Chapter 23

Borel-Lebesgue Integration

In this chapter, we present an extension of the theory of integration that overcomes some of the issues associated with Riemann integration, and show how to integrate multi-variate functions in this new framework.

One of the problems associated with Riemann integration (see Chapters 4 and 5) is that some functions that should be integrable in any reasonable theory of integration fail to be so, for a variety of reasons.

Examples

1. Consider the Dirichlet function $\chi_{\mathbb{Q}} : \mathbb{R} \to \mathbb{R}$ defined by

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 0 & x \in \mathbb{R} \setminus \mathbb{Q} \\ 1 & x \in \mathbb{Q} \end{cases}$$

   We have seen in Chapter 4 that this function is not Riemann-integrable over any interval $[a, b]$, but ...it should be, right? $\mathbb{R} \setminus \mathbb{Q}$ is so much "bigger" than $\mathbb{Q}$ that the first branch should dominate and give us an integral of 0. Unfortunately, it doesn't.

2. Consider the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{1}{\sqrt{x}} & x \in (0, 1] \\ 1 & x = 0 \end{cases}$$

   It is not Riemann-integrable on $[0, 1]$ as it is not bounded on $[0, 1]$, but it is Riemann-integrable of $[a, 1]$ for all $1 \geq a > 0$ since it is continuous on $[a, 1]$ for all $1 \geq a > 0$.

Furthermore
$$\int_a^1 f \, \mathrm{d}x = \left[2\sqrt{x}\right]_a^1 = 2(1-\sqrt{a}).$$
As $a \to 0^+$, we see that
$$\int_a^1 f \, \mathrm{d}x \to 2(1-\sqrt{0}) = 2,$$
and we would at the very least consider an extension of Riemann integration for which $\int_0^1 f \, \mathrm{d}x = 2$.

3. The function $g : [0,\infty) \to \mathbb{R}$ defined by $g(x) = e^{-x}$ is not Riemann-integrable on $[0,\infty)$ since the domain of integration cannot even be partitioned. But it is clearly Riemann-integrable on $[0,n]$, $n > 0$, since it is continuous on $[0,n]$; in fact,
$$\int_0^n e^{-x} \, \mathrm{d}x = [-e^{-x}]_0^n = 1 - e^{-n}.$$
Since
$$\lim_{n\to\infty} \int_0^n e^{-x} \, \mathrm{d}x = \lim_{n\to\infty} (1-e^{-n}) = 1 - 0 = 1;$$
any extension of Riemann integration should at least give us $\int_0^\infty g = 1$. $\square$

In this chapter, we will introduce an **extension** of the Riemann integral in which all of these examples will work out as we think they should. The **Lebesgue-Borel** approach to integration views the problem from a different example:[1] fundamentally, instead of building **vertical** boxes under the graph of f, we stack **horizontal** boxes under it. This conceptual shift has far-ranging consequences.[2]

We will also extend our definition of the integral to **multivariate domains** (which is to say, the functions we consider will be functions of $\mathbb{R}^n$ to $\mathbb{R}$). To help illustrate the concepts, we will often work with functions $f : \mathcal{A} \subseteq \mathbb{R}^2 \to \mathbb{R}_+$, where f is **bounded** (as a function), as is A (as a set). By analogy to the 1-dimensional case, we will want to define
$$I = \iint_A f(x,y) \, \mathrm{d}x \, \mathrm{d}y$$
so that
$$I = \text{Vol}\Big(\{(x,y,t) \mid (x,y) \in A,\ 0 \leq t \leq f(x,y)\}\Big).$$

[1]There are other approaches: **improper Riemann integration** and **generalized Riemann integration**, say, but we will not be touching on those.

[2]It does not resolve all difficulties, however: there are differentiable functions $F : [a,b] \to \mathbb{R}$ for which F' is not Lebesgue-integrable and some important improper integrals do not exist, for instance.

23.1 Borel Sets and Borel Functions

Generally speaking, the **Borel subsets** of $\mathbb{R}^n$ are the $\sigma-$algebra of subsets for which we know how to compute the **length**, and/or the **surface area**, and/or the **volume**, and so on.[3]

Formally, a $\sigma-$**algebra** $\mathfrak{S}$ of $\mathbb{R}^n$ is a collection of subsets of $\mathbb{R}^n$ such that

1. $A_1, A_2, \dots, A_n, \dots \in \mathfrak{S} \implies \bigcup_{n\geq 1} A_n \in \mathfrak{S}$, and
2. $A \in \mathfrak{S} \implies A^c = \mathbb{R}^n \backslash A \in \mathfrak{S}$.

Consequently (see exercises),

1. $A_1, A_2, \dots, A_n, \dots \in \mathfrak{S} \implies \bigcap_{n\geq 1} A_n \in \mathfrak{S}$;
2. $A, B \in \mathfrak{S} \implies A \cap B^c \in \mathfrak{S}$, and
3. $\varnothing, \mathbb{R}^n \in \mathfrak{S}$.

Examples

1. The **power set** $\wp(\mathbb{R}^n)$ is the **largest** $\sigma-$algebra of $\mathbb{R}^n$, since the union of any collection subsets of $\mathbb{R}^n$ is itself a subset of $\mathbb{R}^n$, and since the complement of any subset of $\mathbb{R}^n$ is also a subset of $\mathbb{R}^n$.
2. The **standard topology** $\tau = \{U \subseteq \mathbb{R}^n \mid U \subseteq_O \mathbb{R}^n\}$ is **not** a $\sigma-$algebra of $\mathbb{R}^n$ since the complement of the open ball of radius 1 centered at the origin, say, is not open in $\mathbb{R}^n$ (see Part IV).
3. $\mathfrak{S}_0(\mathbb{R}^n) = \{\mathbb{R}^n, \varnothing\}$ is the smallest $\sigma-$algebra of $\mathbb{R}^n$. $\square$

Note that $\mathfrak{S}$ of $\mathbb{R}^n$ is a subset of $\wp(\mathbb{R}^n)$.

Proposition 286
If $(\mathfrak{S}_i)_{i\geq 1}$ is a collection of $\sigma-$algebras of $\mathbb{R}^n$ then $\mathfrak{S} = \bigcap_{i\geq 1} \mathfrak{S}_i$ is a $\sigma-$algebra of $\mathbb{R}^n$.

Proof:

1. Suppose $A_1, \dots, A_n \dots \in \mathfrak{S}$.Then, $A_1, \dots, A_n, \dots \in \mathfrak{S}_i\ \forall i$. But, $\mathfrak{S}_i$ is a $\sigma-$algebra for all i so that $\bigcup_{n\geq 1} A_n \in \mathfrak{S}_i\ \forall i$. Then, $\bigcup_{n\geq 1} A_n \in \bigcap_{i\geq 1} \mathfrak{S}_i = \mathfrak{S}$.
2. Suppose $A \in \mathfrak{S}$ then we have that $A \in \mathfrak{S}_i\ \forall\, i$. But $\mathfrak{S}_i$ is a $\sigma-$ algebra so that $A^c \in \mathfrak{S}_i\ \forall\, i \implies A^c \in \bigcap_{i\geq 1} \mathfrak{S}_i = \mathfrak{S}$. $\blacksquare$

[3] We only present a restricted version of the Borel-Lebesgue theory of integration; the full version is built on **measurable subsets** of $\mathbb{R}^n$, where the **measure** generalizes the notions of of length, surface area, volume, etc. to "not-as-nice" geometric subsets of $\mathbb{R}^n$ (a feature of the theory is that not every subset of $\mathbb{R}^n$ is measurable).

The standard topology is not a $\sigma-$algebra of $\mathbb{R}^n$, but since $\tau \in \wp(\mathbb{R}^n)$, there is at least one σ-algebra containing the open sets of $\mathbb{R}^n$. The **Borel $\sigma-$algebra of $\mathbb{R}^n$** is the intersection of all $\sigma-$algebras containing the open sets of $\mathbb{R}^n$, we denote it by:

$$\mathcal{B} = \mathcal{B}(\mathbb{R}^n) = \bigcap_{\tau \subseteq \mathfrak{S} \in \wp(\mathbb{R}^n)} \mathfrak{S}.$$

An element of $\mathcal{B}$ is called a **Borel set of $\mathbb{R}^n$**.

Just about every subset of $\mathbb{R}^n$ that we encounter **in practice** is a Borel set:

- every open subset of $\mathbb{R}^n$ is a Borel set of $\mathbb{R}^n$;
- every closed subset of $\mathbb{R}^n$ is a Borel set of $\mathbb{R}^n$;
- any set built *via* unions, intersections, and complements with open sets and/or closed sets is a Borel set of $\mathbb{R}^n$.

Theorem 287
Let $\mathcal{B} = \mathcal{B}(\mathbb{R}^2)$. There exists a unique function Area : $\mathcal{B} \to [0, \infty]$ such that:

1. *Area$(A) \geq 0, \ \forall A \in \mathcal{B}$*
2. *if $A_1, \ldots, A_n, \ldots \in \mathcal{B}$ are pairwise disjoint then:*

$$\textit{Area}\left(\bigcup_{n\geq 1} A_n\right) = \sum_{n\geq 1} \textit{Area}(A_n)$$

3. *Area$([a, a'] \times [b, b']) = (a' - a)(b' - b)$.*

The area function whose existence is guaranteed by theorem 287 corresponds to our intuition of area in $\mathbb{R}^2$, but such a function cannot be defined on the entirety of $\wp(\mathbb{R}^2)$ (see the Banach-Tarski paradox).[4]

Theorem 288
Let $A, B \in \mathcal{B}(\mathbb{R}^2)$ such that $A \subseteq B$, then Area$(A) \leq$ Area(B).

Proof: by definition $B = (A \cap B) \cup (A^c \cap B) = A \cup (B \backslash A^c)$ where $B \backslash A^c \in \mathcal{B}(\mathbb{R}^n)$. Hence, we have

$$\text{Area}(B) = \text{Area}(A) + \text{Area}(B \backslash A^c) \geq \text{Area}(A),$$

which completes the proof. ■

We can extend Theorem 287.2 to not necessarily pairwise disjoint Borel sets.

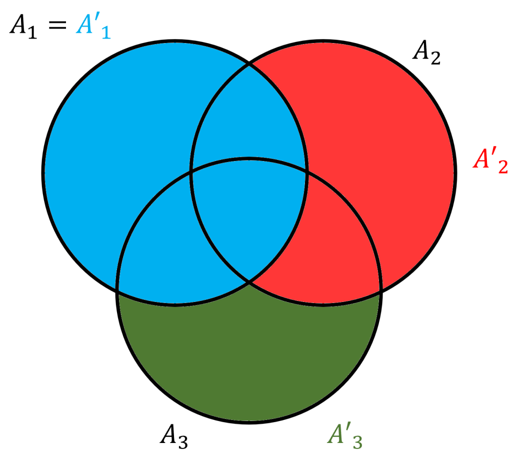
[4] Proving the existence of the function and of a set whose area cannot be measured is rather difficult and is properly tackled in advanced measure theory courses.

Theorem 289
Let $A_1, A_2, \ldots, A_n, \ldots \in \mathcal{B}(\mathbb{R}^2)$. *Then Area*$(\bigcup_{n\geq 1} A_n) \leq \sum_{n\geq 1}$ *Area*(A_n).

Proof: construct the sequence $A'_n \in \mathcal{B}(\mathbb{R}^2)$ as follows:

1. $A'_1 = A_1$;
2. $A'_2 = A_2 \cap A_1^c$;
3. $A'_3 = A_3 \cap (A_1 \cup A_2)^c$, etc.

The process is illustrated below on A_1, A_2, A_3.



Then $A'_1, \ldots, A'_n, \ldots \in \mathcal{B}(\mathbb{R}^2)$ are pairwise disjoint and

$$A_1 \cup A_2 \cup \ldots \cup A_n = A'_1 \cup A'_2 \cup \ldots \cup A'_n$$

for all $n \geq 1$. Since $A'_n \subseteq A_n \; \forall n \geq 1$. Then

$$\text{Area}\left(\bigcup_{n\geq 1} A_n\right) = \text{Area}\left(\bigcup_{n\geq 1} A'_n\right) = \sum_{n\geq 1} \text{Area}(A'_n) \leq \sum_{n\geq 1} \text{Area}(A_n),$$

which completes the proof. ■

We say that $B \subseteq \mathbb{R}^2$ has a $(2D)$ **measure** 0 if $\forall \varepsilon > 0$, there is a **cover**

$$\{R_1, R_2, \ldots, R_n, \ldots\}$$

of B by rectangles $R_n = [a_n, a'_n] \times [b_n, b'_n]$ with $a_n < a'_n$ and $b_n < b'_n$ for all $n \geq 1$, such that

$$\sum_{n\geq 1} \text{Area}(R_n) < \varepsilon.$$

Examples

1. Show that $B = \mathbb{R} \times \{b\}$ has a $2D$ measure 0 for any choice of $b \in \mathbb{R}$.

   Proof: let $\varepsilon > 0$ and set
   $$R_n = [-n, n] \times \left[b - \frac{\varepsilon}{2n2^{n+2}}, b + \frac{\varepsilon}{2n2^{n+2}}\right].$$
   Then Area$(R_n) = 2n \cdot \frac{\varepsilon}{n2^{n+2}} = \frac{\varepsilon}{2^{n+1}}$ for all $n \in \mathbb{N}$, and $B \subseteq \bigcup_{n\geq 1} R_n$, so that
   $$0 \leq \text{Area}(B) \leq \sum_{n\geq 1}(R_n) = \varepsilon \sum_{n\geq 1} \frac{1}{2^{n+1}} < \varepsilon.$$
   As $\varepsilon > 0$ is arbitrary, Area$(B) = 0$. ■

2. $S^1 = \{(x, y) \mid x^2 + y^2 = 1\}$ has $2D$ measure 0.

