Chapter 20

Introduction to Algebraic Topology

While there are tons of other interesting results and counter-examples in **point set topology**, we have touched upon most of the important ideas of the discipline in Chapters 15-19. In this chapter, we introduce the basic concepts of **algebraic topology**, which is both a precursor and an application of **category theory**, and which provides a stepping stone to **homology theory**, a fascinating (but out-of-scope) offshoot of general topology.

20.1 Fundamental Groups

A **path** in a space X from x to y is a continuous function $p : I = [0,1] \to X$ where $p(0) = x$ and $p(1) = y$. A **path homotopy** between 2 paths p_0 and p_1 from x_0 to x_1 is a continuous function $F : I \times I \to X$, where

$$F(t,0) \ = p_0(t), \quad F(t,1) = p_1(t), \quad F(0,s) = x_0, \quad F(1,s) = x_1.$$

If such an F exists, we say that p_0 is **(path) homotopic to** p_1 **under** F, which we denote by $p_0 \sim_F p_1$, or $p_0 \sim p_1$ if the dependence on F does not need to be emphasized. Path homotpy is an **equivalence relation** on the set of paths.

Reflexivity: if p is a path from x_0 to x_1 in X, set $F(t,s) = p(t)$ for all s,t. Then $p \sim_F p$.

Symmetry: if p_0, p_1 are homotopic paths from x_0 to x_1 with $p_0 \sim_F p_1$, set $G(t,s) = F(t, 1-s)$ for all s,t. Then $p_1 \sim_G p_0$.

Transitivity: let p_0 and p_1 be paths from x_0 to x_1 with $p_0 \sim_F p_1$, and let p_1 and p_2 be paths from x_1 to x_2 with $p_1 \sim_G p_2$. Then $p_0 \sim_H p_2$, where

$$H(s,t) = \begin{cases} F(t,2s) & s \in [0,1/2] \\ G(t,2s-1) & s \in [1/2,1] \end{cases}$$

for all $t,s \in I$. By the pasting lemma (Lemma 213), H is continuous since $F(t,1) = G(t,0)$ for all $t \in I$.

Examples (PATH HOMOTOPIES)

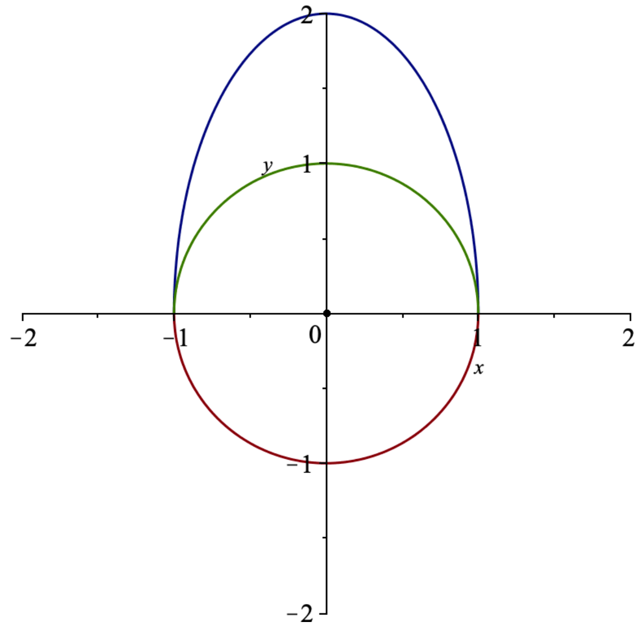
1. Let p and q be any paths with the same endpoints in $\mathbb{R}^n$. Then $p \sim_F q$ where

$$F(t,s) = (1-s)p(t) + sq(t).$$

This path homotopy is called the **straight-line homotopy**.

2. Let p, q, and r be paths from $x_0 = (1,0)$ to $x_1 = (-1,0)$ in the punctured plane $\mathbb{R}^2 \setminus \{0\}$, defined by:

$$\begin{aligned} p(s) &= (\cos \pi s, \sin \pi s) \quad \text{(in green)}, \\ q(s) &= (\cos \pi s, 2\sin \pi s) \quad \text{(in blue)}, \\ r(s) &= (\cos \pi s, -\sin \pi s) \quad \text{(in red)}. \end{aligned}$$



Then p and q are path homotopic (through the straight-line homotopy, say). But p and r are not path homotopic – we will prove this at a later point. □

The **equivalence class of a path** p is denoted by $[p]$. We show that the equivalence classes of paths behave very much like the elements of a **group**. Let X be a topological space.

Composition If p, q are paths in X from x_0 to x_1 and from x_1 to x_2, respectively, then pq is a path from x_0 to x_2, and we have:

$$pq(t) = \begin{cases} p(2t) & t \in [0, 1/2], \\ q(2t-1) & t \in [1/2, 1]. \end{cases}$$

If $p_0 \sim_F p_1$ from x_0 to x_1 and $q_0 \sim_G q_1$ from x_1 to x_2, define $H : I \times I \to X$ by

$$H(t,s) = \begin{cases} F(2t,s) & t \in [0,1/2] \\ G(2t-1,s) & t \in [1/2,1]. \end{cases}$$

By the pasting lemma, H is continuous since $F(1,s) = G(0,s) = x_1$. Hence $p_0q_0 \sim_H p_1q_1$. Whenever the composition pq is defined, we can define the product of the path classes by $[p][q] = [pq]$.

Associativity If p, q, r are paths in X from x_0 to x_1, x_1 to x_2 and x_2 to x_3 respectively, then $(pq)r$ and $p(qr)$ are paths from x_0 to x_3, and we have:

$$(pq)r(t) = \begin{cases} p(4t) & t \in [0,1/4], \\ q(4t-1) & t \in [1/4,1/2], \\ r(2t-1) & t \in [1/2,1]. \end{cases}$$

$$p(qr)(t) = \begin{cases} p(2t) & t \in [0,1/2], \\ q(4t-2) & t \in [1/2,3/4], \\ r(4t-3) & t \in [3/4,1]. \end{cases}$$

Clearly, $(pq)r \neq p(qr)$. But $(pq)r \sim_F p(qr)$, where

$$F(t,s) = \begin{cases} p\left(\frac{4t}{s+1}\right) & 0 \leq t \leq \frac{1}{4}(s+1), \\ q(4t-1-s) & \frac{1}{4}(s+1) \leq t \leq \frac{1}{4}(s+2), \\ r\left(\frac{4t-s-2}{2-s}\right) & \frac{1}{4}(s+2) \leq t \leq 1. \end{cases}$$

Hence $([p][q])[r] = [p]([q][r])$ whenever these multiplications are defined.

Identities The **constant path** c_x at x is defined by $c_x(t) = x$ for all $t \in I$. If p is a path from x to y, then $c_x \sim_F p \sim_G pc_y$. One gets

$$c_xp(t) = \begin{cases} x & t \in [0,1/2], \\ q(2t-1) & t \in [1/2,1]. \end{cases}$$

$$pc_y(t) = \begin{cases} p(2t) & t \in [0,1/2], \\ y & t \in [1/2,1]. \end{cases}$$

Then

$$F(t,s) = \begin{cases} x & t \in [0,(1-s)/2], \\ p\left(\frac{2t+s-1}{s+1}\right) & t \in [0,(1-s)/2], \end{cases}$$

$$G(t,s) = \begin{cases} p\left(\frac{2t}{2-s}\right) & t \in [0,1-s/2], \\ y & t \in [1-s/2,1]. \end{cases}$$

Then F and G are the required homotopies. Hence for any path p from x to y, $[c_x][p] = [p]$ and $[p] = [p][c_y]$.

Inverses If p is a path in X from x to y, then $\overline{p}$ is a path from y to x defined by $\overline{p}(t) = p(1-t)$ with $p\overline{p} \sim_F c_x$ and $\overline{p}p \sim_G c_y$, where

$$F(t,s) = \begin{cases} p\left(2t\right) & 0 \leq t \leq \frac{s}{2}, \\ p(s) & \frac{s}{2} \leq t \leq 1 - \frac{s}{2}, \\ p(2-2t) & 1 - \frac{s}{2} \leq t \leq 1. \end{cases}$$

Note that $\overline{\overline{p}} = p$, so we get

$$\begin{aligned} G(t,s) &= \begin{cases} \overline{p}\left(2t\right) & 0 \leq t \leq \frac{s}{2}, \\ \overline{p}(s) & \frac{s}{2} \leq t \leq 1 - \frac{s}{2}, \\ \overline{p}(2-2t) & 1 - \frac{s}{2} \leq t \leq 1. \end{cases} \\ &= \begin{cases} p\left(1-2t\right) & 0 \leq t \leq \frac{s}{2}, \\ p(1-s) & \frac{s}{2} \leq t \leq 1 - \frac{s}{2}, \\ p(2t-1) & 1 - \frac{s}{2} \leq t \leq 1. \end{cases} \end{aligned}$$

Hence $[p][\overline{p}] = [c_x]$ and $[\overline{p}][p] = [c_y]$, which means that $[\overline{p}] = [p]^{-1}$.

