty} a_n - \frac{\varepsilon}{2} + \limsup_{n\to\infty} b_n.$$

We conclude with the desired reverse inequality,

$$\limsup_{n\to\infty}(a_n+b_n) \geq \liminf_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

For the second question, consider the sequences

$$a_n = (-1)^n,\ b_n = (-1)^{n+1}.$$

Thus $a_n + b_n = 0$ for all n, so $\limsup_{n\to\infty}(a_n+b_n) = 0$. However,

$$\limsup_{n\to\infty} a_n = \limsup_{n\to\infty} b_n = 1,$$

which completes the proof. ■

[7] As an aside, if I_-, I_+ are both infinite, then we have

$$\limsup_{n\to\infty} a_n = \infty, \quad \liminf_{n\to\infty} a_n = -\infty,$$

which you can check holds for sequences such as $a_n = (-n)^n$, say.

2.8 Exercises

1. Prepare a 2-page summary of this chapter, with important definitions and results.
2. Is the converse of Theorem 13 true?
3. Let $|q| < 1$. Compute $\lim_{n\to\infty} nq^n$, if the limit exists.
4. Let (x_n) be a decreasing sequence, bounded below. Show that $x_n \to \inf\{x_n \mid n \in \mathbb{N}\}$.
5. Find a divergent sequence with convergent subsequences.
6. Show directly that the sum of two Cauchy sequences is a Cauchy sequence.
7. Show directly that every bounded Cauchy sequence admits at least one convergent subsequence.
8. Complete the induction argument that allows you to compute the limit of the sequence defined by $x_n = \frac{1}{2}(x_{n-2} + x_{n-1})$, $n > 2$, with $x_1 = 1$ and $x_2 = 2$.
9. Show that $(x_n) = \frac{1}{n}$ and $(x_n) = (n)$ are both positive real sequences with $x_n^{1/n} \to 1$, even though one converges and one diverges.
10. Complete the proof of solved problem 21 (do the lim inf case). Consider the sequence given by the recursion $a_{n+1} = \frac{1}{2}(a_n + a_n^{-1})$, with some initial condition $a_1 \in (-\infty, 0) \cup (0, \infty)$. Find and prove the limit, if it exists.