3. Show that Area$((a, a') \times (b, b')) =$ Area$([a, a'] \times [b, b'])$.

   Proof: write
   $$[a, a'] \times [b, b'] = (a, a') \times (b, b') \sqcup \{a\} \times [b, b'] \sqcup \{a'\} \times [b, b'] \sqcup [a, a'] \times \{b\} \sqcup [a, a'] \times \{b'\}.$$
   Each of the components $[*, *] \times \{*\}$ are subsets of $\mathbb{R} \times \{*\}$, so that they have 2D area 0 (and similarly for the components $\{*\} \times [*, *]$). Thus
   $$\text{Area}([a, a'] \times [b, b']) \leq \text{Area}((a, a') \times (b, b')) + 0 + 0 + 0 + 0 = \text{Area}((a, a') \times (b, b')).$$
   But Area$((a, a') \times (b, b')) \leq$ Area$([a, a'] \times [b, b'])$ since $(a, a') \times (b, b') \subseteq [a, a'] \times [b, b']$, so Area$((a, a') \times (b, b')) =$ Area$([a, a'] \times [b, b'])$. ■

4. Show that every finite subset $B \subseteq \mathbb{R}^2$ has $2D$ measure 0.

   Proof: let $B = \{(x_1, y_1), \ldots (x_n, y_n)\}$ and $\varepsilon > 0$. Pick:

   - a closed rectangle R_1 with Area$(R_1) = \frac{\varepsilon}{2}$ and $(x_1, y_1) \in R_1$;
   - a closed rectangle R_2 with Area$(R_2) = \frac{\varepsilon}{2^2}$ and $(x_2, y_2) \in R_2$;

     $\ldots$
   - a closed rectangle R_n with Area$(R_n) = \frac{\varepsilon}{2^n}$ and $(x_n, y_n) \in R_n$;
   - for $m > n$, any closed rectangle with Area$(R_m) = \frac{\varepsilon}{2^{m+1}}$ will do.

   Then $B \subseteq \bigcup_{m\geq 1} R_m$ and
   $$\sum_{m\geq 1} \text{Area}(R_m) = \sum_{m\geq 1} \frac{\varepsilon}{2^{m+1}} < \varepsilon,$$
   which completes the proof. ■

5. Every countable subset of $\mathbb{R}^2$ has $2D$ measure 0.
6. Let $\varphi : [0,1] \to \mathbb{R}^2$ be continuous and such that there exists $M > 0$ with

$$\|\varphi(s) - \varphi(t)\|_\infty \leq M|s-t| \quad \forall s,t \in [0,1].$$

Then $\varphi([0,1])$ has $2D$ measure 0.

Proof: recall that $\|(x_1,x_2)\|_\infty = \max\{|x_1|,|x_2|\}$. For all $N \geq 1$, let

$$0 = t_0 < t_1 < \cdots < t_N = 1,$$

with $t_i = \frac{i}{N}$. Let $s_i, s_i' \in [t_{i-1}, t_i]$. By hypothesis,

$$\|\varphi(s_i) - \varphi(s_i')\|_\infty \leq M|s_i - s_i'| \leq M|t_{i-1} - t_i| \leq M\left|\frac{i-1}{N} - \frac{i}{N}\right| \leq \frac{M}{N}.$$

Thus, there exists a square $I_i \subseteq \mathbb{R}^2$ whose sides have length $\frac{2M}{N}$ such that $\varphi([t_{i-1},t_i]) \subseteq I_i$. By construction, for all $1 \leq i \leq N$ we have

$$\text{Area}(I_i) = \frac{4M^2}{N^2} \quad \text{and} \quad \sum_{i=1}^N \text{Area}(I_i) = \frac{4M^2}{N}.$$

Let $\varepsilon > 0$ and select $N > \frac{4M^2}{\varepsilon}$. Going through the above procedure yields a sequence of rectangles $R_i = I_i$ for $1 \leq i \leq N$; for $n > N$, set $R_n = \{*\} \subseteq \mathbb{R}^2$, a singleton square of area 0. Then

$$\varphi([0,1]) \subseteq \bigcup_{i=1} R_i \implies \sum_{i\geq 1} \text{Area}(R_i) = \frac{4M^2}{N} < \varepsilon,$$

which completes the proof. ■

In the rest of this section, we introduce the class of functions $f : \mathbb{R}^2 \to \overline{\mathbb{R}}$ for which we may expect that

$$\iint_{\mathbb{R}^2} f(x,y)\, \mathrm{d}x\, \mathrm{d}y \in \overline{\mathbb{R}}$$

exists.[5] As we see below, we cannot untangle the function rule from its domain. If $A \in \mathcal{B}(\mathbb{R}^2)$, let the **characteristic function** $\chi_A : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$\chi_A(x,y) = \begin{cases} 0 & \text{if } (x,y) \notin A \\ 1 & \text{if } (x,y) \in A \end{cases}$$

[5] The set $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ is the one-point compactification of $\mathbb{R}$ (see Section 17.4).

Characteristic functions are the building blocks of **Borel-Lebesgue integrable functions**; their integral is easy to obtain. Let $k \in \mathbb{R}$; if $f : \mathbb{R}^2 \to \overline{\mathbb{R}}$ is defined by $f(x, y) = k \cdot \chi_A(x, y)$, then the **Borel-Lebesgue integral of f over $\mathbb{R}^2$ is**

$$\iint_{\mathbb{R}^2} f \, \mathrm{d}x \, \mathrm{d}y = k \cdot \text{Area}(A) \in \overline{\mathbb{R}}.$$

A function $f : \mathbb{R}^2 \to \mathbb{R}$ is **simple** if $\exists A_1, \ldots, A_n \in \mathcal{B}(\mathbb{R}^2)$ and $a_1, \ldots, a_n \in \mathbb{R}$ such that

$$\mathbb{R}^2 = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_n \quad \text{and} \quad f|_{A_i} \equiv a_i;$$

in that case, $f = \sum_{i=1}^n a_i \chi_{A_i}$.

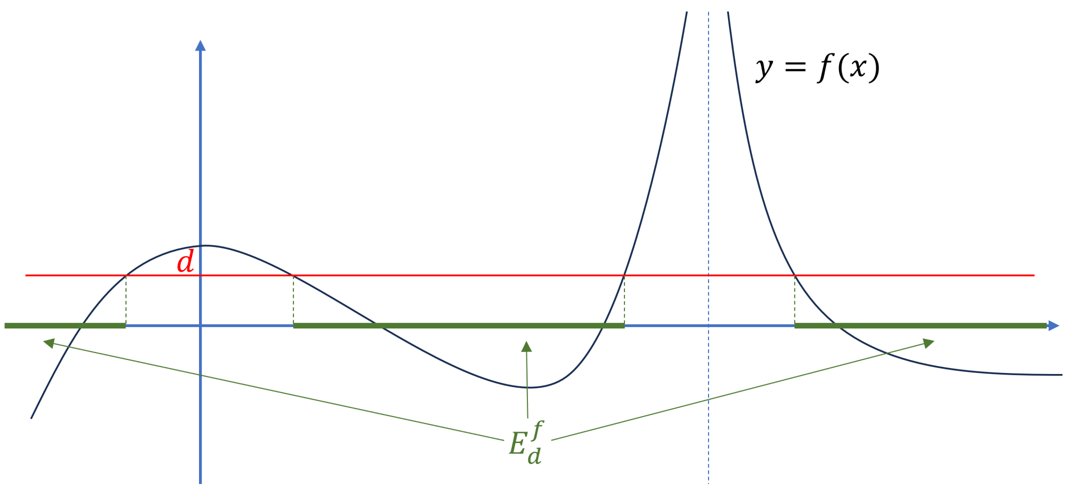
Examples (Simple Functions)

1. If $f(x, y) = k$ for all $(x, y) \in \mathbb{R}^2$, then f is a simple function.
2. If $f = \sum_{i=1}^n a_i \chi_{A_i}$, then $|f| = \sum_{i=1}^n |a_i| \chi_{A_i}$ is a simple function.
3. If $f = \sum_{i=1}^n a_i \chi_{A_i}$ and $g = \sum_{j=1}^m b_j \chi_{B_j}$ are simple functions, then
   a) $\mathbb{R} = \bigsqcup_{i=1}^n \bigsqcup_{j=1}^m A_i \cap B_j$;
   b) $f + g = \sum_{i=1}^n \sum_{j=1}^m (a_i + b_j) \chi_{A_i \cap B_j}$ is a simple function, and
   c) $fg = \sum_{i=1}^n \sum_{j=1}^m a_i \, b_j \, \chi_{A_i \cap B_j}$ is a simple function. $\square$

A **Borel function** is a function $f : \mathbb{R}^2 \to \overline{\mathbb{R}}$ for which

$$E_d^f = \{(x, y) \mid f(x, y) \leq d\} \in \mathcal{B}(\mathbb{R}^2), \quad \forall d \in \mathbb{R}.$$

We illustrate the concept below, for a function over $\mathbb{R}$.



Since every subset of $\mathbb{R}^2$ we encounter in practice is a Borel set, every function $f : \mathbb{R}^2 \to \overline{\mathbb{R}}$ we encounter in practice is a Borel function.[6]

[6] It is in fact rather difficult to construct a non-Borel function, although they do exist.

Proposition 289
Let $f, g : \mathbb{R}^2 \to \overline{\mathbb{R}}$ be Borel functions. Then, $|f|, f+g, fg$ are also Borel functions.

Proof: we prove the result only for $|f|$; the proof for the other two functions is left as an exercise. Write $z = (x, y) \in \mathbb{R}^2$. Then, we want to show

$$E_d^{|f|} = \{z \in \mathbb{R}^2 \mid |f(z)| \leq d\} \in \mathcal{B}(\mathbb{R}^2 \quad \forall d \in \mathbb{R}$$

1. if $d < 0$, then $E_d^{|f|} = \varnothing \in \mathcal{B}(\mathbb{R}^2)$;
2. if $d \geq 0$, then

$$\begin{aligned} E_d^{|f|} &= \{z \mid -d \leq f(z) \leq d\} = \{z \mid -d \leq f(z)\} \cap \{z \mid f(z) \leq d\} \\ &= \{z \mid -d \leq f(z)\} \cap E_d^f = \{z \mid f(z) < -d\}^c \cap E_d^f \\ &= \left(\bigcup_{n \geq 1} E^f_{-d-\frac{1}{n}} \right)^c \cap E_d^f. \end{aligned}$$

But f is a Borel function, so $E_d^f, E^f_{-d-\frac{1}{n}} \in \mathcal{B}(\mathbb{R}^2)$ for all $n \geq 1$. This implies that

$$\bigcup_{n \geq 1} E^f_{-d-\frac{1}{n}} \in \mathcal{B}(\mathbb{R}^2),$$

as $\mathcal{B}(\mathbb{R}^2)$ is a $\sigma-$algebra, and so that

$$\mathbb{R}^2 \setminus \left(\bigcup_{n \geq 1} E^f_{-d-\frac{1}{n}} \right) \in \mathcal{B}(\mathbb{R}^2),$$

for the same reason; hence $E_d^{|f|} \in \mathcal{B}(\mathbb{R}^2)$. ■

We can approximate positive-valued Borel functions with simple functions.

Theorem 290
Let $f : \mathbb{R}^2 \to [0, \infty]$ be a Borel function; then there is a sequence (f_n) of simple functions such that:

1. *$\forall z \in \mathbb{R}^2$, $f_n(z) \to f(z)$, and*
2. *$0 \leq f_n \leq f$, for all $n \geq 1$.*

Proof: we provide a proof for $f : \mathbb{R} \to [0,\infty]$; the proof for functions on $\mathbb{R}^k$ is identical, but the simpler case is easier to illustrate.

We build the sequence (f_n) as follows.

1. For f_1, write

$$\mathbb{R} = \underbrace{\left\{x \,\middle|\, 0 \le f(x) < \frac{1}{2^1}\right\}}_{A_1^1} \sqcup \underbrace{\left\{x \,\middle|\, \frac{1}{2} \le f(x) < 1\right\}}_{A_2^1} \sqcup \underbrace{\{x \mid f(x) \ge 1\}}_{A^1},$$

   and set

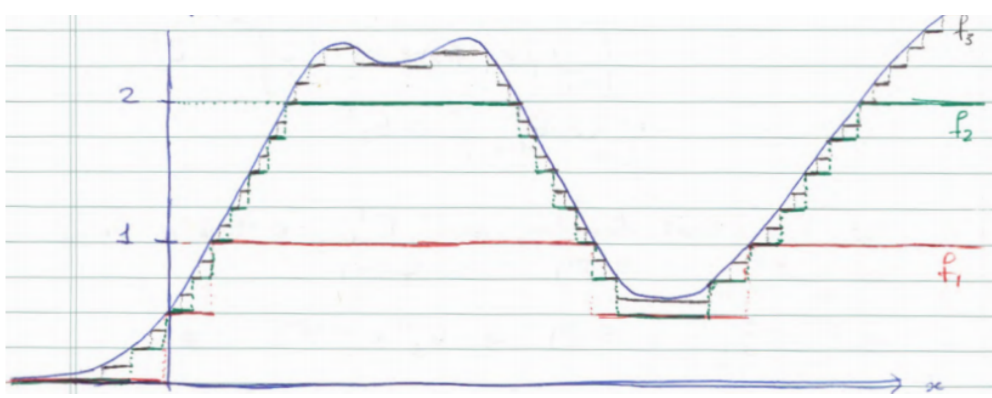
$$f_1 = 0 \cdot \chi_{A_1^1} + \frac{1}{2}\chi_{A_2^1} + 1 \cdot \chi_{A^1}, \quad A_1^1, A_2^1, A^1 \in \mathcal{B}(\mathbb{R}^2).$$

2. For f_2, write

$$\mathbb{R} = \left(\bigsqcup_{i=1}^{8} \underbrace{\left\{x \,\middle|\, \frac{i-1}{2^2} \le f(x) < \frac{i}{2^2}\right\}}_{A_i^2} \right) \sqcup \underbrace{\{x \mid f(x) \ge 2\}}_{A^2} = \left(\bigsqcup_{i=1}^{8} A_i^2 \right) \cup A^2,$$

   and set

$$f_2 = \sum_{i=1}^{8} \frac{i-1}{2^2}\chi_{A_i^2} + 2\chi_{A^2}.$$



...

n. For f_n, write $A^n = \{x \mid f(x) \ge n\}$ and

$$A_i^n = \left\{x \,\middle|\, \frac{i-1}{2^n} \le f(x) < \frac{i}{2^n}\right\}, \quad \text{for } 1 \le i \le n \cdot 2^n.$$

We then have $\mathbb{R} = \left(\bigsqcup_{i=1}^{n\cdot 2^n} A_i^n\right) \sqcup A^n$. Set $f_n = \sum_{i=1}^{n\cdot 2^n} \frac{i-1}{2^n}\chi_{A_i^n} + n \cdot \chi_{A^n}$.