But it is not always possible to multiply path classes, as two paths may not have matching endpoints, so the group idea is **not complete**. To remedy the situation, we introduce a new concept. A path in X from x to x is a **loop** in X based at x. When p is a loop at x we call the path class $[p]$ a loop at x.

For a fixed $x_0 \in X$, if we consider only loops based at x_0, then pq is always defined. This means that the composition of path classes is always defined and so, for any path classes α, β, γ, with ε the path class of the constant path c_{x_0}, we have

$$(\alpha\beta)\gamma = \alpha(\beta\gamma), \quad \alpha\varepsilon = \varepsilon\alpha = \alpha, \quad \alpha\alpha^{-1} = \alpha^{-1}\alpha = \varepsilon;$$

the path classes of loops in X at x_0 thus form a group, the **fundamental group of X based at** x_0, denoted by $\pi(X, x_0)$. It is also sometimes known as the **first homotopy group of X at** x_0, denoted by $\pi_1(X, x_0)$. The fundamental group does depend on the chosen base point.

Examples (Fundamental Group)

1. If $X = \mathbb{R}^n$ and $x_0 = 0$, then $\pi(\mathbb{R}^n, 0) = \{\varepsilon\}$, as every loop at 0 is path homotopic to the constant loop c_0.

2. If X is any convex subset of $\mathbb{R}^n$ and $x_0 \in X$, then $\pi(X, x_0) = \{\varepsilon\}$, as every loop at x_0 is path homotopic to the constant loop c_{x_0} through the straight-line homotopy.

3. If $X = \mathbb{R}^n \setminus \{0\}$ and p, q and r are defined as in the 2nd example on p. 458, then $p\overline{q}$ and $p\overline{r}$ are two loops based at $(-1, 0)$. But these loops are not path homotopic and so their path classes differ, which means that $\pi(X, (-1, 0))$ is not the trivial group. The fundamental group of the punctured plane will be computed in Section 20.3.

If X is a path-connected space for which there exists $x_0 \in X$ such that $\pi(X, x_0) = \{\varepsilon\}$, we say that X is **simply connected**. The reason why we only need one $x_0 \in X$ is that the fundamental groups of path-connected spaces are independent of the chosen base point.

Theorem 270
If X is path-connected, then $\pi(X,x) \cong \pi(X,y)$ for $x, y \in X$.

Proof: As X is path-connected, there is a path class γ from x to y. Define $\hat{\gamma} : \pi(X,x) \to \pi(X,y)$ by $\hat{\gamma}(\alpha) = \gamma^{-1}\alpha\gamma$. We show that $\hat{\gamma}$ is the desired isomorphism. First, let $\alpha, \beta \in \pi(X,x)$. Then

$$\hat{\gamma}(\alpha)\hat{\gamma}(\beta) = \gamma^{-1}\alpha\gamma\gamma^{-1}\beta\gamma = \gamma^{-1}\alpha\beta\gamma = \hat{\gamma}(\alpha\beta),$$

so $\hat{\gamma}$ is a homomorphism. The reverse class $\overline{\gamma}$ also provides a fundamental group homomorphism $\widehat{\overline{\gamma}} : \pi(X,y) \to \pi(X,x)$ defined by $\widehat{\overline{\gamma}}(\xi) = \gamma\xi\gamma^{-1}$. Then $\hat{\gamma}^{-1} = \widehat{\overline{\gamma}}$, which implies that γ is an isomorphism. ■

In the proof of Theorem 270, if we use a different path class δ from x to y, we get a different isomorphism $\widehat{\delta} : \pi(X,x) \to \pi(X,y)$. But

$$\widehat{\delta}^{-1}\widehat{\gamma}(\alpha) = \delta\gamma^{-1}\alpha\gamma\delta^{-1} = (\delta\gamma^{-1})\alpha(\delta\gamma^{-1})^{-1}$$

for all $\alpha \in \pi(X,x)$. Hence $\widehat{\delta}$ and $\widehat{\gamma}$ differ by an **inner automorphism**.

Suppose $\varphi : X \to Y$ is a continuous function and $p : I \to X$ is a path, then $\varphi \circ p : I \to Y$ is a path, denoted φp. If the composition pq is defined, then $\varphi(pq) = (\varphi p)(\varphi q)$. Thus, if $p \sim_F q$, then $\varphi p \sim_{\varphi F} \varphi q$, and φ induces a homomorphism of path classes

$$\varphi^* : \pi(X,x) \to \pi(Y, \varphi(x)),$$

defined by $\varphi^*([p]) = [\varphi p]$ for all $[p] \in \pi(X,x)$. If furthermore $\psi : Y \to Z$ is a continuous function, then $(\psi\varphi)^* = \psi^*\varphi^*$. From this, if φ is a homeomorphism, $(\varphi^{-1})^* = (\varphi^*)^{-1}$ and φ^* is an isomorphism. As a result, if X is homeomorphic to Y, then $\pi(X,x)$ is isomorphic to $\pi(Y,\varphi(x))$, where φ is the homeomorphism between X and Y.

Corollary 271
If $\pi(X,x) \ncong \pi(Y,y)$, then X and Y are not homeomorphic.

Note that φ^* need not be surjective (injective) when φ is surjective (injective).

1. Let $X = \mathbb{R}$, $Y = S^1$ and define $\varphi : \mathbb{R} \to S^1$ by $\varphi(x) = e^{2\pi i x}$. Then φ is continuous and surjective, and $\varphi(0) = 1$. But $\pi(\mathbb{R}, 0) = \{\varepsilon_0\}$, so $\varphi^*(\pi(\mathbb{R}, 0)) = \{\varepsilon_1\}$. As we shall see in Section 20.3, $\pi(S^1, 1) = \mathbb{Z}$. Hence φ^* is not surjective.

2. Let $X = S^1$, $Y = \mathbb{C}$, and $\varphi : S^1 \to \mathbb{C}$ with $\varphi(z) = z$. Then φ is continuous and injective, and $\varphi(1) = 1$. But $\pi(S^1, 1) = \mathbb{Z}$ and $\pi(\mathbb{C}, 1) = \{\varepsilon_1\}$, so ker $\varphi^* = \pi(S^1, 1) \neq \{\varepsilon_1\}$ and φ^* is not injective.

20.2 Covering Spaces

Suppose $p : \widetilde{X} \to X$ is a continuous map. Let V be a neighbourhood of $x \in X$. We say that V **is evenly covered by** p **at** x if $p^{-1}(V)$ can be written as a disjoint union of sets $\widetilde{V}$ (the **slices** of $p^{-1}(V)$) such that the restriction $p|_{\widetilde{V}} : \widetilde{V} \to V$ is a homeomorphism. If for every $x \in X$, there is some neighbourhood V of x that is evenly covered by p, then p is a **covering map** and $(\widetilde{X}, p)$ is a **covering space of** X. Note that a covering map is automatically surjective.

Example (Covering Spaces)

1. Let $\widetilde{X} = \mathbb{R}$, $X = S^1$ and define $p : \mathbb{R} \to S^1$ by $p(\widetilde{x}) = e^{2\pi i \widetilde{x}}$. Let $z \in S^1$. Then there exists $\theta_z \in \mathbb{R}$ such that $z = e^{2\pi i \theta_z}$ and $p^{-1}(z) = \{\theta_z + n \mid n \in \mathbb{Z}\}$. Let $V_z = \left\{ e^{2\pi i \phi} \mid |\phi - \theta_z| < \frac{1}{2} \right\}$. We show that V_z is evenly covered by p and so that $(\mathbb{R}, p)$ is a covering space of S^1. Note that $p^{-1}(V_z) = \bigsqcup_{n \in \mathbb{Z}} \widetilde{V}_n$, where $\widetilde{V}_n = (\theta_z + n - \frac{1}{2}, \theta_z + n - \frac{1}{2})$ for all $n \in \mathbb{Z}$. But, for all $n \in \mathbb{Z}$,

$$p(\widetilde{V}_n) \quad = \{e^{2\pi i \phi} \mid \phi \in \widetilde{V}_n\} = \{e^{2\pi i \phi} \mid |\phi - \theta_z| < 1/2\} = V_z,$$

   and $p|^{-1}_{\widetilde{V}_n}(V_z) = \widetilde{V}_n$, so $p|_{\widetilde{V}_n}$ is an homeomorphism and V_z is evenly covered.