By construction, each f_n is simple and

$$0 \leq f_1(x) \leq f_2(x) \leq \cdots \leq f_n(x) \leq \cdots \leq f(x) \quad \forall x \in \mathbb{R}.$$

1. If $f(x) = \infty$, then $x \in A^n$ for all $n \geq 1$, whence $f_n(x) = n \to \infty = f(x)$
2. If $f(x) < \infty$, then for $n > f(x)$, there exists $1 \leq i \leq u \leq n \cdot 2^n$ such that

$$\frac{i-1}{2^n} \leq f(x) < \frac{i}{2^n}.$$

In that case $x \in A_i^n$ and

$$|f(x) - f_n(x)| = \left| f(x) - \frac{i-1}{2^n} \right| < \frac{1}{2^n} \to 0,$$

which completes the proof. ■

23.2 Integral of Simple Functions

Let $f = \sum_{i=1}^k \alpha_i \chi_{A_i}$ be a simple function $\mathbb{R}^2 \to [0,\infty]$, that is, $\alpha_i \in [0,\infty]$ for $1 \leq i \leq k$ and $\mathbb{R}^2 = A_1 \sqcup \cdots \sqcup A_k$. Since simple functions are finite linear combinations of characteristic functions, we define the **integral of a simple function** as

$$\iint_{\mathbb{R}^2} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \sum_{i=1}^k \alpha_i \cdot \text{Area}(A_i) \in [0,\infty]$$

(in the Borel-Lebesgue theory of integration, we have $0 \cdot (+\infty) = 0$, by convention). But there might be multiple ways to write a simple function as a sum of characteristic functions: if

$$f = \sum_{i=1}^k \alpha_i \chi_{A_i} = \sum_{j=1}^m \beta_j \chi_{B_j},$$

is the integral the same in both cases? For each $1 \leq i \leq k$, let $J_i = \{j \mid \beta_j = \alpha_i\}$. Then

$$\begin{aligned}\sum_{j=1}^m \beta_j \cdot \text{Area}(B_j) &= \sum_{i=1}^k \sum_{j \in J_i} \beta_j \cdot \text{Area}(B_j) = \sum_{i=1}^k \alpha_i \sum_{j \in J_i} \text{Area}(B_j) \\ &= \sum_{i=1}^k \alpha_i \cdot \text{Area}\left(\bigsqcup_{j \in J_i} B_j \right) = \sum_{i=1}^k \alpha_i \cdot \text{Area}(A_i).\end{aligned}$$

In what follows, we denote the **set of simple functions on** $\mathbb{R}^n$ by $\zeta^{(n)}$ and the **set of positive simple functions on** $\mathbb{R}^n$ by $\zeta_+^{(n)}$.

Lemma 291
Let $f, g \in \zeta_+^{(2)}, \alpha \geq 0$. *Then:*

1. $\iint_{\mathbb{R}^2} \alpha f \, \mathrm{d}x \, \mathrm{d}y = \alpha \iint_{\mathbb{R}^2} f \, \mathrm{d}x \, \mathrm{d}y$;
2. $\iint_{\mathbb{R}^2} (f+g) \, \mathrm{d}x \, \mathrm{d}y = \iint_{\mathbb{R}^2} f \, \mathrm{d}x \, \mathrm{d}y + \iint_{\mathbb{R}^2} g \, \mathrm{d}x \, \mathrm{d}y$, *and*
3. if $f \leq g$ on $\mathbb{R}^2$, then $\iint_{\mathbb{R}^2} f \, \mathrm{d}x \, \mathrm{d}y \leq \iint_{\mathbb{R}^2} g \, \mathrm{d}x \, \mathrm{d}y$.

Proof: note that the results hold over general multi-dimensional spaces, but we restrict the demonstration to $\mathbb{R}^2$.

1. The first statement is clear; its proof is left as an exercise.
2. If $f = \sum_{i=1}^n \alpha_i \chi_{A_i}$ and $g = \sum_{j=1}^m \beta_j \chi_{B_j}$ then $f + g = \sum_{i,j} (\alpha_i + \beta_j) \chi_{A_i \cap B_j}$ and

$$\begin{aligned}
\iint_{\mathbb{R}^2} (f+g) \, \mathrm{d}x \, \mathrm{d}y &= \sum_{i,j} (\alpha_i + \beta_j) \cdot \text{Area}(A_i \cap B_j) \\
&= \sum_{i,j} \alpha_i \cdot \text{Area}(A_i \cap B_j) + \sum_{i,j} \beta_j \cdot \text{Area}(A_i \cap B_j) \\
&= \sum_{i=1}^n \alpha_i \sum_{j=1}^m \text{Area}(A_i \cap B_j) + \sum_{j=1}^m \beta_j \sum_{i=1}^n \text{Area}(A_i \cap B_j) \\
&= \sum_{i=1}^n \alpha_i \cdot \text{Area}\left[A_i \cap \left(\bigsqcup_{j=1}^m B_j\right)\right] + \sum_{j=1}^m \beta_j \cdot \text{Area}\left[B_j \cap \left(\bigsqcup_{i=1}^n A_i\right)\right] \\
&= \sum_{i=1}^n \alpha_i \cdot \text{Area}(A_i) + \sum_{j=1}^m \beta_j \cdot \text{Area}(B_j) = \iint_{\mathbb{R}^2} f \, \mathrm{d}x \, \mathrm{d}y + \iint_{\mathbb{R}^2} g \, \mathrm{d}x \, \mathrm{d}y
\end{aligned}$$

3. If $f \leq g$ on $\mathbb{R}^2$, then $g - f \in \zeta_+^{(2)}$ and

$$\iint_{\mathbb{R}^2} = \iint_{\mathbb{R}^2} [f + (g-f)] \, \mathrm{d}x \, \mathrm{d}y = \iint_{\mathbb{R}^2} f \, \mathrm{d}x \, \mathrm{d}y + \underbrace{\iint_{\mathbb{R}^2} (g-f) \, \mathrm{d}x \, \mathrm{d}y}_{\geq 0} \geq \iint_{\mathbb{R}^2} f \, \mathrm{d}x \, \mathrm{d}y,$$

since $g - f \geq 0$. ■

The first two properties of Lemma 291 indicate that the integral of a simple function behaves as a **linear operator** on the set of positive simple functions on $\mathbb{R}^n$.[7]

[7]We cannot say "over the vector space of positive simple functions" since $\zeta_+^{(n)}$ is not a vector space over $\mathbb{R}$... but $\zeta^{(n)}$ is, however.

Furthermore, if $f = \chi_A$, $A \in \mathcal{B}(\mathbb{R}^2)$, then $\iint f\,\mathrm{d}x\,\mathrm{d}y = \text{Area}(A)$.[8]

As mentioned in the proof of Lemma 291, we can generalize the notion of the integral of positive simple functions directly to higher dimensions. For instance, if $f : \mathbb{R}^3 \to \mathbb{R} \in \zeta_+^{(3)}$, then

$$\iiint_{\mathbb{R}^3} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_{\mathbb{R}^3} \sum_{k=1}^{\ell} \gamma_k \chi_{A_k}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \sum_{k=1}^{\ell} \gamma_k \cdot \text{Vol}(A_k),$$

and so on with $n \geq 3$:

$$\int \cdots \int f(x_1,\ldots,x_n)\,\mathrm{d}x_1 \ldots \mathrm{d}x_n$$

if $f : \mathbb{R}^n \to \mathbb{R}$ is in $\zeta_+^{(n)}$.

23.3 Integral of Positive Borel Functions

Of course, the overwhelming majority of functions on $\mathbb{R}^n$ are not simple positive functions; but large classes of non-negative functions can be approximated by simple functions (as we have seen Theorem 290). If f is a **positive Borel function** of $\mathbb{R}^2$ to $[0,\infty]$, its Borel-Lebesgue integral

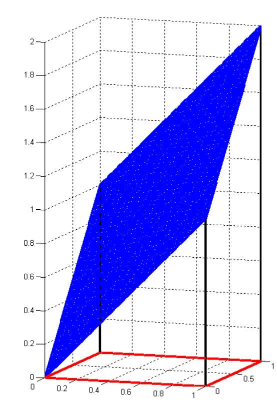
$$\iint f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \sup_{s \in \zeta_+^{(2)}} \left\{ \iint s\,\mathrm{d}x\,\mathrm{d}y \;\middle|\; s \leq f \right\};$$

this definition can be extended to higher-dimensional domains in the obvious way. We illustrate how it applies in practice with a deceptively complicated example.

Example: using the definition, find $\iint f\,\mathrm{d}x\,\mathrm{d}y$, where

$$f(x,y) = \begin{cases} x+y & \text{if } (x,y) \in [0,1]^2 \\ 0 & \text{otherwise} \end{cases}$$

Solution: the function is shown below.



[8]When the context is clear, we may omit the domain of integration.

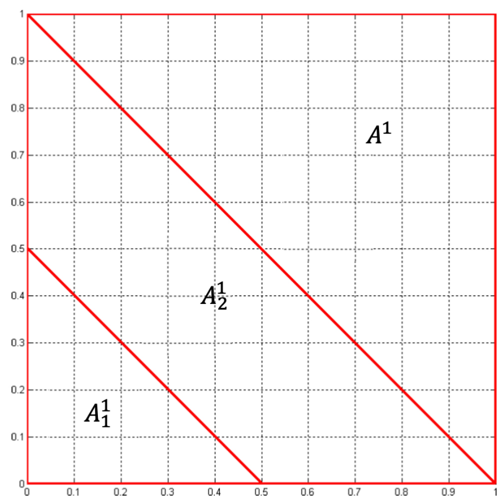
We start by building the sequence of positive simple functions

$$s_1 \leq \ldots \leq s_n \leq \ldots \leq f$$

from Theorem 290.

For $n = 1$, we have:

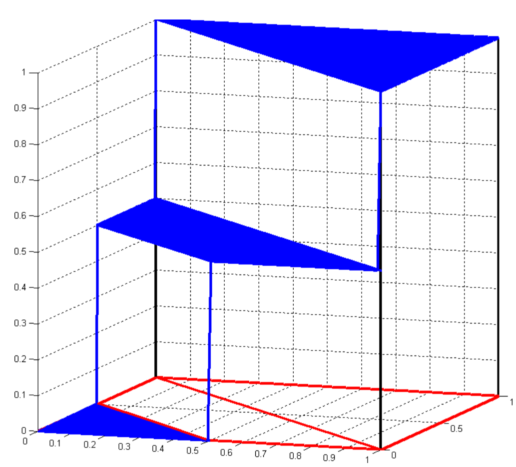
- $A_1^1 = \{(x, y) \mid 0 \leq f(x, y) < \frac{1}{2}\} = (\{(x, y) \mid 0 \leq x + y < \frac{1}{2}\} \cap [0, 1]^2) \cup (\mathbb{R}^2 \setminus [0, 1]^2)$,
- $A_2^1 = \{(x, y) \mid \frac{1}{2} \leq f(x, y) < 1\} = \{(x, y) \mid \frac{1}{2} \leq x + y < 1\} \cap [0, 1]^2$, and
- $A^1 = \{(x, y) \mid f(x, y) \geq 1\} = \{(x, y) \mid x + y \geq 1\} \cap [0, 1]^2$ (see below).



The first simple approximation is thus

$$s_1 = 0 \cdot \chi_{A^1} + \frac{1}{2} \cdot \chi_{A_2^1} + 1 \cdot \chi_{A^1},$$

whose graph is shown below:



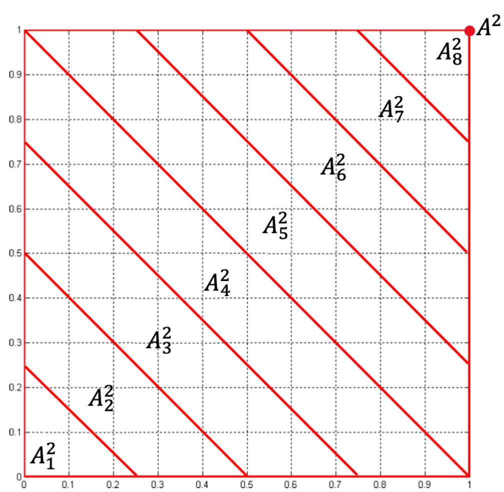
We then have

$$\iint s_1(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0\cdot \text{Area}(A_1^1) + \frac{1}{2}\cdot \text{Area}(A_2^1) + 1\cdot \text{Area}(A^1),$$

whose value we leave un-evaluated.

For $n=2$, we have

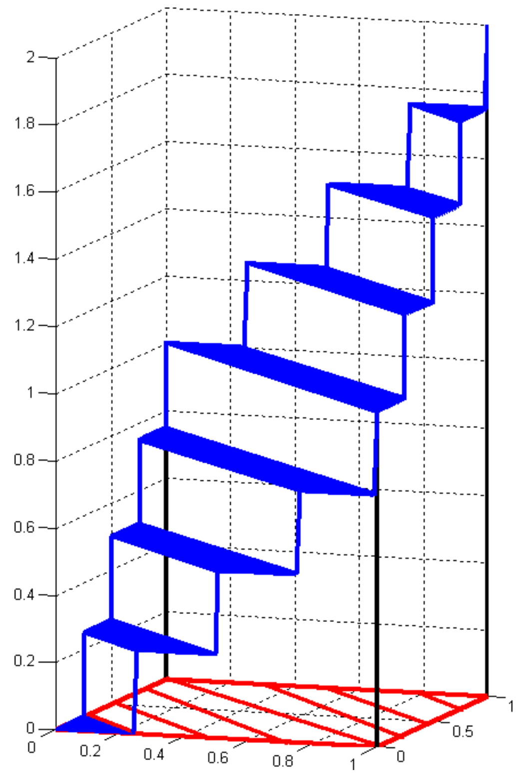
- $A_1^2 = \{(x,y) \mid 0 \leq f(x,y) < \frac{1}{4}\} = (\{(x,y) \mid 0 \leq x+y < \frac{1}{4}\} \cap [0,1]^2) \cup (\mathbb{R}^2 \setminus [0,1]^2)$,
- $A_2^2 = \{(x,y) \mid \frac{1}{4} \leq f(x,y) < \frac{2}{4}\} = \{(x,y) \mid \frac{1}{4} \leq x+y < \frac{1}{2}\} \cap [0,1]^2$,
- $A_3^2 = \{(x,y) \mid \frac{2}{4} \leq f(x,y) < \frac{3}{4}\} = \{(x,y) \mid \frac{1}{2} \leq x+y < \frac{3}{4}\} \cap [0,1]^2$,
- $A_4^2 = \{(x,y) \mid \frac{3}{4} \leq f(x,y) < \frac{4}{4}\} = \{(x,y) \mid \frac{3}{4} \leq x+y < 1\} \cap [0,1]^2$,
- $A_5^2 = \{(x,y) \mid \frac{4}{4} \leq f(x,y) < \frac{5}{4}\} = \{(x,y) \mid 1 \leq x+y < \frac{5}{4}\} \cap [0,1]^2$,
- $A_6^2 = \{(x,y) \mid \frac{5}{4} \leq f(x,y) < \frac{6}{4}\} = \{(x,y) \mid \frac{5}{4} \leq x+y < \frac{3}{2}\} \cap [0,1]^2$,
- $A_7^2 = \{(x,y) \mid \frac{6}{4} \leq f(x,y) < \frac{7}{4}\} = \{(x,y) \mid \frac{3}{2} \leq x+y < \frac{7}{4}\} \cap [0,1]^2$,
- $A_8^2 = \{(x,y) \mid \frac{7}{4} \leq f(x,y) < \frac{8}{4}\} = \{(x,y) \mid \frac{7}{4} \leq x+y < 8\} \cap [0,1]^2$, and
- $A^2 = \{(1,1)\}$ (see below).



The second simple approximation is thus

$$s_2 = \sum_{i=1}^{2(2^2)} \frac{i-1}{2^2}\cdot \chi_{A_i^2} + 2\cdot \chi_{A^2} = \sum_{i=1}^{8} \frac{i-1}{4}\cdot \chi_{A_i^2} + 2\cdot \chi_{A^2},$$

whose graph is shown on the next page:

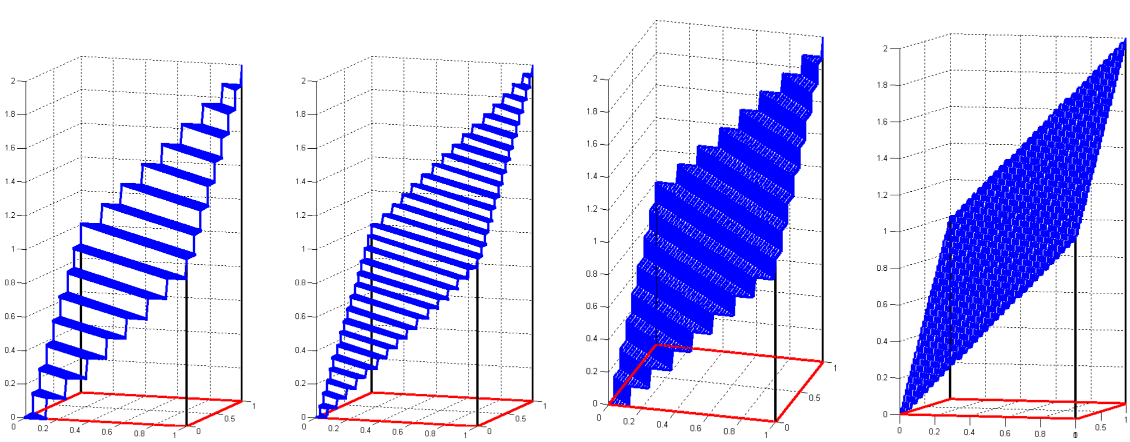


We then have

$$\iint s_2(x,y)\,\mathrm{d}x\,\mathrm{d}y = \sum_{i=1}^{8} \frac{i-1}{4} \cdot \mathrm{Area}(A_i^2) + 2 \cdot \mathrm{Area}(A^2),$$

whose value we again leave un-evaluated.

The process continues in the same way for all n, yielding a sequence of positive simple functions.



At step n, we have:

- $A_1^n = \left(\{(x,y) \mid 0 \leq x+y < \frac{1}{2^n}\} \cap [0,1]^2\right) \cup (\mathbb{R}^2 \setminus [0,1]^2)$,
- $A_i^n = \{(x,y) \mid \frac{i-1}{2^n} \leq x+y < \frac{i}{2^n}\} \cap [0,1]^2$ for $2 \leq i \leq 2^{n+1}$,
- $A_{2^{n+1}+1}^n = \{(1,1)\}$ and $A^n = A_j^n = \varnothing$ for $j > 2^{n+1}+1$.

Then the nth simple approximation is

$$s_n = \sum_{i=1}^{n\cdot 2^n} \frac{i-1}{2^n} \cdot \chi_{A_i} + n \cdot \chi_{A^n} = \sum_{i=1}^{2^{n+1}} \frac{i-1}{2^n} \cdot \chi_{A_i^n} + 2 \cdot \chi_{A_{2^{n+1}+1}^n},$$

so that

$$\begin{aligned}\iint s_n(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \sum_{i=1}^{2^{n+1}} \frac{i-1}{2^n} \cdot \text{Area}(A_i^n) + 2 \cdot \underbrace{\text{Area}(A_{2^{n+1}+1}^n)}_{=0} \\ &= \sum_{i=1}^{2^n} \frac{i-1}{2^n} \cdot \text{Area}(A_i^n) + \sum_{i=2^n+1}^{2^{n+1}} \frac{i-1}{2^n} \cdot \text{Area}(A_i^n).\end{aligned}$$

We can show (see Exercises) that

$$\text{Area}(A_i^n) = \begin{cases} \frac{1}{4^n}\left(i-\frac{1}{2}\right) & \text{for } 1 \leq i \leq 2^n \\ \frac{1}{4^n}\left(2^{n+1}-i-\frac{1}{2}\right) & \text{for } 2^n+1 \leq i \leq 2^{n+1} \end{cases}$$

In general, then, we have:

$$\begin{aligned}\iint s_n(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \sum_{i=1}^{2^n} \frac{i-1}{2^n} \cdot \frac{1}{4^n}\left(i-\frac{1}{2}\right) + \sum_{i=2^n+1}^{2^{n+1}} \frac{i-1}{2^n} \cdot \frac{1}{4^n}\left(2^{n+1}-i-\frac{1}{2}\right) \\ &= \frac{1}{2^n r^n}\left[\sum_{i=1}^{2^n}(i-1)(i-1/2) + \sum_{i=1}^{2^{n+1}}(i-1)(2^{n+1}-i-1/2)\right] \\ &= 1 - \frac{1}{2^{n-1}} + \frac{1}{2\cdot 4^n}.\end{aligned}$$

Write $B_n = \iint s_n\,\mathrm{d}x\,\mathrm{d}y$; we clearly have $B_n < 1$ for all n, and $B_n \to 1$. Then

$$\iint f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \sup\left\{\iint s(x,y)\,\mathrm{d}x,\ \mathrm{d}y \;\middle|\; s \in \zeta_+^{(2)}, s \leq f\right\} \geq 1 = \lim_{n\to\infty} B_n.$$

For $s \in \zeta_+^{(2)}$, we have seen that

$$\iint s(x,y)\,\mathrm{d}x\,\mathrm{d}y = \sum_{j=1}^{m} \alpha_j \cdot \text{Area}(A_i),$$

and so the integral represents the volume of a collection of m prisms with base area A_j and height α_j. By construction,

$$\iint s(x,y)\,\mathrm{d}x\,\mathrm{d}y \leq \text{Volume}(\text{solid bounded by } 0 \leq x, y \leq 1 \text{ and } 0 \leq z \leq x+y).$$

We cannot compute the volume using integrals as we have not yet established that the integral of a general positive Borel function over a domain A is the volume of the solid bounded by f over A, but we see easily that the solid in question is exactly the bottom half of the prism defined by $0 \leq x, y \leq 1$ and $0 \leq z \leq 2$, whose volume we know to be 2, from geometry (see the bottom image on p. 501).

By definition, we must then have

$$\sup\left\{\iint s(x,y)\,\mathrm{d}x,\ \mathrm{d}y \,\middle|\, s \in \zeta_+^{(2)}, s \leq f\right\} \leq \frac{1}{2}(2) = 1,$$

which, combined with the previous inequality, shows that

$$\iint f(x,y)\,\mathrm{d}x\,\mathrm{d}y = 1.$$

Phew! □

If $f \in \zeta_+^{(2)}$, both definitions **coincide**: i.e, if $f = \sum \alpha_i \chi_{A_i}$, with $\alpha_i \in \overline{\mathbb{R}}$, $A_i \in \mathcal{B}(\mathbb{R}^2)$, and $A_1 \sqcup \cdots \sqcup A_n = \mathbb{R}^2$, then

$$\iint f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \sum_{i=1}^{n} \alpha_i \cdot \text{Area}(A_i) = I(f) = \sup\left\{\iint s(x,y)\,\mathrm{d}x\,\mathrm{d}y \,\middle|\, s \in \zeta_+^{(2)}, s \leq f\right\}.$$

Indeed, if $f \in \zeta_+^{(2)}$, we have $\iint f(x,y)\,\mathrm{d}x\,\mathrm{d}y \leq I(f)$. On the other hand, if $s \in \zeta_+^{(2)}$, with $s \leq f$, then

$$\iint s(x,y)\,\mathrm{d}x\,\mathrm{d}y \leq \iint f(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$

according to Lemma 291.3, from which we conclude that

$$I(f) = \sup\left\{\iint s(x,y)\,\mathrm{d}x\,\mathrm{d}y \,\middle|\, s \in \zeta_+^{(2)}, s \leq f\right\} \leq \iint f(x,y)\,\mathrm{d}x\,\mathrm{d}y \leq I(f).$$

The next result shows that Lemma 291.3 also applies to positive Borel functions.

Proposition 292
If f, g are positive Borel functions and if $f \leq g$, then

$$\iint f\,\mathrm{d}x\,\mathrm{d}y \leq \iint g\,\mathrm{d}x\,\mathrm{d}y.$$

Proof: if $f \leq g$, then $\{s \in \zeta_+^{(2)} \mid s \leq f\} \subseteq \{s \in \zeta_+^{(2)} \mid s \leq g\}$ whence

$$\left\{ \iint s(x,y)\,\mathrm{d}x\,\mathrm{d}y \;\middle|\; s \in \zeta_+^{(2)}, s \leq f \right\} \subseteq \left\{ \iint s(x,y)\,\mathrm{d}x\,\mathrm{d}y \;\middle|\; s \in \zeta_+^{(2)}, s \leq g \right\}$$

So that

$$\sup \left\{ \iint s(x,y)\,\mathrm{d}x\,\mathrm{d}y \;\middle|\; s \in \zeta_+^{(2)}, s \leq f \right\} \subseteq \sup \left\{ \iint s(x,y)\,\mathrm{d}x\,\mathrm{d}y \;\middle|\; s \in \zeta_+^{(2)}, s \leq g \right\}$$

■

One might wonder why exactly we bothered to introduce the Borel-Lebesgue integral – while going from Riemann sums to simple functions does change our viewpoint of integration, are the corresponding integrals equivalent, or is one "preferable" over the other?

Theorem 293 (Lebesgue Monotone Convergence Theorem)
Let $(f_n)_{n\geq 1}$ be a sequence of Borel functions on $\mathbb{R}^2$ such that

1. $0 \leq f_1(x,y) \leq f_2(x,y) \leq \cdots \leq f_n(x,y) \leq \cdots \quad \forall (x,y) \in \mathbb{R}^2$, *and*
2. $f_n(x,y) \to f(x,y) \quad \forall (x,y) \in \mathbb{R}^2$.

Then f is a Borel function on $\mathbb{R}^2$ and $\iint f_n(x,y)\,\mathrm{d}x\,\mathrm{d}y \to \iint f(x,y)\,\mathrm{d}x\,\mathrm{d}y$. In particular, $\iint f\,\mathrm{d}x\,\mathrm{d}y = \lim_{n\to\infty} \iint s_n\,\mathrm{d}x\,\mathrm{d}y$, whenever (s_n) is a monotonically increasing sequence of positive simple functions bounded above by f, with $s_n \to f$ (pointwise).

Proof: left as a (difficult) exercise. ■.

Theorem 293 suggests that the new definition has a clear advantage: what additional constraint does the equivalent limit interchange theorem 69 of Riemann integration require?