2. Let $p : \widetilde{X} \to X$ be a homeomorphism. Then every open set $U \subseteq X$ is evenly covered by p since $p^{-1}(U) \simeq U$. Hence $(\widetilde{X}, p)$ is a covering space of X.

3. Let $\widetilde{X} = S^1$, $X = S^1$ and define $p : S^1 \to S^1$ by $p(z) = z^n$, for all $z \in S^1$ and for some $n \in \mathbb{Z}$. Let $z \in S^1$. Then there exists $\theta_z \in \mathbb{R}$ such that $z = e^{2\pi i \theta_z}$. By definition, $p^{-1}(z) = \left\{ e^{\frac{2\pi i m}{n} \theta_z} \mid 0 \le m \le n-1 \right\}$. Let $U_z = \left\{ e^{2\pi i \phi} \mid |\phi - \theta_z| < \frac{1}{4n} \right\}$. We show that U_z is evenly covered by p and so that (S^1, p) is a covering space of S^1. But $p^{-1}(U_z) = \bigsqcup_{m=0}^{n-1} \widetilde{U}_m$, where $\widetilde{U}_m = \left\{ e^{2\pi i \phi} \mid |\phi + m - \theta_z| < \frac{1}{4n} \right\}$ for all $0 \le m \le n-1$, and so

$$p(\widetilde{U}_m) = \{e^{2\pi i \phi} \mid |\phi - \theta_z| < 1/(4n)\} = U_z$$

   for all $0 \le m \le n-1$, hence $p|^{-1}_{\widetilde{U}_m}(U_z) = \widetilde{U}_n$, so $p|_{\widetilde{U}_n}$ is an homeomorphism and U_z is evenly covered.

4. Let $\widetilde{X} = S^2$ and $X = \mathbb{R}P^2$ be the real projective plane. Then the quotient map $p : S^2 \to \mathbb{R}P^2$ where $p(v) = p(-v)$ for all $v \in S^2$ is a covering map.

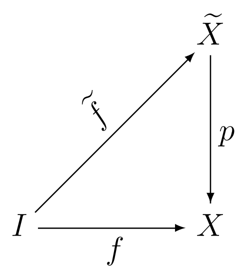
A continuous function $f : X \to Y$ is a **local homeomorphism** if for each $x \in X$, there is a neighbourhood V of x such that $f|_V : V \to f(V)$ is a homeomorphism. Consequently, every covering map is a local homeomorphism. But the converse is not necessarily true.

Example let $X = \mathbb{R}^+$, $Y = S^1$ and define $p : \mathbb{R}^+ \to S^1$ by $p(x) = e^{2\pi i x}$ for $x \in \mathbb{R}^+$. Then p is continuous and surjective. Let $x \in \mathbb{R}^+$. Any basic neighbourhood $(x - \varepsilon, x + \eta)$ in $\mathbb{R}^+$, where $\varepsilon + \eta < 1/2$ is mapped homeomorphically to $\left\{e^{2\pi i \phi} \mid -\varepsilon < \phi - x < \eta\right\}$ by p. This makes p a local homeomorphism.

But p is not a covering map. Indeed, if U is an evenly covered neighbourhood of $e^{2\pi i}$ *via* p, then $p^{-1}(U) = \bigsqcup_{n=0}^{\infty} V_n$, where V_n is a small neighbourhood around n when $n > 0$ and $V_0 = (0, \varepsilon)$ for some small ε. But $p(V_0)$ is not homeomorphic to U. So there is no neighbourhood of $e^{2\pi i}$ which is evenly covered by p. □

Suppose $p : \widetilde{X} \to X$ is a covering map and $f : Y \to X$ is a continuous function. A **lift** of f is a map $\widetilde{f} : Y \to \widetilde{X}$ such that $p\widetilde{f} = f$. The following theorems show that paths and path homotopies can be lifted.

Theorem 272 (Path Lifting Property)
Suppose $p : \widetilde{X} \to X$ is a covering map and $f : I \to X$ is a path with $f(0) = x_0$. For each $\widetilde{x}_0 \in p^{-1}(x_0)$, there is a unique path $\widetilde{f} : I \to \widetilde{X}$ such that $\widetilde{f}(0) = \widetilde{x}_0$ and $p\widetilde{f} = f$.



Proof: the sets $f^{-1}(V)$ where V is a **canonical** (which is to say, evenly covered) neighbourhood of a point in $f(I)$ give an open covering $\mathfrak{F}$ of I. As I is a compact metric space, Theorem 245 guarantees the existence of a Lebesgue number ε of $\mathfrak{F}$. Let $n \in \mathbb{N}$ be such that $\frac{1}{n} < \varepsilon$. Let $t_m = \frac{m}{n}$ for $1 \leq m \leq n$ and set $t_0 = 0$. Then $I_m = [t_{m-1}, t_m]$ has diameter less than ε, so it lies in $f^{-1}(V_m)$ for some canonical V_m and $f(I_m) \subseteq V_m$ for $1 \leq m \leq n$.

But V_1 is a canonical neighbourhood of x_0. Let $\widetilde{V}_1$ be the slice of $p^{-1}(V_1)$ containing $\widetilde{x}_0$. Define $\widetilde{f}$ on I_1 by

$$\widetilde{f}(t) = p_1^{-1} f(t),$$

where $p_1 = p|_{\widetilde{V}_1}$. As f is continuous and p_1 is a homeomorphism, $\widetilde{f}$ is continous on I_1.

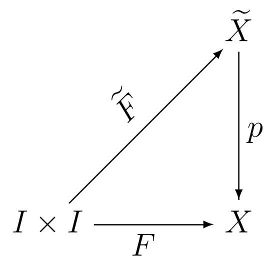
Now suppose $\widetilde{f}$ is defined on $[0, t_m]$ and let $x_m = f(t_m)$ and $\widetilde{x}_m = \widetilde{f}(t_m)$. Take $\widetilde{V}_{m+1}$ to be the slice of $p^{-1}(V_m)$ containing $\widetilde{x}_m$ and let $p_{m+1} = p|_{\widetilde{V}_{m+1}}$. Define $\widetilde{f}$ on I_{m+1} by

$$\widetilde{f}(t) = p_{m+1}^{-1} f(t).$$

Since $\widetilde{f}$ is defined at t_m, the pasting lemma guarantees that $\widetilde{f}$ is continuous on $[0, t_{m+1}]$. After n steps, the continuous function $\widetilde{f}$ is defined on I and, by construction, $p\widetilde{f} = f$.

Now suppose $g : I \to \widetilde{X}$ is another path such that $g(0) = \widetilde{x}_0$ and $pg = f$. By construction $p_1 g = p_1 \widetilde{f}$ on I_1. Since p_1 is a homeomorphism, $g = \widetilde{f}$ on I_1. Using an argument identical to that used in the construction of $\widetilde{f}$, if $g = \widetilde{f}$ on $[0, t_m]$, then $g = \widetilde{f}$ on $[t_m, t_{m+1}]$. Recursively, $g = \widetilde{f}$ on I. ■

Theorem 273 (Square Lifting Property)
Suppose $p : \widetilde{X} \to X$ is a covering map and $F : I \times I \to X$ is a continuous function with $F(0,0) = x_{0,0}$. For each $\widetilde{x}_{0,0} \in p^{-1}(x_{0,0})$, there is a unique lift of F to $\widetilde{F} : I \times I \to \widetilde{X}$ where $\widetilde{F}(0,0) = \widetilde{x}_{0,0}$.



Proof: the sets $F^{-1}(V)$ where V is a canonical neighbourhood of a point in $F(I \times I)$ form an open covering $\mathfrak{F}$ of $I \times I$ with Lebesgue number ε. Subdivide $I \times I$ into n^2 small squares of diameter less than ε. Using arguments similar to that of the previous proof, lift F to $\widetilde{F}$ on $I_1 \times I_1$, then across the base of $I \times I$ on $I \times I_1$. Next, fill the square one layer at a time. Special care has to be taken to extend $\widetilde{F}$ to $I_k \times I_{l+1}$ from the previous rectangles. This hinges on the fact that the union of the bottom and leftmost edges is connected. Then $\widetilde{F} : I \times I \to \widetilde{X}$ is uniquely defined. ■

Theorem 274
If $f_0, f_1 : I \to X$ are paths with initial point x_0, $p : \widetilde{X} \to X$ is a covering map and $p(\widetilde{x}_0) = x_0$, then the lifts $\widetilde{f}_0, \widetilde{f}_1 : I \to \widetilde{X}$ with initial point $\widetilde{x}_0$ are path homotopic under $\widetilde{F}$ if and only if f_0, f_1 are path homotopic under F, where $\widetilde{F}$ is the unique lift of F based at $\widetilde{x}_0$.