Corollary 294
Let $f, g : \mathbb{R}^2 \to [0,\infty]$ be Borel functions and $\alpha \geq 0$. Then

1. $\iint (f+g)\,\mathrm{d}x\,\mathrm{d}y = \iint f\,\mathrm{d}x\,\mathrm{d}y + \iint g\,\mathrm{d}x\,\mathrm{d}y$,
2. $\iint \alpha f\,\mathrm{d}x\,\mathrm{d}y = \alpha \iint f\,\mathrm{d}x\,\mathrm{d}y$.

Proof: left as an exercise. ■.

From this point on, in order to not have to rely on the notation of **iterated integrals**, we write

$$\int f\,\mathrm{d}m = \int \cdots \int f(x_1,\ldots,x_n)\,\mathrm{d}x_1 \cdots \mathrm{d}x_n$$

and $m(B)$ for the **measure** of $B \subseteq \mathbb{R}^n$ (a generalization of the length, area, volume).

Theorem 295
Let f be a positive Borel function, taking on the value 0 outside of a Borel set A with Area$(A) = 0$. Then $\iint f \, \mathrm{d}x \, \mathrm{d}y = 0$.

Proof: let

$$k(x,y) = \begin{cases} \infty & (x,y) \in A \\ 0 & (x,y) \notin A \end{cases}$$

THen $k \in \zeta_+^{(2)}$ and

$$\int k \, \mathrm{d}m == 0 \cdot \text{Area}(\mathbb{R}^2 \setminus A) + \infty \cdot \text{Area}(A) = 0 \cdot \infty + \infty \cdot 0 = 0,$$

by convention. Since $f \leq k$, then

$$0 \leq \int f \, \mathrm{d}m \leq \int k \, \mathrm{d}m = 0,$$

which completes the proof. ■

We say that a positive Borel function f is **(Borel-Lebesgue) integrable** if $\int f \, \mathrm{d}m < \infty$. If $f \geq 0$ is integrable and $g \leq f$ is a Borel function, then

$$\infty > \int f \, \mathrm{d}m = \int g \, \mathrm{d}m + \int (f - g) \, \mathrm{d}m \geq \int g \, \mathrm{d}m,$$

and so g is also integrable. This result definitely does not hold in general for Riemann integration.[9]

Theorem 296
Let g be a bounded positive Borel function, taking on the value 0 outside a bounded Borel set A. Then g is integrable.

Proof: let M be such that $g(z) \leq M$. By definition, $\exists B = [a_1, a_1'] \times [a_2, a_2']$ such that $A \subseteq B$ and $g(z) = 0$ if $z \notin B$. Then $g \leq M\chi_B$ and

$$\int g \, \mathrm{d}m \leq \int M\chi_B \, \mathrm{d}m = M \cdot \text{Area}(\chi_B) < \infty,$$

which completes the proof. ■

We can extend the idea to general Borel functions using the positive and negative parts.

Note that the Riemann and Borel-Lebesgue integral **coincide** when the former exists.

[9] Can you think of a counterexample?

23.4 Integral of Borel Functions

For a general function $f : \mathbb{R}^n \to \mathbb{R}$, define the **positive part of** f by

$$f_+(x) = \begin{cases} f(x) & \text{when } f(x) \geq 0 \\ 0 & \text{when } f(x) < 0, \end{cases}$$

and the **negative part of** f by

$$f_-(x) = \begin{cases} -f(x) & \text{when } f(x) \leq 0 \\ 0 & \text{when } f(x) > 0. \end{cases}$$

Then $f = f_+ - f_-$ and $|f| = f_+ + f_-$.

If $f : \mathbb{R}^n \to \mathbb{R}$ is a finite Borel function, then f_+, f_- are positive Borel functions, by definition. A Borel function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is **integrable** if both f_+ and f_- are integrable. In this case, we define

$$\int f \, \mathrm{d}m = \int f_+ \, \mathrm{d}m - \int f_- \, \mathrm{d}m.$$

We see now that Lemma 291 has a counterpart for Borel functions.

Theorem 297
Let f, g be integrable functions and $\lambda \in \mathbb{R}$. Then

1. $\int \lambda f \, \mathrm{d}m = \lambda \int f \, \mathrm{d}m$,
2. $\int (f+g) \, \mathrm{d}m = \int f \, \mathrm{d}m + \int g \, \mathrm{d}m$, and
3. If $f \leq g$ then $\int f \, \mathrm{d}m \leq \int g \, \mathrm{d}m$.

Proof: since f, g are integrable, we have

$$\int f \, \mathrm{d}m = \int f_+ \, \mathrm{d}m - \int f_- \, \mathrm{d}m < \infty, \quad \text{and} \quad \int g \, \mathrm{d}m = \int g_+ \, \mathrm{d}m - \int g_- \, \mathrm{d}m < \infty.$$

1. Assume $\lambda \geq 0$. Then

$$\begin{aligned} \infty > \lambda \int f \, \mathrm{d}m &= \lambda \left(\int f_+ \, \mathrm{d}m - \int f_- \, \mathrm{d}m \right) = \lambda \int f_+ \, \mathrm{d}m - \lambda \int f_- \, \mathrm{d}m \\ \boxed{\text{Corollary 294}} &= \int \lambda f_+ \, \mathrm{d}m - \int \lambda f_- \, \mathrm{d}m = \int (\lambda f)_+ \, \mathrm{d}m - \int (\lambda f)_+ \, \mathrm{d}m \\ &= \int \lambda f \, \mathrm{d}m, \end{aligned}$$

which simultaneously shows that λf is integrable.

The only thing left to do is to show that the property holds for $\lambda - 1$. Note that $(-f)_+ = f_-$ and that $(-f_-) = f_+$, so that $-f$ is itself integrable. Then

$$\begin{aligned} -\int f \, \mathrm{d}m &= -\int f_+ \, \mathrm{d}m + \int f_- \, \mathrm{d}m = \int f_- \, \mathrm{d}m - \int f_+ \, \mathrm{d}m \\ &= \int (-f)_+ \, \mathrm{d}m - \int (-f)_- \, \mathrm{d}m = \int (-f) \, \mathrm{d}m, \end{aligned}$$

because $-f$ is integrable.

2. By definition, we have

$$f + g = (f_+ - f_-) + (g_+ - g_-) = (f_+ + g_+) - (f_- + g_-).$$

According to the second solved problem (see p. 520), $f + g$ is thus integrable and

$$\begin{aligned} \int (f+g) \, \mathrm{d}m &= \int [(f_+ + g_+) - (f_- + g_-)] \, \mathrm{d}m \\ &= \int (f_+ + g_+) \, \mathrm{d}m - \int (f_- + g_-) \, \mathrm{d}m \\ \boxed{\text{Corollary 294}} &= \int f_+ \, \mathrm{d}m + \int g_+ \, \mathrm{d}m - \int f_- \, \mathrm{d}m - \int g_- \, \mathrm{d}m = \int f \, \mathrm{d}m + \int g \, \mathrm{d}m. \end{aligned}$$

3. Since $g - f \geq 0$ and $g = f + (g - f)$, we have

$$\int g \, \mathrm{d}m = \int f \, \mathrm{d}m + \int g - f \, \mathrm{d}m \geq \int f \, \mathrm{d}m,$$

according to Corollary 294 and Proposition 292. ■

The set $\mathcal{V}_n = \{f : \mathbb{R}^n \to \mathbb{R} \mid f \text{ finite, Borel, integrable}\}$ is a **vector space over** $\mathbb{R}$; the integral of f over $\mathbb{R}^n$ is a **linear functional**, which is to say that

$$\int_{\mathbb{R}^n} \_\_ \, \mathrm{d}m : \mathcal{V}_n \to \mathbb{R}$$

is a **linear functional**.

Theorem 298
Let $B \in \mathcal{B}(\mathbb{R}^n)$, with $m(B) = 0$. If f, g are Borel functions such that $f = g$ on $\mathbb{R}^n \setminus B$ and if f is integrable, then g is integrable and $\int f \, \mathrm{d}m = \int g \, \mathrm{d}m$.

Proof: the functions $f-g$ is a Borel function with $f-g\equiv 0$ on $\mathbb{R}^n\setminus B$. Since $f=g+(f-g)$, we have

$$\int f\,\mathrm{d}m=\int g\,\mathrm{d}m+\int (f-g)\,\mathrm{d}m.$$

Write $h=f-g$; then $\int h\,\mathrm{d}m=0$. Since $h_+,h_-=0$ on $\mathbb{R}^n\setminus B$, we must have

$$\int h_+\,\mathrm{d}m=\int h_-\,\mathrm{d}m=0,$$

according to Theorem 295. Then

$$\int h\,\mathrm{d}m=\int h_+\,\mathrm{d}m-\int h_-\,\mathrm{d}m=0\quad\text{and}\quad\int f\,\mathrm{d}m-\int g\,\mathrm{d}m=0\Longrightarrow\int f\,\mathrm{d}m=\int g\,\mathrm{d}m,$$

which completes the proof. ■

23.5 Integration Over a Subset

To this point, we have studied integration over $\mathbb{R}^n$ in its entirety:

$$\int f\,\mathrm{d}m=\int f\,\mathrm{d}mA.$$

But we can also integrate functions over substes of $\mathbb{R}^n$. Let $A\in\mathcal{B}(\mathbb{R}^n)$ and $f:A\subseteq\mathbb{R}^n\to\mathbb{R}$. If the function $f\chi_A:\mathbb{R}^n\to\mathbb{R}$ defined by

$$(f\chi_A)(x)=\begin{cases}f(x) & x\in A\\ 0 & x\notin A\end{cases}$$

is a Borel function and if $f\chi_A\geq 0$ or $f\chi_A$ is integrable, we define

$$\int_A f\,\mathrm{d}m=\int f\chi_A\,\mathrm{d}m.$$

We can show (see Exercises and Theorem 296) that if f is bounded on A and $f\chi_A$ is a Borel function, then $f\chi_A$ is **integrable**. When $\int_A f\,\mathrm{d}m<\infty$, we say that f is **integrable on** A.

Theorem 299
Let $A,B\in\mathcal{B}(\mathbb{R}^n)$, $A\cap B=\varnothing$. If f is a Borel function on $A\cup B$, then

1. *if $f\geq 0$, $\int_{A\cup B} f\,dm=\int_A f\,dm+\int_B f\,dm$, and*
2. *f is integrable over $A\cup B$ if and only if f is integrable over A and B.*

Proof: left as an exercise. ■

If $m(B)=0$, then $\int_B f\,\mathrm{d}m=0$. In that case $\displaystyle\int_{A\cup B} f\,\mathrm{d}m=\int_A f\,\mathrm{d}m.$

23.6 Multiple Integrals

The example of Section 21.4 shows that while we can compute the (Borel-Lebesgue) integral of a relatively straightforward integrand f, the process can leave a lot to be desired.[10] Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a bounded Borel function, that is 0 outside of a bounded region. For all $y \in \mathbb{R}$, $x \mapsto f(x, y)$ is a Borel bounded function that is 0 outside of a bounded subset of $\mathbb{R}$, hence $x \mapsto f(x, y)$ is integrable.

Theorem 300 (FUBINI'S THEOREM)
Let $f : \mathbb{R}^2 \to [0, \infty]$ be a Borel function. For every y, let $F(y) = \int_{\mathbb{R}} f(x, y)\, dx$. Then F is a Borel function and

$$\int_{\mathbb{R}^2} f \,\mathrm{d}m = \iint_{\mathbb{R}^2} f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}} F(y)\,\mathrm{d}y = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x, y)\,\mathrm{d}x \right) \mathrm{d}y.$$

Proof: left as an exercise. ■

Similarly, if $G(x) = \int_{\mathbb{R}} f(x, y)\,\mathrm{d}x$, we have

$$\int_{\mathbb{R}^2} f \,\mathrm{d}m = \int_{\mathbb{R}} G(x)\,\mathrm{d}x = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x, y)\,\mathrm{d}y \right) \mathrm{d}x,$$

Example: let $f : \mathbb{R}^2 \to [0, \infty]$ be defined by $f(x, y) = (x + y)^{-4}$, where $A \subseteq \mathbb{R}^2$ is the triangle bounded by $x = 1$, $y = 1$, and $x + y = 4$. Compute $\int_A f \,\mathrm{d}m$.

Solution: the triangle's three vertices are located at $(1, 1)$, $(1, 3)$, and $(3, 1)$. For a fixed $x \in \mathbb{R}$, we have

$$F(x) = \int_{\mathbb{R}} f(x, y)\,\mathrm{d}y = \begin{cases} 0 & \text{if } x \notin [1, 3] \\ \int_{[1,4-x]} (x + y)^{-4}\,\mathrm{d}y & \text{otherwise} \end{cases}$$

But

$$\int_{[1,4-x]} \frac{\mathrm{d}y}{(x + y)^4} = \int_1^{4-x} (x + y)^{-4}\,\mathrm{d}y = \left[\frac{(x + y)^{-3}}{-3} \right]_{y=1}^{y=4-x} = \frac{(x + 1)^{-3}}{3} - \frac{1}{192},$$

from which we have

$$\int_A f \,\mathrm{d}m = \int_{[1,3]} F(x)\,\mathrm{d}x = \int_1^3 \left[\frac{(x + 1)^{-3}}{3} - \frac{1}{192} \right] \mathrm{d}x = \left[\frac{(x + 1)^{-2}}{3(-2)} - \frac{x}{192} \right]_{x=1}^3 = \frac{1}{48}.$$ □

[10] As in the previous sections, we will provide the important details for functions $\mathbb{R}^2 \to \mathbb{R}$; the process is easy to generalize to $\mathbb{R}^n$.

If f is a positive Borel function, we can interchange the order of integration (as in Theorem 300); for general functions, there are complications. One way out of the quagmire is to decompose $f = f_+ - f_-$ and to integrate f_+ and f_- separately, but that can quickly get cumbersome.