Proof: suppose $\widetilde{f}_0 \sim_{\widetilde{F}} \widetilde{f}_1$, then let $F = p\widetilde{F}$, so $f_0 \sim_F f_1$. Conversely, suppose $f_0 \sim_F f_1$ and let $\widetilde{F}$ be the lift of F obtained by the previous theorem. Then

$$p\widetilde{F}(t,0) = F(t,0) = f_0(t),$$

so $\widetilde{F}(t,0)$ is a lift of f_0 at $\widetilde{F}(0,0) = \widetilde{x}_0$. By uniqueness of lifts, $\widetilde{F}(t,0) = \widetilde{f}_0(t)$.

Similarly, $\widetilde{F}(t,1)=\widetilde{f}_1(t)$. Now

$$p\widetilde{F}(0,s)=f(0,s)=x_0$$

and $\widetilde{F}(0,s)$ is a lift of the constant path $e_{x_0}(s)$. But the constant path $e_{\widetilde{x}_0}(s)=\widetilde{x}_0$ is a lift of e_{x_0}. By uniqueness of lifts,

$$\widetilde{F}(0,s)=e_{\widetilde{x}_0}(s)=\widetilde{x}_0.$$

Similarly $\widetilde{F}(1,s)$ is a constant path and $\widetilde{F}$ is a path homotopy. ■

Corollary 275
If X and $\widetilde{X}$ are path-connected, then $p^{-1}(x)$ has the same cardinality at every point $x\in X$.

Proof: for any path f in X from x to y, if $\widetilde{x}\in p^{-1}(x)$, then the lift of f to $\widetilde{f}$ with initial point $\widetilde{x}$ gives a path in $\widetilde{X}$ from $\widetilde{x}$ to $\widetilde{f}(1)=\widetilde{y}$. Define $\varphi:p^{-1}(x)\to p^{-1}(y)$ by $\varphi(\widetilde{x})=\widetilde{y}$.

For $\overline{f}$ the reverse path of f from y to x, we get a unique lift from $\widetilde{y}$ to some terminal point. But that terminal point has to be $\widetilde{x}$, since $\widetilde{\overline{f}}=\overline{\widetilde{f}}$. Thus $\overline{\varphi}:p^{-1}(y)\to p^{-1}(x)$ and $\overline{\varphi}=\varphi^{-1}$. ■

The cardinality of $p^{-1}(x)$ is the number of **sheets** of the covering.

Examples (SHEETS)

1. The map $p:S^1\to S^1$ defined by $p(z)=z^n$ is an n-sheeted covering.
2. The map $p:\mathbb{R}\to S^1$ defined by $p(x)=e^{2\pi ix}$ is an ω-sheeted covering. □

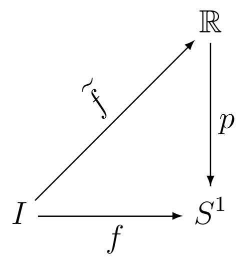
20.3 Fundamental Groups of S^1 and $\mathbb{R}^2\setminus\{\mathbf{0}\}$

In this section we show how to compute the fundamental group of the circle and of the punctured plane, using techniques introduced in the previous section.

Theorem 276
The fundamental group of S^1 is infinite cyclic, that is it is isomorphic to the additive group $\mathbb{Z}$.

Proof: since S^1 is path-connected, the fundamental group can be based at any point of S^1. For convenience, take $z=e^{2\pi i}=1$. The map $p:\mathbb{R}\to S^1$ defined by $p(x)=e^{2\pi ix}$ is a covering map.

Let $\alpha \in \pi(S^1, 1)$. Then $\alpha = [f]$, where f is a loop in S^1 based at 1. Then, by the path lifting property, there exists a unique $\widetilde{f}$ with initial point $0 \in p^{-1}(1)$ such that the following diagram commutes.



Then

$$p\widetilde{f}(0) = p(0) = 1 \quad \text{and} \quad p\widetilde{f}(1) = f(1) = 1.$$

Hence $\widetilde{f}(1) \in \mathbb{Z}$, say $\widetilde{f}(1) = n$. This integer is independent of the choice of the representative f, by Theorem 274. Define a map $\varphi : \pi(S^1, 1) \to \mathbb{Z}$ by $\varphi(\alpha) = \widetilde{f}(1)$. We show that φ is an isomorphism, which yields the desired result.

φ **is a homomorphism:** Let $\alpha = [f], \beta = [g] \in \pi(S^1, 1)$. By construction, $\varphi(\alpha) = \widetilde{f}(1) = n$ and $\varphi(\beta) = \widetilde{g}(1) = m$ for some $m, n \in \mathbb{Z}$. Define $\widetilde{h}$ by $\widetilde{h}(t) = n + \widetilde{g}(t)$. Then $\widetilde{f}\widetilde{h}$ is a path from 0 to $n + m$. Then

$$p(\widetilde{h}(t)) = e^{2\pi i (n + \widetilde{g}(t))} = p(\widetilde{g}(t)) = g(t)$$

and $p(\widetilde{f}.\widetilde{h}) = p(\widetilde{f}).p(\widetilde{h}) = fg$, so $\widetilde{f}.\widetilde{h}$ is a lift of fg starting at 0. Consequently,

$$\varphi(\alpha\beta) = \widetilde{f}.\widetilde{h}(1) = n + m = \varphi(\alpha) + \varphi(\beta).$$

φ **is injective:** Suppose $\varphi(\alpha) = 0$ for $\alpha = [f]$. Then, if $\widetilde{f}$ is a lift of f starting at 0, $\widetilde{f}(1) = 0$ and so $\widetilde{f}$ is a loop in $\mathbb{R}$ based at 0. But $\mathbb{R}$ is simply connected, so $\widetilde{f} \sim e_0$. By Theorem 274, $f \sim e_1$, or $\alpha = \varepsilon_1$. Then $\ker \varphi = \{\varepsilon_1\}$.

φ **is surjective:** For any $n \in \mathbb{Z}$, let $\widetilde{f}(t) = nt$. Then $\widetilde{f}$ is a path from 0 to n and $f = p\widetilde{f}$ is a loop in S^1. Let $\alpha = [f]$. Then $\varphi(\alpha) = \widetilde{f}(1) = n$. ■

Interestingly, the punctured plane has the same fundamental group as the circle.

Theorem 277
The fundamental group of $\mathbb{R}^2 \setminus \{0\}$ *is infinite cyclic.*

Proof: the point $b = (1, 0)$ belongs to both S^1 and $X = \mathbb{R}^2 \setminus \{0\}$. Let $i : S^1 \to X$ be the inclusion map and $r : X \to S^1$ be the **radial map** defined by $r(z) = z/|z|$ on X.

Both i and r are continuous, and these maps induce the homomorphisms

$$i^* : \pi(S^1, b) \to \pi(X, b) \quad \text{and} \quad r^* : \pi(X, b) \to \pi(S^1, b).$$

Note that $ri = \text{id}_{S^1}$ and so that $r^* i^* = \text{id}_{\pi(S^1,b)}$. Then i^* is injective and r^* is surjective. It remains only to show that $i^* r^* = \text{id}_{\pi(X,b)}$.

Let $\alpha = [f] \in \pi(X, b)$ and define $F : I \times I \to X$ by

$$F(t, s) = (1 - s)f(t) + s\frac{f(t)}{|f(t)|}.$$

Then F is continuous and defined everywhere since $|f(t)| \neq 0$ in X. Furthermore $F(t, s) \neq 0$, as can be easily verified.

$$F(0, s) = F(1, s) = b \quad \text{and} \quad F(t, 0) = f(t),\ F(t, 1) = \frac{f(t)}{|f(t)|}.$$

Then if $g = f/|f|$, F is a path homotopy between f and g. Hence $\alpha = [g]$. But g is a loop in S^1 based at b, so $r(g(t)) = g(t)$ and

$$i^* r^*(\alpha) = i^*([r(g)]) = i^*([g]) = \alpha.$$

Then $i^* r^* = \text{id}_{\pi(X,b)}$ and i^* and r^* are isomorphisms. Consequently, $\pi(X, b)$ is isomorphic to the additive group $\mathbb{Z}$. ■

This last result tells us that puncturing the plane changes the topological nature of $\mathbb{R}^2$.

Corollary 278
$\mathbb{R}^2 \setminus \{0\}$ *and* $\mathbb{R}^2$ *are not homeomorphic.*

Note that $\mathbb{R}^n \setminus \{0\}$ and $\mathbb{R}^n$ **are** homeomorphic when $n > 2$, however.