Theorem 301 (SPECIAL FUBINI THEOREM)
Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a bounded Borel function taking on the value 0 outside of a bounded region. For all y, $x \mapsto f(x,y)$ is a bounded Borel function taking on the value 0 outside of a bounded subset of $\mathbb{R}$. Set $F(y) = \int_{\mathbb{R}} f(x,y)\,dx$. Then F is a bounded Borel function and

$$\int_{\mathbb{R}^2} f\,\mathrm{d}m = \iint_{\mathbb{R}^2} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}} F(y)\,\mathrm{d}y = \int_{\mathbb{R}} G(x)\,\mathrm{d}x.$$

Proof: by hypothesis, $\exists M, N > 0$ such that $|f(x,y)| \leq M$ for all $(x,y) \in \mathbb{R}^2$ and $f(x,y) = 0$ for all $(x,y) \notin [-N,N]^2$.

For a fixed $y = y_0$, $x \mapsto f(x,y_0)$ is a Borel function, with $|f(x,y_0)| \leq M$ for all x (and y_0) and $f(x,y_0) = 0$ when $|x| > N$. If $|y_0| > N$, $F(y_0) = 0$; more generally,

$$|F(y_0)| \leq \int_{-N}^{N} M\,\mathrm{d}x = 2MN,$$

so it is bounded.

It remains to see that F is a Borel function and that conclusion of the theorem holds. Using the decomposition $f = f_+ - f_-$, we reduce the problem to the case $f \geq 0$; it then suffices to apply Theorem 300 to each of the positive and negative parts of f, completing the proof. ■

The result generalizes to $\mathbb{R}^n$ in the natural way.

Example: Let $f : A \subseteq \mathbb{R}^3 \to \mathbb{R}$ be defined by $f(x,y,z) = 2xyz \cdot \chi_A(x,y,z)$, where

$$A = \{(x,y,z) \mid x \geq 0, y \geq 0, z \geq 0, x^2 + y^2 + z^2 \leq 1\}.$$

Compute

$$I = \int f\,\mathrm{d}m = \iiint_{\mathbb{R}^3} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Solution: let $B = \{(x,y,z) \mid x^2 + y^2 \leq 1, x \geq 0, y \geq 0, z = 0\}$. For fixed $x, y \in \mathbb{R}^2$, we have

$$F(x,y) = \int_{\mathbb{R}} 2xyz \cdot \chi_A(x,y,z)\,\mathrm{d}z = \begin{cases} 0 & \text{if } (x,y,0) \notin B \\ \int_{[0,\sqrt{1-x^2-y^2}]} 2xyz\mathrm{d}z & \text{if } (x,y,0) \in B \end{cases}$$

Since

$$\int_0^{\sqrt{1-x^2-y^2}} 2xyz\,\mathrm{d}z = 2xy\left[\frac{z^2}{2}\right]_{z=0}^{z=\sqrt{1-x^2-y^2}} = xy(1-x^2-y^2),$$

the desired integral is

$$I = \iint_{\mathbb{R}^2} F(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_B xy(1-x^2-y^2)\,\mathrm{d}x\,\mathrm{d}y.$$

We can decompose this double integral as follows: for $0 \leq x \leq 1$, set

$$G(x) = \int_0^{\sqrt{1-x^2}} xy(1-x^2-y^2)\,\mathrm{d}y = \frac{x}{4}(1-x^2)^2;$$

otherwise, set $G(x) = 0$. Then

$$I = \int_{\mathbb{R}} G(x)\,\mathrm{d}x = \frac{1}{4}\int_{[0,1]} x(1-x^2)^2\,\mathrm{d}x = \frac{1}{24}. \qquad \square$$

In general, if $D \subseteq \mathbb{R}^n$ is a Borel set, then

$$m(D) = \int \chi_D\,\mathrm{d}m.$$

If $n = 2$, this takes the form

$$\text{Area}(D) = \iint_{\mathbb{R}^2} \chi_D(x,y)\,\mathrm{d}x\,\mathrm{d}y;$$

if $n = 3$, we have

$$\text{Vol}(D) = \iiint_{\mathbb{R}^3} \chi_D(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Examples

1. Let $a, b > 0$. Find the area of the ellipse $A = \{(x,y) \in \mathbb{R}^2 \mid x^2/a^2 + y^2/b^2 \leq 1\}$.

   Solution: rewrite

   $$A = \left\{(x,y) \in \mathbb{R}^2 \,\middle|\, -a \leq x \leq a, -\frac{b}{a}\sqrt{a^2-x^2} \leq y \leq \frac{b}{a}\sqrt{a^2-x^2}\right\}.$$

   Then

   $$\text{Area}(A) = \iint_{\mathbb{R}^2} \chi_A(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{-a}^{a}\left(\int_{\mathbb{R}} \chi_A(x,y)\,\mathrm{d}y\right)\mathrm{d}x$$

But

$$\int_{\mathbb{R}} \chi_A(x,y)\,\mathrm{d}y = \begin{cases} 0 & \text{if } x \notin [-a,a] \\ \int_{-b/a\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}} \mathrm{d}y = \frac{2b}{a}\sqrt{a^2-x^2} & \text{if } x \in [-a,a] \end{cases}$$

Then

$$\begin{aligned} \text{Area}(A) &= \frac{2b}{a}\int_{x=-a}^{x=a} \sqrt{a^2-x^2}\,\mathrm{d}x \\ \boxed{x = a\cos\varphi,\ \mathrm{d}x = -a\sin\varphi\,\mathrm{d}\varphi} &= \frac{2b}{a}\int_{\varphi=\pi}^{\varphi=0} \sqrt{a^2(1-\cos^2\varphi)}(-a\sin\varphi)\,\mathrm{d}\varphi \\ &= -\frac{2b}{a}\int_{\pi}^{0} a^2\sin^2\varphi\,\mathrm{d}\varphi = 2ab\int_0^{\pi}\sin^2\varphi\,\mathrm{d}\varphi \\ &= 2ab\int_0^{\pi}\left(\frac{1-\cos 2\varphi}{2}\right)\mathrm{d}\varphi = ab\left[\varphi - \frac{\sin 2\varphi}{2}\right]_0^{\pi} = \pi ab. \end{aligned}$$

2. Let $a,b,c>0$ and $E = \{(x,y,z) \mid x^2/a^2 + y^2/b^2 + z^2/c^2 \leq 1\}$. Find $\text{Vol}(E)$.

Solution: we have

$$\text{Vol}(E) = \iiint_{\mathbb{R}^3} \chi_E(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_{-c}^{c} \underbrace{\left(\iint_{\mathbb{R}^2} \chi_E(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\right)}_{=\text{Area}(E_z)} \mathrm{d}z,$$

where

$$E_z = \left\{(x,y) \,\middle|\, \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1\right\} = \left\{(x,y) \,\middle|\, \frac{x^2}{(ah)^2} + \frac{y^2}{(bh)^2} \leq 1\right\},$$

where $h = \sqrt{1 - z^2/c^2} > 0$.

According to the preceding example, we know that

$$\text{Area}(E_z) = \pi(ah)(bh) = \pi abh^2 = \pi ab(1 - z^2/c^2)$$

when $|z| \leq c$, so that

$$\text{Vol}(E) = \int_{-c}^{c} \pi ab\left(1 - \frac{z^2}{c^2}\right)\mathrm{d}z = \pi ab\left[z - \frac{z^3}{3c^2}\right]_{z=-c}^{z=c} = \frac{4\pi}{3}abc. \qquad \square$$

We finish the chapter with some detail regarding one of the most commonly-used integration shortcuts: **changes of variables**.

23.7 Change of Variables and/or Coordinates

In the preceding section's example where we compute the area of an ellipse, we encounter an integral in x which we cannot compute directly; instead we introduce a new variable φ and a relation between x and φ that we leverage to easily compute the integral. We formalize the process in this section.

Let $\Psi : U \subseteq_O \mathbb{R}^n \to V \subseteq_O \mathbb{R}^n$ be a **diffeomorphism**; thus, Ψ and Ψ^{-1} are C^1, $\Psi \circ \Psi^{-1}(v) = v$, $\Psi^{-1} \circ \Psi(u) = u$, the **Jacobians** $\mathrm{d}\Psi(u), \mathrm{d}\Psi^{-1}(v) : \mathbb{R}^n \to \mathbb{R}^n$ are linear maps and

$$\mathrm{d}(\Psi \circ \Psi^{-1})(v) = \mathrm{d}\Psi(\Psi^{-1}(v))\mathrm{d}\Psi^{-1}(v) = I_n,$$

for all $u \in U, v \in V$, which means that $\mathrm{d}\Psi(u)$ and $\mathrm{d}\Psi^{-1}(v)$ are invertible for all $u \in U, v \in V$.

Examples

1. For $n = 1$, define $\Psi : U = (0, \pi) \to V = (-1, 1)$ by $\Psi(u) = \cos u$. Then $\mathrm{d}\Psi(u) = -\sin u < 0$ for all $u \in (0, \pi)$, i.e., Ψ is decreasing on $(0, \pi)$, with $\Psi(0) = 1$ and $\Psi(\pi) = -1$.

2. For $n = 1$, let $U = V = (0, 1)$ and define $\Psi : U \to V$ by $\Psi(u) = u^2$. Then $\mathrm{d}\Psi(u) = 2u > 0$ for all $u \in U$, i.e., Ψ is increasing on U, with $\Psi(0) = 0$ and $\Psi(1) = 1$.

3. For $n = 2$, let $U = \{(r, \theta) \mid r > 0 \text{ and } -\pi < \theta < \pi\}$, $V = \mathbb{R}^2 \setminus \{(x, 0) \mid x \leq 0\}$, and define $\Psi(r, \theta) = (r \cos \theta, r \sin \theta)$. Then

$$\mathrm{d}\Psi(r, \theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -r \sin \theta & r \cos \theta \end{pmatrix}.$$

Note that $J_\Psi(r, \theta) = \det(\mathrm{d}\Psi) = r \cos^2 \theta + r \sin^2 \theta = r > 0$ and that Ψ is:

- **injective** since if $\Psi(r_1, \theta_1) = \Psi(r_2, \theta_2)$, then

$$r_1 = \|\Psi(r_1, \theta_1)\|_2 = \|\Psi(r_2, \theta_2)\|_2 = r_2$$

and $\cos \theta_1 = \cos \theta_2$ and $\sin \theta_1 = \sin \theta_2$ yields $\theta_1 = \theta_2 \in (-\pi, \pi)$;

- **surjective** since if $(x, y) \in V$, set $r = \sqrt{x^2 + y^2} > 0$; then

$$1 = \frac{x^2 + y^2}{r^2} = \frac{x^2}{r^2} + \frac{y^2}{r^2} \implies x = r \cos \theta, y = r \sin \theta \quad \text{for some } \theta \in (-\pi, \pi].$$

But if $\theta = \pi$, then $x = -r$ and $y = 0$, so that $(x, y) \notin V$, a contradiction; thus $\theta \in (-\pi, \pi)$.

Thus $\Psi : U \to V$ is a bijection; its inverse is $\Psi^{-1} : V \to U$ is defined by $\Psi^{-1}(x,y) = (r,\theta)$, as given on the previous page. It is easy to verify that $\Psi \circ \Psi^{-1} : V \to V$ is the identity, as

$$\Psi(\Psi^{-1}(x,y)) = \Psi(\sqrt{x^2+y^2},\theta) = \Psi(r,\theta) = (r\cos\theta, r\sin\theta) = (x,y).$$

Both Ψ and Ψ^{-1} are C^1 and the Jacobians $\mathrm{d}\Psi(r,\theta)$ and $\mathrm{d}\Psi^{-1}(x,y)$ are invertible (see Exercises); as such, Ψ is a diffeomorphism between U and V. In this particular case, we can express θ explicitly in terms of (x,y):

$$\theta \in (-\pi,\pi) \Longrightarrow \frac{\theta}{2} \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right) \Longrightarrow \cos(\theta/2) \neq 0;$$

then

$$\begin{aligned}\tan(\theta/2) &= \frac{\sin(\theta/2)}{\cos(\theta/2)} = \frac{\sin\theta}{1+\cos\theta} = \frac{r\sin\theta}{r(1+\cos\theta)} = \frac{y}{\sqrt{x^2+y^2}+x} \\ &\Longrightarrow \theta = 2\mathrm{Arctan}\left(\frac{y}{\sqrt{x^2+y^2}+x}\right). \qquad \square\end{aligned}$$

If $f : V \to \overline{\mathbb{R}}$ is a Borel function, let $J_\Psi(z) = \det(\mathrm{d}\Psi(z))$; then $J_\Psi(z) \neq 0$ since Ψ is a diffeomorphism, and the composition $f \circ \Psi : U \to \overline{\mathbb{R}}$ is also a Borel function. In $\mathbb{R}^2$, for instance, if $\Psi(s,t) = (x,y) = (x(s,t), y(s,t))$, then

$$J_\Psi(s,t) = \det\begin{pmatrix} \frac{\partial x(s,t)}{\partial s} & \frac{\partial x(s,t)}{\partial t} \\ \frac{\partial y(s,t)}{\partial s} & \frac{\partial y(s,t)}{\partial t}\end{pmatrix} = \frac{\partial x(s,t)}{\partial s}\cdot\frac{\partial y(s,t)}{\partial t} - \frac{\partial x(s,t)}{\partial t}\cdot\frac{\partial y(s,t)}{\partial s} \neq 0.$$

Theorem 301 (Change of Variables)

1. *Let $f : V \to [0,\infty]$ be a positive Borel function. Then*

$$\iint_V f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_U f(x(s,t),y(s,t))\,|J_\Psi(s,t)|\,\mathrm{d}s\,\mathrm{d}t.$$

2. If $f : V \to \overline{\mathbb{R}}$ is an integrable Borel function, then $f \circ \Psi|J_\Psi|$ is Borel and integrable on U and

$$\iint_V f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_U f\circ\Psi(s,t)\,|J_\varphi(s,t)|\,\mathrm{d}s\,\mathrm{d}t.$$

Proof: left as an exercise. ■

As usual, this result easily generalizes to $\mathbb{R}^n$.