A subspace A of X is a **retract of** X if there is a continuous function $r : X \to A$ such that $r(a) = a$ for all $a \in A$. Such a function is called a **retraction**. If $r : X \to A$ is a retraction, $ri = \text{id}_A$ where $i : A \to X$ is the inclusion mapping. If $a \in A$, this induces $r^* i^* = \text{id}_{\pi(A,a)}$, so that r^* is surjective and i^* is injective.

Examples (RETRACTS)

1. S^1 is a retract of $\mathbb{R}^2 \setminus \{0\}$ with the radial map $r : \mathbb{R}^2 \setminus \{0\} \to S^1$.
2. Since $\pi(\mathbb{R}^2, 0) = \{\varepsilon_0\}$ and $\pi(S^1, 1) \cong \mathbb{Z}$, there is no surjective homomorphism $r^* : \pi(\mathbb{R}^2, 0) \to \pi(S^1, 1)$. Hence there cannot be a retraction $r : \mathbb{R}^2 \to S^1$, so S^1 is not a retract of $\mathbb{R}^2$.

3. The disc $D = \{z \mid |z| \leq 1\}$ is a retract of $\mathbb{C}$ with the continuous map $r : \mathbb{C} \to D$ defined by

$$r(z) = \begin{cases} z & \text{if } |z| \leq 1, \\ z/|z| & \text{if } |z| > 1. \end{cases}$$

□

Two continuous maps $f, g : X \to Y$ are **homotopic** if $\exists$ a continuous map $F : X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all $x \in X$. A subset A of X is a **strong deformation retract** if there is a retraction $r : X \to A$ and a homotopy $F : X \times I \to X$ such that $F(x, 0) = x$ and $F(x, 1) = r(x)$ for all $x \in X$ and $F(a, t) = a$ for all $a \in A$, that is if $ir \sim_F \text{id}_X$. The importance of strong deformation retracts is explained by the following theorem.

Theorem 279
If A is a strong deformation retract of X, then $\pi(X, a) \simeq \pi(A, a)$ for $a \in A$.

Proof: suppose $r : X \to A$ is a retraction. Then the induced homomorphisms

$$r^* : \pi(X, a) \to \pi(A, a) \quad \text{and} \quad i^* : \pi(A, a) \to \pi(X, a)$$

are respectively surjective and injective. It will be sufficient to show that i^* is also surjective. Let f be a loop in X based at $a \in A$. Then $rf = g$ is a loop in A based at a. Let F be a homotopy between ir and id_X. Then, setting $F_1(t, s) = F(f(t), s)$ yields $f \sim_{F_1} g$, since

$$F_1(t, 0) = F(f(t), 0) = f(t) \quad \text{and} \quad F_1(t, 1) = F(f(t), 1) = rf(t) = g(t).$$

Therefore $[g] = [f]$ and $i^*([g]) = [f]$. Hence i^* is surjective. ■

Suppose that $f : X \to Y$ and $g : Y \to X$ are continuous functions such that $fg \sim \text{id}_Y$ and $gf \sim \text{id}_X$, then X and Y are said to be **homotopy equivalent**, denoted $X \equiv Y$, and f and g are said to be **homotopy inverses**. The relation $\equiv$ is an equivalence relation. Reflexivity and symmetry are trivially shown. To show that $\equiv$ is transitive, let $X \equiv Y$ and $Y \equiv Z$. Then there exist continuous functions

$$f : X \to Y,\ g : Y \to Z,\ h : Y \to Z \text{ and } k : Z \to Y$$

such that $fg \sim \text{id}_X,\ gf \sim \text{id}_Y,\ hk \sim \text{id}_Z$ and $kh \sim \text{id}_Y$. Then

$$(hf)(gk) \sim h(fg)k \sim h\,\text{id}_Y\,k \sim hk \sim \text{id}_Z \quad \text{and} \quad (gk)(hf) \sim g(kh)f \sim g\,\text{id}_Y\,f \sim gk \sim \text{id}_X,$$

so $X \equiv Z$ through the homotopy inverses hf and gk.

Examples (STRONG DEFORMATION RETRACTS)

1. The figure 8 is a strong deformation retract of the doubly-punctured plane. Intuitively, this is done by representing the figure 8 as two petals teaching the axes at the origin. Puncture each petal once. Points interior to the petal slide radially away from the puncture. Points outside the petals slide radially towards the origin until they reach a petal. Timing it so that each point takes exactly one unit of time to reach the appropriate petal yields the desired homotopy.
2. If A is a strong deformation retract of X, then $A \equiv X$. Indeed, let $r : X \to A$ be a retraction. Then $ri = \text{id}_A$ and $ir \sim \text{id}_X$.

From this point on, the spaces we consider are all path-connected.

Theorem 280
Suppose $f, g : X \to Y$ are continuous functions, $x_0 \in X$ and $f(x_0) = y_0$, $g(x_0) = y_1$. If f and g are homotopic, then there is a path class α from y_0 to y_1 such that $g^* = \widehat{\alpha} f^*$, where $f^* : \pi(X, x_0) \to \pi(Y, y_0)$, $g^* : \pi(X, x_0) \to \pi(Y, y_1)$ and $\widehat{\alpha} : \pi(Y, y_0) \to \pi(Y, y_1)$.

Proof: suppose $F : X \times I \to Y$ is a homotopy between f and g, that is, suppose $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$. Let $q : I \to Y$ be such that $q(s) = F(x_0, s)$. As F is continuous, q is a path from y_0 to y_1 since

$$\begin{aligned} q(0) &= F(x_0, 0) = f(x_0) = y_0 \\ q(1) &= F(x_0, 1) = g(x_0) = y_1. \end{aligned}$$

Let $\alpha = [q]$. For any loop h in X based at x_0, we show that

$$g^*([h]) = \widehat{\alpha} f^*([h]),$$

that is $[g \circ h] = \widehat{\alpha}([f \circ h]) = [\overline{q}][f \circ h][q]$, or $g \circ h \sim (\overline{q}(f \circ h))q$.

Let $e = e_{y_1}$. Then $g \circ h \sim e(g \circ h) \sim (e(g \circ h))e$. We next show that

$$(e(g \circ h))e \sim_G (\overline{q}(f \circ h))q$$

for an appropriate path homotopy G. Define $G : I \times I \to Y$ by

$$G(t, s) = \begin{cases} q(1 - 4t(1 - s)) & t \in [0, 1/4], \\ F(h(4t - 1), s) & t \in [1/4, 1/2], \\ q(2t - 1 + 2(1 - t)s) & t \in [1/2, 1]. \end{cases}$$

At $t = \frac{1}{4}$, $q(s) = F(x_0, s)$ and at $t = \frac{1}{2}$, $F(x_0, s) = q(s)$ so, by the pasting lemma, G is continuous on $I \times I$. Now $G(0, s) = G(1, s) = q(1) = y_1$ and

$$G(t,0) = \begin{cases} q(1-4t) & t \in [0,1/4], \\ F(h(4t-1),0) & t \in [1/4,1/2], \\ q(2t-1) & t \in [1/2,1], \end{cases}$$
$$G(t,1) = \begin{cases} q(1) & t \in [0,1/4], \\ F(h(4t-1),1) & t \in [1/4,1/2], \\ q(1) & t \in [1/2,1]. \end{cases}$$

Then $G(t, 0) = (\overline{q}(f \circ h))q(t)$, $G(t, 1) = (e(g \circ h))e(t)$ and $g^* = \widehat{\alpha} f^*$. ■

The existence of homotopy inverses between X and Y imply that the corresponding fundamental groups are isomorphic.

Corollary 281
If $f : X \to Y, g : Y \to X$ are homotopy inverses, then $f^* : \pi(X, x_0) \to \pi(Y, f(x_0))$ is an isomorphism.

Proof: let $y_0 = f(x_0)$ and $x_1 = g(y_0)$. As f and g are homotopy inverses, $g \circ f \sim \text{id}_X$ and the preceding theorem yields $(g \circ f)^* = \widehat{\alpha}\, \text{id}^*_{\pi(X,x_0)} = \widehat{\alpha}$ for some path class α from x_0 to x_1. Then $g^* f^* = \widehat{\alpha}$. As $\widehat{\alpha}$ is an isomorphism, g^* is surjective and f^* is injective. It is then sufficient to show that g^* is injective.

Let $y_1 = f(x_1)$ and denote by f_1^* the homomorphism induced by f from $\pi(X, x_1)$ to $\pi(Y, y_1)$. As before, $fg \sim \text{id}_Y$ and the preceding theorem yields $(f \circ g)^* = \widehat{\beta}\, \text{id}^*_{\pi(Y,f(x_0))} = \widehat{\beta}$ for some path class β from y_0 to y_1. But this means that g^* is injective as $\widehat{\beta}$ is an isomorphism. Hence g^* is an isomorphism and $f^* = (g^*)^{-1}\widehat{\alpha}$ is an isomorphism. ■

Note that X and Y may have isomorphic fundamental groups yet fail to be homeomorphic and/or homotopy equivalent (compare with Corollary 271).