Examples

1. For $n = 1$, if $\Psi : [\alpha, \beta] \to [a, b]$ is a bijection with $\Psi(\alpha) = a$, $\Psi(\beta) = b$, Ψ is C^1, and $\Psi' > 0$ on (α, β), then Ψ is an increasing diffeomorphism between $[\alpha, \beta]$ and $[a, b]$. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then

$$\int_a^b f(u)\,\mathrm{d}u = \int_{[a,b]} f(u)\,\mathrm{d}u = \int_{(a,b)} f(u)\,\mathrm{d}u = \int_{[\alpha,\beta]} f(\Psi(t))|\Psi'(t)|\,\mathrm{d}t = \int_\alpha^\beta f(\Psi(t))\Psi'(t)\,\mathrm{d}t.$$

2. If Ψ is as in the previous example, but with $\Psi' < 0$ on (α, β), then

$$\int_a^b f(u)\,\mathrm{d}u = -\int_\beta^\alpha f(\Psi(t))\Psi'(t)\,\mathrm{d}t. \qquad \square$$

23.7.1 Polar Coordinates

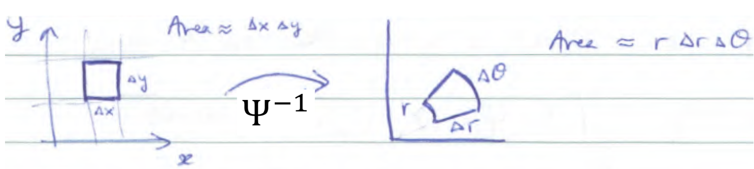
Let U, V, Ψ be as in the example on pp. 516-517. Then $J_\Psi(r, \theta) = r$. If $I = \{(x, 0) \mid x \leq 0\}$, then $\text{Area}(I) = 0$. Then, if $f : \mathbb{R}^2 \to [0, \infty]$ is a positive Borel function, we have

$$\iint_{\mathbb{R}^2} f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \iint_V f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \iint_U f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r\,\mathrm{d}\theta.$$

If f is Borel and integrable over $\mathbb{R}^2$, then $(r, \theta) \mapsto f(r\cos\theta, r\sin\theta)r$ is integrable over U and

$$\iint_{\mathbb{R}^2} f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \iint_U f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r\,\mathrm{d}\theta.$$

This transformation yields **polar coordinates**, as illustrated below.



Example: for the Borel function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = \exp(-x^2 - y^2)$, we have

$$\begin{aligned} I = \iint_{\mathbb{R}^2} \exp(-x^2 - y^2)\,\mathrm{d}x\,\mathrm{d}y &= \iint_U \exp(-r^2) r\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^\infty \int_{-\pi}^{\pi} \exp(-r^2) r\,\mathrm{d}r\,\mathrm{d}\theta \\ &= \pi \int_0^\infty 2r\exp(-r^2)\,\mathrm{d}r = \pi \int_{u=0}^{u=\infty} \exp(-u)\,\mathrm{d}u = \pi. \end{aligned}$$

Since

$$I = \left(\int_{\mathbb{R}} \exp(-x^2)\,\mathrm{d}x\right)\left(\int_{\mathbb{R}} \exp(-y^2)\,\mathrm{d}y\right) = \left(\int_{\mathbb{R}} \exp(-x^2)\,\mathrm{d}x\right)^2 = \pi,$$

then

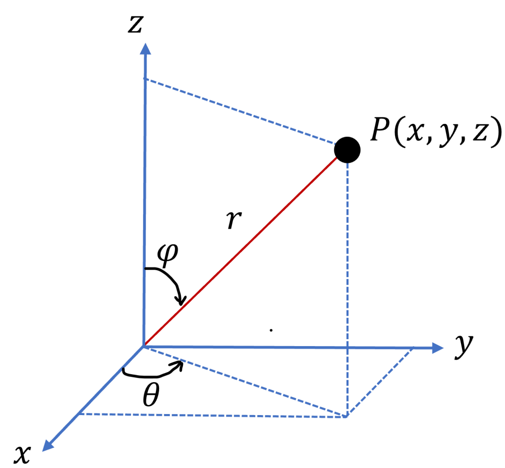
$$\int_{\mathbb{R}} \exp(-x^2)\,\mathrm{d}x = \sqrt{\pi};$$

we can compute the integral even though $\exp(-x^2)$ does not have an elementary anti-derivative. □

23.7.2 Spherical Coordinates

In **spherical coordinates**, we represent the point $P(x, y, z) \in \mathbb{R}^3$ using the coordinates (r, φ, θ):

$$x = r\sin\varphi\cos\theta, \quad y = r\sin\varphi\sin\theta, \quad z = r\cos\varphi.$$



Let $U = \{(r, \varphi, \theta) \mid r > 0, 0 < \varphi < \pi, 0 < \theta < 2\pi\}$ and $V = \mathbb{R}^2 \setminus I_x = \mathbb{R}^3 \setminus \{(x, 0, z) \mid x \geq 0\}$. Set $\Psi : U \to V$, with

$$\Psi(r, \varphi, \theta) = (r\sin\varphi\cos\theta, r\sin\varphi\sin\theta, r\cos\varphi).$$

Then

$$\mathrm{d}\Psi(r, \varphi, \theta) = \begin{pmatrix} \sin\varphi\cos\theta & \sin\varphi\sin\theta & \cos\varphi \\ r\cos\varphi & r\cos\varphi\sin\theta & -r\sin\varphi \\ -r\sin\varphi\sin\theta & r\sin\varphi\cos\theta & 0 \end{pmatrix},$$

so that $|J_\Psi(r, \varphi, \theta)| = r^2\sin\varphi$, because of the restrictions in the definition of U. Furthermore, $\mathrm{Vol}(I_x) = 0$; if $f : \mathbb{R}^3 \to [0, \infty]$ is a positive Borel function, we then have

$$\begin{aligned}\iiint_{\mathbb{R}^3} f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \iiint_V f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= \iiint_U f(r\sin\varphi\cos\theta, r\sin\varphi\sin\theta, r\cos\varphi)\, r^2\sin\varphi\, dr\,\mathrm{d}\varphi\,\mathrm{d}\theta.\end{aligned}$$

More generally, that relationship also holds if $f : \mathbb{R}^3 \to \overline{\mathbb{R}}$ is Borel and integrable.

> **Example:** compute the volume of the ball $B_R = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq R^2\}$, for $R \geq 0$.
>
> **Solution:** according to the definition,
>
> $$\begin{aligned}\text{Vol}(B_R) &= \iiint_{B_R} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_{\mathbb{R}^3} \chi_{B_R}(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= \int_0^R \left(\int_0^\pi \left(\int_0^{2\pi} r^2 \sin\varphi\,\mathrm{d}\theta\,\mathrm{d}\varphi\,dr \right) \right) = 2\pi \int_0^R r^2 \left(\int_0^\pi \sin\varphi\,\mathrm{d}\varphi \right) dr \\ &= 2\pi \int_0^R r^2 [-\cos\varphi]_0^\pi\,dr = 4\pi \int_0^R r^2\,dr = 4\pi \left[\frac{r^3}{3} \right]_0^R = \frac{4}{3}\pi R^3. \qquad \square\end{aligned}$$

23.8 Solved Problems

23.8.1 Borel-Lebesgue Integral on $\mathbb{R}^n$

1. Show that a bounded Borel function which is identically zero outside of a bounded set is integrable.

   Proof: by hypothesis, $\exists M \in \mathbb{R}^+$ such that $|g(z)| < M$ for all $z \in \mathbb{R}^n$. Furthermore, there is a bounded set A such that $g(z) = 0$ for all $z \notin A$. Since A is bounded, there exist $a_i, a_i' \in \mathbb{R}$ such that

   $$A \subseteq B = \prod_{i=1}^n [a_i, a_i']$$

   and $g(z) = 0$ for all $z \notin B$. Finally, $|g| \leq M\chi_B$ and

   $$\left| \int g \right| \leq \int |g| \leq \int M\chi_B = M \int \chi_B = M \cdot m(B) = M \prod_{i=1}^n (a_i' - a_i) < \infty,$$

   that is, g is integrable. ■

2. Let u, v be positive, integrable Borel functions. Show that $u - v$ is integrable and that

   $$\int (u - v)\,\mathrm{d}m = \int u\,\mathrm{d}m - \int v\,\mathrm{d}m.$$

   Proof: by hypothesis, $0 \leq \int u, \int v < \infty$, and so we also have $-\infty \leq \int u, \int v < \infty$. Then,

   $$\infty > \int u = \int (u - v + v) = \int (u - v) + \int v > -\infty$$

so that

$$\infty - \int v > \int (u-v) > -\infty - \int v$$

Since $-\infty < \int v < \infty$, $\infty - \int v = \infty$ and $-\infty - \int v = -\infty$. Finally, this yields

$$\infty > \int (u-v) > -\infty$$

and $u-v$ is integrable. We proved the other required result in the first inequality.■

3. If f is bounded on $A \in \mathcal{B}(\mathbb{R}^2)$, $f\chi_A$ is a Borel function, and Area$(A) < \infty$, show that $f\chi_A$ is integrable.

Proof: let $M > 0$ be such that $|f(x)| < M$ for all $x \in A$. Then, under they hypotheses,

$$\left|\int f\chi_A\right| \leq \int |f\chi_A| = \int |f|\chi_A < \int M\chi_A = M\int \chi_A = M \cdot \text{Area}(A) < \infty,$$

which completes the proof. ■

4. Let $A, B \in \mathcal{B}(\mathbb{R}^n)$, $A \cap B = \varnothing$, and f be a Borel function on $A \cup B$.

   a) If $f \geq 0$, show that
   $$\int_{A\cup B} f \, \mathrm{d}m = \int_A f \, \mathrm{d}m + \int_B f \, \mathrm{d}m.$$
   b) In general, show that f is integrable over $A \cup B$ if and only if f is integrable over A and integrable over B.
   c) If f is integrable over $A \cup B$, show that the equation of part a) holds.

Proof:

a) Let s_n be the sequence of positive simple functions guaranteed by one of the theorems. Then we have
   i. $s_n(z) \to f(z)$ for all z
   ii. $0 \leq s_n(z) \leq f(z)$ for all z
   iii. $s_n(z) \leq s_{n+1}(z)$ for all z

   Let $C \in \mathcal{B}$. Consider the function $f\chi_C$. Then,
   i. $(s_n\chi_C)(z) \to (f\chi_C)(z)$ for all z
   ii. $0 \leq (s_n\chi_C)(z) \leq (f\chi_C)(z)$ for all z
   iii. $(s_n\chi_C)(z) \leq (s_{n+1}\chi_C)(z)$ for all z

   According to the Lebesgue convergence theorem,
   $$\int_C s_n = \int s_n\chi_C \to \int f\chi_C = \int_C f. \tag{23.1}$$
   For any $n \in \mathbb{N}$, we have
   $$s\chi_{A\cup B} = s\chi_A + s\chi_B$$

since $A \cap B = \varnothing$. Then

$$\begin{aligned}\int_{A\cup B} s_n = \int s_n \chi_{A\cup B} \geq \int s_n \chi_{A \cup B} &= \int (s_n\chi_A + s_n\chi_B) \\ = \int s_n\chi_A + \int s_n\chi_B &= \int_A s_n + \int_B s_n.\end{aligned}$$

If we let $C = A \cup B$ in (23.1), we have

$$\int_{A\cup B} s_n \to \int_{A\cup B} f.$$

If we let $C = A$ in (23.1), we have

$$\int_A s_n \to \int_A f.$$

Finally, if we let $C = B$ in (23.1), we have

$$\int_B s_n \to \int_B f.$$

Combining all these results yields

$$\begin{array}{ccc} \displaystyle\int_A s_n + \int_B s_n & = & \displaystyle\int_{A\cup B} s_n \\ \Big\downarrow & & \Big\downarrow \\ \displaystyle\int_A f + \int_B f & & \displaystyle\int_{A\cup B} f \end{array}$$

so that we can conclude that

$$\int_{A\cup B} f = \int_A f + \int_B f$$

as limits are unique.

b) Suppose that f is a general (not necessarily positive) function, integrable over A and B, i.e.

$$\left|\int_A f\right|, \left|\int_B f\right| < \infty.$$

By a remark made in class, this also means that

$$0 \leq \int_A f_+, \int_A f_-, \int_B f_+, \int_B f_- < \infty.$$

Since f_- and f_+ are positive integrable Borel functions, we can apply part a) to obtain

$$\begin{aligned}0 \leq \int_{A\cup B} f_+ &= \int_A f_+ + \int_B f_+ < \infty \\ 0 \leq \int_{A\cup B} f_- &= \int_A f_- + \int_B f_- < \infty\end{aligned}$$

so that f_+ and f_- are both integrable over $A \cup B$. Consequently, f is integrable over $A \cup B$.

Conversely, suppose that f is a general (not necessarily positive) function, integrable over $A \cup B$, i.e.

$$\left| \int_{A\cup B} f \right| < \infty.$$

By a remark made in class, this also means that

$$0 \leq \int_{A\cup B} f_+, \int_{A\cup B} f_- < \infty.$$

Since f_- and f_+ are positive integrable Borel functions, we can apply part a) to obtain

$$\begin{aligned} 0 &\leq \int_A f_+ + \int_B f_+ = \int_{A\cup B} f_+ < \infty \\ 0 &\leq \int_A f_- + \int_B f_- = \int_{A\cup B} f_- < \infty \end{aligned}$$

This implies that

$$0 \leq \int_A f_+, \int_A f_-, \int_B f_+, \int_B f_- < \infty$$

and so that f_+ and f_- are both integrable over A and over B. Consequently, f is integrable over A and over B.

c) Let us assume that f is a general (not necessarily positive) function, integrable over $A \cup B$ (and so also over A and over B, see part b). By construction,

$$\begin{aligned} \int_{A\cup B} f &= \int_{A\cup B} f_+ - \int_{A\cup B} f_- \\ &= \int_A f_+ + \int_B f_+ - \int_A f_- - \int_B f_- \\ &= \int_A f_+ - \int_A f_- + \int_B f_+ - \int_B f_- \\ &= \int_A f + \int_B f \end{aligned}$$

■

5. Show that the area of the circle $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ is zero.

Proof: we use the following intermediary result.