Examples

1. Consider $\mathbb{R}$ in the usual topology and the singleton set $\{*\}$. We have seen that $\pi(\mathbb{R}) = \pi(\{*\}) = \{\varepsilon\}$, but $\mathbb{R}$ and $\{*\}$ are not homeomorphic since $\{*\}$ is compact but $\mathbb{R}$ isn't.
2. Consider S^2 in the usual topology and the singleton set $\{0\} \subseteq \mathbb{R}^3$. We can show (see next section) that $\pi(S^2) = \pi(\{0\}) = \{\varepsilon\}$, but S^2 and $\{0\}$ are not homotopy equivalent (this is harder to prove). □

20.4 Special Seifert-Van Kampen Theorem

The special Seifert-Van Kampen theorem allows us to determine when the fundamental group of a space is . The following lemma will be helpful.

Lemma 282
Suppose $f : I \to X$ *is a path and* $0 = a_0 < a_1 < \ldots < a_n = 1$. *Define* $f_i : I \to X$ *by* $f_i(t) = f((1-t)a_{i-1} + ta_i)$ *for* $1 \leq i \leq n$. *Then*

$$f \sim f_1(f_2(\cdots f_n)\cdots).$$

Proof: left as an exercise.

The main result is stated and proven below.

Theorem 283 (Special Seifert-Van Kampen Theorem)
Let U, V, *and* $U \cap V$ *be non-empty, open, path-connected subsets of* $X = U \cup V$. *Let* $x_0 \in U \cap V$. *If the inclusions* $i : U \to X$ *and* $j : V \to X$ *induce respectively the trivial homomorphisms*

$$\begin{aligned} i^* &: \pi(U, x_0) \to \pi(X, x_0), \\ j^* &: \pi(V, x_0) \to \pi(X, x_0), \end{aligned}$$

then $\pi(X, x_0)$ *is trivial.*

Proof: suppose $f : I \to X$ is a loop based at x_0. The sets $f^{-1}(U)$ and $f^{-1}(V)$ form an open covering of the compact metric space I, so the covering has a Lebesgue number. It is then possible to subdivide I into n intervals of the form $I_i = [a_{i-1}, a_i]$ such that $f(I)$ lies entirely in U or entirely in V for $1 \leq i \leq n$.

Should the image of consecutive intervals I_i and I_{i+1} lie in the same set U or V, amalgamate them to form a single interval. After having done this whenever it was possible to do so, we get a new collection of intervals with images lying entirely either in U or in V, and such that the images of their endpoints lie in $U \cap V$ for all such endpoints. Rename these intervals $I_i = [a_{i-1}, a_i]$ for $1 \leq i \leq m$. Then $f(I_i) \subseteq U$ or $f(I_i) \subseteq V$ and $f(a_i) \in U \cap V$ for $1 \leq i \leq m$.

Let f_i be the image of I_i under f. Then f is a path in U or in V from $f(a_{i-1})$ to $f(a_i)$. Let g_{i-1} be a path in $U \cap V$ from x_0 to $f(a_{i-1})$ and g_i be a path in $U \cap V$ from x_0 to $f(a_i)$. As $U \cap V$ is path connected, the paths g_{i-1} and g_i exist. For consistency, define g_0 and g_m to be the constant paths x_0.

If f_i is a path in V, set $f_i' = (g_{i-1}f_i)\overline{g_i}$. Then f_i' is a loop in V based at x_0.

By hypothesis, $j^*([f_i]) = [\varepsilon]$ in X, so $(g_{i-1}f_i)\overline{g_i} \sim e_{x_0}$ and $f_i \sim \overline{g}_{i-1}g_i$. Define

$$h_i = \begin{cases} \overline{g}_{i-1}g_i & \text{when } f_i \text{ lies in } V, \\ f_i & \text{when } f_i \text{ lies in } U. \end{cases}$$

Then $f_i \sim h_i$ for all i. By the preceding lemma, $f \sim h_1(h_2(\cdots h_m)\cdots)$, which is a loop in U. But loops in U are homotopic to the constant loop e_{x_0} in X, so $f \sim e_{x_0}$ in X and $\pi(X, x_0)$ is trivial as f was arbitrary. ■

We have an easy corollary.

Corollary 284 *If $X = U \cup V$, where U and V are open and simply connected and $U \cap V$ is path-connected, then X is simply connected.*

Using the special Seifert-Van Kampen theorem, we can easily compute the fundamental group of S^n, for $n \geq 2$.

Example: if $n \geq 2$, $\pi(S^n) \simeq \{\varepsilon\}$. Indeed, consider S^n as the unit sphere in $\mathbb{R}^{n+1}$, and let N and S be the north and south pole of S^n, respectively. Let $U = S^n \setminus \{N\}$ and $V = S^n \setminus \{S\}$.

Then U and V are both homeomorphic to $\mathbb{R}^n$ under **stereographic projection**, so U and V are simply connected as $\mathbb{R}^n$ is simply connected for $n \geq 2$. Clearly $S^n = U \cup V$, where U and V are open. But $U \cap V$ is path connected, as it is homeomorphic to $S^{n-1} \times (-1, 1)$, which is path-connected when $n \geq 2$. By the preceding corollary, S^n is simply connected for $n \geq 2$. □

As $\mathbb{R}^n \setminus \{0\}$ and S^n have the same fundamental group (the proof is similar to that of Theorem 277), then $\pi(\mathbb{R}^{n+1} \setminus \{0\})$ is trivial for $n \geq 2$.

Corollary 285
$\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ when $n \geq 3$.

20.5 Solved Problems

1. Suppose that $f, g : I \to X$ are paths in a space X such that $f(t) = g(t)$ for $t \in [a, 1]$. If the paths $f_a, g_a : I \to X$ defined by $f_a(t) = f(at)$ and $g_a(t) = g(at)$ are path homotopic, show that f and g are path homotopic.

   Proof: first, note that if $a = 0$, the result is trivially true. So suppose $a \neq 0$. Let $x_0 = f(0)$, $x_a = f(a)$ and $x_1 = f(1)$. If f_a and g_a are path homotopic, they both start at $f_a(0) = f(0) = x_0$, and they both end at $f_a(1) = f(a) = x_a$. Then, there is a continuous function $H_1 : I \times I \to X$ such that

$$\begin{aligned} H_1(t,0) &= f_a(t) = f(at) \\ H_1(t,1) &= g_a(t) = g(at) \\ H_1(0,s) &= x_0 \\ H_1(1,s) &= x_a. \end{aligned}$$

Let $H_2 : I \times I \to X$ be defined by $H_2(t,s) = f(a + t(1-a))$. Then H_2 is continuous since f is a path, and

$$\begin{aligned} H_2(t,0) &= f(a + t(1-a)) \\ H_2(t,1) &= g(a + t(1-a)) \\ H_2(0,s) &= x_a \\ H_2(1,s) &= x_1. \end{aligned}$$

This makes H_2 into a path homotopy between f and g from x_a to x_1. Now define the map $H : I \times I \to X$ by

$$H(t,s) = \begin{cases} H_1\left(\frac{t}{a}, s\right) & \text{for } t \in [0,a], \\ H_2\left(\frac{t-a}{1-a}, s\right) & \text{for } t \in [a,1]. \end{cases}$$

Then H is continuous by the pasting lemma, as H_1 and H_2 are continuous and at $t = 1$, $H_1(1,s) = H_2(0,s) = x_a$. Furthermore

$$\begin{aligned} H(t,0) &= \begin{cases} H_1\left(\frac{t}{a}, 0\right) & \text{for } t \in [0,a], \\ H_2\left(\frac{t-a}{1-a}, 0\right) & \text{for } t \in [a,1] \end{cases} \\ &= \begin{cases} f_a(t/a) & \text{for } t \in [0,a], \\ f(a + \frac{t-a}{1-a}(1-a)) & \text{for } t \in [a,1] \end{cases} \\ &= \begin{cases} f(t) & \text{for } t \in [0,a], \\ f(t) & \text{for } t \in [a,1] \end{cases} \\ &= f(t), \\ H(t,1) &= \begin{cases} H_1\left(\frac{t}{a}, 1\right) & \text{for } t \in [0,a], \\ H_2\left(\frac{t-a}{1-a}, 1\right) & \text{for } t \in [a,1] \end{cases} \\ &= \begin{cases} g_a(t/a) & \text{for } t \in [0,a], \\ g(a + \frac{t-a}{1-a}(1-a)) & \text{for } t \in [a,1] \end{cases} \\ &= \begin{cases} g(t) & \text{for } t \in [0,a], \\ g(t) & \text{for } t \in [a,1] \end{cases} \\ &= g(t), \\ H(0,s) &= H_1(0,s) = x_0, \\ H(1,s) &= H_2(1,s) = x_1. \end{aligned}$$

Hence H is a path homotopy from f to g between x_0 and x_1. ■

2. Let x_0 and x_1 be two given points of the path-connected space X. Show that $\pi_1(X, x_0)$ is abelian if and only if for every pair α and β of paths from x_0 to x_1, the induced isomorphisms $\widehat{\alpha}$ and $\widehat{\beta}$ are equal.

   Proof: suppose $\pi_1(X, x_0)$ is abelian, and let α and β be two paths from x_0 to x_1. Then $\beta\overline{\alpha}$ is a loop at x_0, so $[\beta\overline{\alpha}] \in \pi_1(X, x_0)$ and

   $$[\beta\overline{\alpha}][f] = [f][\beta\overline{\alpha}]$$

   for all $[f] \in \pi_1(X, x_0)$. Then

   $$\begin{aligned} [f] &= [\alpha\overline{\beta}][f][\beta\overline{\alpha}] \\ &= [\alpha]\widehat{\beta}([f])[\overline{\alpha}] \\ &= \widehat{\widehat{\alpha}}(\widehat{\beta}([f])). \end{aligned}$$

   Hence $\widehat{\alpha}([f]) = \widehat{\beta}([f])$ for all $[f] \in \pi_1(X, x_0)$, so $\widehat{\alpha} = \widehat{\beta}$.

   Conversely, suppose the induced isomorphisms of any two paths in X from x_0 to x_1 are equal. Let α be such a path, and let f be a loop at x_0. Then $f\alpha$ is a path from x_0 to x_1 and $\widehat{\alpha} = \widehat{f\alpha}$. Let $[g] \in \pi_1(X, x_0)$. Then

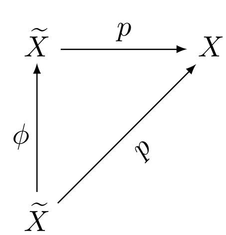
   $$[\overline{\alpha}][g][\alpha] = \widehat{\alpha}([g]) = \widehat{f\alpha}([g]) = [\overline{f\alpha}][g][f\alpha] = [\overline{\alpha}][\overline{f}][g][f][\alpha],$$

   thus $[g] = [\overline{f}][g][f]$ for all loops f and g at x_0, and $\pi_1(X, x_0)$ is abelian. ■

3. Suppose that $\widetilde{X}$ is a two-sheeted covering space of X, that is for each $x \in X$, there are two values $\widetilde{x}_1$ and $\widetilde{x}_2$ with $p^{-1}(x) = \{\widetilde{x}_1, \widetilde{x}_2\}$. Prove that the map $\phi : \widetilde{X} \to \widetilde{X}$, which interchanges the values $\widetilde{x}_1$ and $\widetilde{x}_2$ is a homeomorphism.

   Proof: the map ϕ is clearly a bijection and $\phi^2 = \mathrm{id}$, so ϕ is its own inverse. Furthermore, $\phi(z) \neq z$ for all $z \in \widetilde{X}$. To show ϕ is a homeomorphism, it is then sufficient to show that ϕ is a continuous map. To do so, we find a collection $\{Z_\alpha\}$ of open sets in $\widetilde{X}$ such that $\bigcup_{\alpha\in A} Z_\alpha = \widetilde{X}$ and such that $\phi|_{Z_\alpha}\colon Z_\alpha \to \widetilde{X}$ is continuous for all $\alpha \in A$. Then ϕ will be a continuous map.

   First note that



   is a commutative diagram, since for every $x \in X$, there exists $\tilde{x} \in \widetilde{X}$ such that $p^{-1}(x) = \{\tilde{x}, \phi(\tilde{x})\}$. Thus $p\phi = p$. As $(\widetilde{X}, p)$ is a two-sheeted covering of X there exists, for every $x \in X$, a neighbourhood V_x of x in X and two disjoint open sets U_x and W_x in $\widetilde{X}$ such that $p^{-1}(V_x) = U_x \cup W_x$ and such that the mappings

   $$p|_{U_x}\colon U_x \to V_x \quad \text{and} \quad p|_{W_x}\colon W_x \to V_x$$

are homeomorphisms.[1] Then U_x is homeomorphic to W_x. We show that $\phi(U_x) = W_x$, and so that $\phi(U_x)$ is homeomorphic to U_x. Suppose however that $\phi(U_x) \neq W_x$, that is, suppose there is $y \in U_x$ such that $\phi(y) \notin W_x$. Then

$$p(y) = p(\phi(y)) \in V_x,$$

and so $\phi(y) \in p^{-1}(V_x) \cup U_x$, since $\phi(x) \neq W_x$. But this would mean that $p|_{U_x}\colon U_x \to V_x$ is not injective as $y \neq \phi(y)$ and $p(y) = p(\phi(y))$. Then $\phi(U_x) \subseteq W_x$, and so $\phi(U_x) = W_x$ since U_x and W_x have the same cardinality and since ϕ is a bijection. Thus $\phi|_{U_x}\colon U_x \to W_x$ is a homeomorphism and $\phi|_{U_x}\colon U_x \to \widetilde{X}$ is continuous. Similarly, $\phi(W_x) = U_x$ and $\phi|_{W_x}\colon W_x \to \widetilde{X}$ is continuous. But

$$\widetilde{X} = p^{-1}(X) = p^{-1}\left(\bigcup_{x\in X} V_x\right) = \bigcup_{x\in X} p^{-1}(V_x) = \bigcup_{x\in X}(U_x \cup W_x),$$

where U_x and W_x are open in $\widetilde{X}$. By the argument in the first paragraph, ϕ is a homeomorphism. ■

4. If $(\widetilde{X}, p)$ and $(\widetilde{Y}, q)$ are covering spaces of X and Y respectively, show that $(\widetilde{X}\times\widetilde{Y}, (p, q))$ is a covering space of $X \times Y$.

Proof: let $h = (p, q)$. We need to show that h is a continuous surjective map and that for every $(x, y) \in X \times Y$, there exists a neighbourhood V of (x, y) such that $h^{-1}(V)$ is a disjoint union of open sets in $\widetilde{X} \times \widetilde{Y}$ and that each of these open sets is homeomorphic to V *via* h.

h **is continuous** Let $U_1 \times U_2$ be a basic neighbourhood of $X \times Y$. Then

$$\begin{aligned} h^{-1}(U_1 \times U_2) &= \{(\tilde{x}, \tilde{y}) \in \widetilde{X} \times \widetilde{Y} : (p(\tilde{x}), q(\tilde{y})) \in U_1 \times U_2\} \\ &= p^{-1}(U_1) \times q^{-1}(U_2). \end{aligned}$$

But p and q are continuous, so $p^{-1}(U_1) \times q^{-1}(U_2)$ is a basic neighbourhood of $\widetilde{X} \times \widetilde{Y}$, so h is continuous.

h **is surjective** Let $(x, y) \in X \times Y$. As p and q are surjective, there exist $\tilde{x} \in \widetilde{X}$ and $\tilde{y} \in \widetilde{Y}$ such that $p(\tilde{x}) = x$ and $q(\tilde{y}) = y$. Then we have $h(\tilde{x}, \tilde{y}) = (x, y)$ and h is surjective.

h **is a covering map** If $(x, y) \in X \times Y$, as p and q are covering maps, there exist neighbourhoods V_x of x in X and V_y of y in Y that are evenly covered by p and q respectively. That is $p^{-1}(V_x)$ is a disjoint union of open sets $\widetilde{V}_x$ in $\widetilde{X}$, each homeomorphic to V_x *via* p, and $q^{-1}(V_y)$ is a disjoint union of open sets $\widetilde{V}_y$ in $\widetilde{Y}$, each homeomorphic to V_y *via* q. Set $V = V_x \times V_y$. Then $(x, y) \in V$ and

$$h^{-1}(V) = p^{-1}(V_x) \times q^{-1}(V_y) = \left(\bigcup \widetilde{V}_x\right) \times \left(\bigcup \widetilde{V}_y\right) = \bigcup(\widetilde{V}_x \times \widetilde{V}_y),$$

[1] Strictly speaking, $p^{-1}(V)$ should be the disjoint union of an arbitrary collection of homeomorphic open sets in $\widetilde{X}$. But there cannot be more than two of them, since this would violate the condition that $(\widetilde{X}, p)$ be a two-sheet covering of X. Similarly, there cannot be less than two of them, since p has to be a homeomorphism when restricted to $p^{-1}(V)$.

that is $h^{-1}(V)$ is a disjoint union of open sets $\widetilde{V}_x \times \widetilde{V}_y$. But

$$h(\widetilde{V}_x \times \widetilde{V}_y) = p(\widetilde{V}_x) \times q(\widetilde{V}_y) \simeq V_x \times V_y,$$

so $\widetilde{V}_x \times \widetilde{V}_y$ is homeomorphic to $V_x \times V_y$ *via* h.

Then $(\widetilde{X} \times \widetilde{Y}, h)$ is a covering space of $X \times Y$. ■

5. a) For X as in the previous problem, if (X, p') is an n-sheeted covering space of X_1, show that $(\widetilde{X}, p'p)$ is a covering space of X_1.

   b) If X is either i. Hausdorff or ii. completely regular, show that $\widetilde{X}$ has the same property.

Proof:

a) That $p'p$ is a continuous surjective mapping is clear, as it is the composition of two such mappings. It remains only to show that it is a covering map of X.

Let $x \in X_1$. We show that we can find an open neighbourhood V of x in X_1 evenly covered by p'. We then show that the disjoint open sets in X making up $(p')^{-1}(V)$, each of which is homeomorphic to V *via* p', are themselves evenly covered by p. Then there is a disjoint union of open sets in $\widetilde{X}$ making up

$$p^{-1}\left((p')^{-1}(V)\right) = (p'p)^{-1}(V),$$

each of which is homeomorphic to V *via* $p'p$. It is going to be messy, so let's get down to it methodically.

Let $x \in X_1$. Then $(p')^{-1}(x) = \{y_1, \ldots, y_n\}$ in X, as (X, p') is an n-sheeted covering of X_1. First, the *dramatis personæ*.

- V_x is a neighbourhood of x in X_1 evenly covered by p';
- $(p')^{-1}(V_x) = \bigsqcup_{j=1}^{n} W_j$, where $\sqcup$ denotes a disjoint union, W_j is open in X and homeomorphic to V_x *via* p' and $y_j \in W_j$ for all $1 \leq j \leq n$.
- For $1 \leq i \leq n$, U_i is a neighbourhood of y_i in X evenly covered by p;
- For $1 \leq i \leq n$, $p^{-1}(U_i) = \bigsqcup_{\alpha} Z(i)_\alpha$, where $Z(i)_\alpha$ is open in $\widetilde{X}$ and homeomorphic to U_i *via* p for all α;
- $V = (\bigcap_{i=1}^{n} p'(U_i) \cap V_x)$;
- For $1 \leq j \leq n$, $K_j = \left(p'|_{W_j}\right)^{-1}(V) \subseteq W_j$ and $y_j \in K_j$;
- For $1 \leq j \leq n$, $M_j = K_j \cap U_j$ and $y_j \in M_j$;
- For $1 \leq j \leq n$ and for α, $N(j)_\alpha = \left(p|_{Z(j)_\alpha}\right)^{-1}(M_j) \subseteq Z(j)_\alpha$.

Since p' is a covering map, it is an open mapping. Then V is an open subset of X contained in V_x, since it is a finite intersection of open sets in X. As

$$p'|_{W_j} \colon W_j \to V_x$$

is a homeomorphism, K_j is homeomorphic to V *via* p' for $1 \leq j \leq n$. Note that K_j is open for $1 \leq j \leq n$ since V is open and that the K_j are disjoint since the

W_j are disjoint. Then M_j is open, $M_j \subseteq U_j$ for $1 \leq j \leq n$. Note further that the M_j are disjoint since the K_j are disjoint. As

$$p|_{Z(j)_\alpha} \colon Z(j)_\alpha \to U_j$$

is a homeomorphism, M_j is homeomorphic to $N(j)_\alpha$ *via* p for α and $1 \leq j \leq n$. Note that $N(j)_\alpha$ is open for $1 \leq j \leq n$ and α since M_j is open for $1 \leq j \leq n$ and that the $N(j)_\alpha$ are disjoint since the $Z(j)_\alpha$ are disjoint.

Then $N(j)_\alpha$ is homeomorphic to, say, the open subset $p'(M_1) \subseteq V_x$ *via* $p'p$ for $1 \leq j \leq n$. But $p'(M_1)$ is a neighbourhood of x in X_1 so that $p'p$ evenly covers $p'(M_1)$ at x. Hence $(\widetilde{X}, p'p)$ is a covering space of X_1.

b) i. Let $\tilde{x} \neq \tilde{y} \in \widetilde{X}$ and set $x = p(\tilde{x})$, $y = p(\tilde{y})$. Suppose V_x and V_y are neighbourhoods of x and y respectively, who are evenly covered by p. Let W_x and W_y be the (open) slices of $p^{-1}(V_x)$ and $p^{-1}(V_y)$ containing $\tilde{x}$ and $\tilde{y}$ respectively.

A. If $x = y$, W_x and W_y meet $p^{-1}(x) = p^{-1}(y)$ in exactly one point respectively, namely $\tilde{x}$ and $\tilde{y}$. Hence $\tilde{y} \notin W_x$ and $\tilde{x} \notin W_y$.

B. If $x \neq y$, let U_x and U_y be the Hausdorff neighbourhoods of x and y in X. Then $U_x \cap V_x$ is a neighbourhood of x in X disjoint from the neighbourhood $U_y \cap V_y$ of y in X. Furthermore, $O_x = (p|_{W_x})^{-1}(U_x \cap V_x)$ and $O_y = (p|_{W_y})^{-1}(U_y \cap V_y)$ are open in $\widetilde{X}$, as p is a covering map. Then $\tilde{x} \in O_x$, $\tilde{y} \in O_y$ and $O_x \cap O_y = \varnothing$ as

$$U_x \cap V_x \cap U_y \cap V_y = \varnothing.$$

In both cases, $\widetilde{X}$ is Hausdorff.

ii. Suppose $\widetilde{X}$ is non-empty and completely regular. If $\widetilde{W}$ is a neighbourhood of $\tilde{x} \in \widetilde{X}$, let U be a neighbourhood of $p(\tilde{x})$ evenly covered by p such that at least one of the slices, say M, of $p^{-1}(U)$ lies in $\widetilde{W}$.

As X is completely regular, there is a neighbourhood V of $p(\tilde{x})$ such that $\overline{V} \subseteq U$. Take $Z = p^{-1}(V) \cap M$. Then Z is homeomorphic to the slice of $p^{-1}(V)$ in M. By complete regularity of X, there is a continuous function $f : X \to [0,1]$ such that $f(p(\tilde{x})) = 1$ and $f(X - V) = \{0\}$.

Define $g_1 : \overline{Z} \to [0,1]$ by $g_1 = fp$ and $g_2 : \widetilde{X} - Z \to [0,1]$ to be the constant zero-function. As $p(\widetilde{X} - Z) \subseteq X - V$, g_1 and g_2 are both constantly zero on $\overline{Z} \cap \widetilde{X} - Z$. Then, by the pasting lemma, g_1 and g_2 define a continuous function g from $\overline{Z} \cup (\widetilde{X} - Z) = \widetilde{X}$ to $[0,1]$.

By construction, $g(\tilde{x}) = f(p(\tilde{x})) = 1$ and $g(\widetilde{X} - \widetilde{W}) = \{0\}$ since $y \notin \widetilde{W}$ implies $y \notin Z$. Hence $\widetilde{X}$ is completely regular. ■

20.6 Exercises

1. Prepare a 2-page summary of this chapter, with important definitions and results.
2. Let $h : I \to I$ be a continuous such that $h(0) = 0$ and $h(1) = 1$. For any path $f : I \to X$, prove that f and fh are path homotopic.
3. For a product space $X = \prod X_\alpha$, let $f : I \to X$ be a path, and define $f_\alpha = \pi_\alpha f$ for all α. Prove that
   a) two paths f and g in X are path homotopic if and only if $f_\alpha \sim g_\alpha$ for every α;
   b) if $f(1) = g(0)$ for paths $f, g : I \to X$, and $h = fg$, then $h_\alpha = f_\alpha g_\alpha$ for every α.
4. Prove that the fundamental group of X is isomorphic to the direct product of the fundamental groups $\pi(X_\alpha, x_\alpha)$.
5. Let $T = S^1 \times S^1$ denote the torus. For $z_0 \in S^1$, show that $S^1 \times \{z_0\}$ is a retract of T but not a strong deformation retract.
6. Show that $\varphi : X \to Y$ induces a homomorphism of path classes φ^*, as in the discussion on p. 461.
7. Show that $\mathbb{R}^3$ and $\mathbb{R}^3 \setminus \{0\}$ are homeomorphic.
8. Prove Lemma 282.
9. Show that $\mathbb{R}^n \setminus \{0\}$ and S^n have the same fundamental group.