LEMMA: let $\varphi : [0,T] \to \mathbb{R}^2$ be continuous, with $T > 0$. If $\exists M > 0$ such that

$$\|\varphi(s) - \varphi(t)\|_\infty \leq M|s-t| \tag{23.2}$$

for all $s, t \in [0, T]$, then $\varphi([0,1])$ has $2D$ measure 0.

Proof: for all $N \geq 1$, let

$$0 = t_0 < t_1 < \cdots < t_N = 1, \quad t_i = \frac{i}{N}.$$

Recall that $\|\vec{x}\|_\infty = \max\{|x_1|, |x_2|\}$. Then, according to (23.2), $\varphi([t_{i-1}, t_i]) \subseteq I_i$ for some square I_i of length $\frac{2M}{N}$ (think about this for a second). Then, $\text{Area}(I_i) = \frac{4M^2}{N^2}$ and

$$\sum_{i=1}^{N} \text{Area}(I_i) = \frac{4M^2}{N}.$$

Now, let $\varepsilon > 0$ and select $N > \frac{4M^2}{\varepsilon}$. QED

Let $\varphi : [0, 2\pi] \to \mathbb{R}^2$ be defined by $\varphi(t) = (\cos t, \sin t)$. Then φ is continuous and $\varphi([0, 2\pi]) = S^1$. According to the mean value theorem,

$$\begin{aligned}
\|\varphi(s) - \varphi(t)\|_\infty &\leq \max\{\sup_\eta |D\varphi_1(\eta)|, \sup_\eta |D\varphi_2(\eta)\}|s - t| \\
&\leq \max\{\sup_\eta |\sin \eta|, \sup_\eta |\cos \eta\}|s - t| \\
&\leq |s - t|
\end{aligned}$$

We can then apply the preceding Lemma to obtain $\text{Area}(S^1) = 0$. ■

6. Show that if $f, g : \mathbb{R}^2 \to \mathbb{R}$ are Borel functions, then so is $f + g$.

Proof: let $d \in \mathbb{R}$. For any $r, s \in \mathbb{Q}$ such that $r + s < d$, we have

$$\{z \mid f(z) < r\} \cap \{z \mid g(z) < s\} \subseteq \{z \mid f(z) + g(z) < d\},$$

or

$$E_r^f \cap E_s^g \subseteq E_d^{f+g}.$$

Then

$$\bigcup_{\substack{r,s\in\mathbb{Q}\\ r+s<d}} \left(E_r^f \cap E_s^g\right) \subseteq E_d^{f+g}.$$

If $z_0 \in E_d^{f+g}$, i.e. if $f(z_0) + g(z_0) < d$, then $\exists r, s \in \mathbb{Q}$ such that $f(z_0) < r$, $g(z_0) < s$ and $r + s < d$ (because $\mathbb{Q}$ is dense in $\mathbb{R}$), so that $z_0 \in E_r^f \cap E_s^g$. Then

$$\bigcup_{\substack{r,s\in\mathbb{Q}\\ r+s<d}} \left(E_r^f \cap E_s^g\right) = E_d^{f+g}.$$

But f, g are Borel functions; as a result,$E_r^f, E_s^g \in \mathcal{B}$ for all $r, s \in \mathbb{Q}$. Since $\mathcal{B}$ is a $\sigma-$algebra,

$$E_d^{f+g} = \bigcup_{\substack{r,s\in\mathbb{Q}\\ r+s<d}} \left(E_r^f \cap E_s^g\right) \in \mathcal{B}$$

and $f + g$ is a Borel function. ■

7. Show that every countable subset of $\mathbb{R}^2$ has $2D$ measure zero.

Proof: let $\varepsilon > 0$. List the elements of the countable subset as $A = \{a_1, a_2, \ldots, a_n, \ldots\}$. Let R_n be a square centered at a_n with Area$(R_n) = \frac{\varepsilon}{2^{n+1}}$. Then

$$\sum_{n\in\mathbb{N}} \text{Area}(R_n) = \sum_{n\in\mathbb{N}} \frac{\varepsilon}{2^{n+1}} = \frac{\varepsilon}{2}\sum_{n\in\mathbb{N}} \frac{1}{2^n} = \frac{\varepsilon}{2} < \varepsilon.$$

Thus, Area$(A) = 0$. ■

8. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \sin(x)$ and set $A = [0, 2\pi] \times [0, 1]$. Compute $\int_A f \, \mathrm{d}m$.

Solution: we have

$$\int_A f = \int f\chi_A = \int (f\chi_A)_+ - \int (f\chi_A)_- = \int f_+\chi_A - \int f_-\chi_A$$

where

$$f_+(x,y)\chi_A(x,y) = \begin{cases} \sin x & \text{if } x \in [0,\pi] \\ 0 & \text{otherwise} \end{cases}$$
$$f_-(x,y)\chi_A(x,y) = \begin{cases} -\sin x & \text{if } x \in [\pi,2\pi] \\ 0 & \text{otherwise} \end{cases}$$

Clearly, $\int f_+\chi_A = \int f_-\chi_A$, so that $\int_A f = 0$. □

9. Show that the set

$$\mathcal{I} = \{f : \mathbb{R}^n \to \mathbb{R} : f \text{ finite, Borel, integrable}\}$$

is a vector space over $\mathbb{R}$.

Proof: since $\mathcal{I}$ is a subset of the vector space of all functions from $\mathbb{R}^n$ to $\mathbb{R}$ over the scalar field $\mathbb{R}$, it suffices to verify that the three subspace conditions hold:

a) $\mathcal{O} \in \mathcal{I}$: this is the case since the function defined by $\mathcal{O}(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbb{R}^n$ is Borel as I was able to write it down, finite since $|\mathcal{O}(\mathbf{x})| = 0 < \infty$ for all $\mathbf{x} \in \mathbb{R}^n$, and integrable as $\int \mathcal{O} = 0 < \infty$.

b) $f, g \in \mathcal{I} \implies f + g \in \mathcal{I}$: if f, g are Borel, finite and integrable, then $f + g$ is clearly Borel and finite. It is also clearly integrable, albeit I have to use Theorem 25 (in disguise) to show this:

$$-\infty < -\left|\int f\right| - \left|\int g\right| \leq \left|\int (f+g)\right| \leq \left|\int f\right| + \left|\int g\right| < \infty.$$

Thus, $f + g \in \mathcal{I}$.

c) $f \in \mathcal{I}, \alpha \in \mathbb{R} \implies \alpha f \in \mathcal{I}$: if f is Borel, finite and integrable, and $\alpha \in \mathbb{R}$, then αf is clearly Borel and finite (since $|\alpha| \neq \infty$). It is also clearly integrable, albeit I have to use Theorem 25 (once again in disguise) to show this:

$$-\infty < -\alpha \left| \int f \right| \leq \left| \int \alpha f \right| \leq \alpha \left| \int f \right| < \infty.$$

Thus, $\alpha f \in \mathcal{I}$.

Consequently, $\mathcal{I}$ is a vector space. ■

10. Show that $I : \mathcal{I} \to \mathbb{R}$ defined by $I(f) = \int f \,\mathrm{d}m$ is a linear functional.

Proof: now that we know that $\mathcal{I}$ is a vector space over $\mathbb{R}$, it suffices to show that $I : \mathcal{I} \to \mathbb{R}$ acts linearly on $\mathcal{I}$, i.e. that

$$I(\alpha f + \beta g) = \alpha I(f) + \beta I(g)$$

for all $f, g \in \mathcal{I}$, $\alpha, \beta \in \mathbb{R}$.

But that is the content of Theorem 25 (since f, g are integrable):

$$\begin{aligned} I(\alpha f + \beta g) &= \int (\alpha f + \beta g) = \int (\alpha f) + \int (\beta g) \\ &= \alpha \int f + \beta \int g = \alpha I(f) + \beta I(g), \end{aligned}$$

which completes the proof. ■

11. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

Is f integrable? If so, what value does $\int f \,\mathrm{d}m$ take? If not, where does the problem lie?

Proof: note that $f(x) \geq 0$ for all $x \in \mathbb{R}$ and $\mathbb{Q} \in \mathcal{B}(\mathbb{R})$ with $\text{Length}(\mathbb{Q}) = 0$. Thus,

$$\int_{\mathbb{R}} f = \int_{\mathbb{R}-\mathbb{Q}} f = \int_{\mathbb{R}-\mathbb{Q}} 0 = 0 < \infty$$

and f is integrable. ■

12. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = x + y$ Is f integrable? If so, what value does $\int f \,\mathrm{d}m$ take? If not, where does the problem lie?

Proof: a function is integrable if and only if both its positive part and negative part are integrable. Here, $f_+, f_- : \mathbb{R}^2 \to \mathbb{R}$ are defined by

$$f_+(x, y) = \begin{cases} x + y & \text{if } y \geq -x \\ 0 & \text{else} \end{cases}$$

$$f_-(x, y) = \begin{cases} -x - y & \text{if } y \leq -x \\ 0 & \text{else} \end{cases}$$

Consider the positive simple functions

$$s_1(x,y) = \begin{cases} 1 & \text{if } x,y \geq 1 \\ 0 & \text{else} \end{cases}$$
$$s_2(x,y) = \begin{cases} 1 & \text{if } x,y \leq -1 \\ 0 & \text{else} \end{cases}$$

Then

$$0 \leq s_1(x,y) \leq f_+(x,y)$$
$$0 \leq s_2(x,y) \leq f_-(x,y)$$

for all $(x,y) \in \mathbb{R}^2$. Consequently,

$$0 \leq \int s_1 \leq \int f_+$$
$$0 \leq \int s_2 \leq \int f_-$$

But $\int s_1, \int s_2 = \infty$, so $\int f_+, \int f_- = \infty$ and f is not integrable, as neither its positive part nor its negative part is integrable. ■

13. Suppose that f is R-integrable over $[a,b]$. Is f integrable over $[a,b]$? What relation is there between $\int_{[a,b]} f\,\mathrm{d}m$ and $\int_a^b f(x)\,\mathrm{d}x$, if any?

Proof: if f is R-integrable over $[a,b]$, then on the one hand we have $\int_a^b f(x)\,\mathrm{d}x = \int_{[a,b]} f\,\mathrm{d}m$ and on the other hand we have $\infty > \left|\int_a^b f(x)\,\mathrm{d}x\right|$. Consequently, $\left|\int_{[a,b]} f\,\mathrm{d}m\right| < \infty$ and f is integrable over $[a,b]$. ■

14. Suppose that f is integrable over $[a,b]$. Is f R-integrable over $[a,b]$? What relation is there between $\int_{[a,b]} f\,\mathrm{d}m$ and $\int_a^b f(x)\,\mathrm{d}x$, if any?

Proof: there is no relation in this case. There are instances of integrable functions which are also R-integrable, such as $f:[0,1]\to\mathbb{R}$ defined by $f(x)=x^2$. Then

$$\int_{[0,1]} f\,\mathrm{d}m = \int_0^1 f(x)\,\mathrm{d}x = \frac{1}{3} < \infty.$$

But there are also instances of integrable functions which are not R-integrable.

Consider the function $f:[0,1]\to[0,\infty]$ defined by

$$f(x) = \begin{cases} \infty & x \in \mathbb{Q}\cap[0,1] \\ 0 & \text{else} \end{cases}.$$

We have seen that $\int_{[0,1]} f\,\mathrm{d}m = 0 < \infty$ so that f is integrable. We have also seen that $\int_0^1 f(x)\,\mathrm{d}x$ does not exist, so that it is not R-integrable.

The moral of the story: Lebesgue integration is more general than Riemann integration. But you already knew that. ■

23.8.2 Multivariate Calculus

1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be independent of y, that is, there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) \equiv g(x)$ for all $(x, y) \in \mathbb{R}^2$.

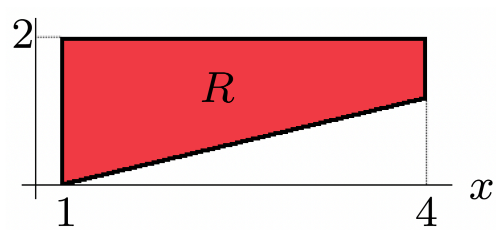
   a) What general property does the surface $z = f(x, y)$ possess?

   b) Let $R = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$. By interpreting the integral as a volume and by using the answer from part a), write $\int_R f \, \mathrm{d}A$ using a function of one variable.

   Solution: if f is independent of y, the surface $z = f(x, y)$ is constant in the $y-$direction, that is, for any $x \in \mathbb{R}$, $f(x, y_1) = f(x, y_2)$ for all y_1, y_2. As such,

$$\int_R f \, \mathrm{d}A = \left(\int_a^b g(x) \, \mathrm{d}x \right) (d - c). \qquad \square$$

2. Let $f : R \subset \mathbb{R}^2 \to \mathbb{R}$ be an integrable function and R be as below.



   Write $\int_R f \, \mathrm{d}A$ as an iterated integral.

   Solution: the vertices of R are: $(1, 0)$, $(2, 1)$, $(4, 2)$ and $(4, a)$, where $1 < a < 2$. The line from $(1, 2)$ to $(4, a)$ is $y = \frac{a}{3}(x - 1)$. Thus, R is the region defined by

$$\frac{a}{3}(x - 1) \leq y \leq 2, \quad 1 \leq x \leq 4,$$

   and $\int_1^4 \int_{\frac{a}{3}(x-1)}^2 f(x, y) \, \mathrm{d}y \, \mathrm{d}x$ is one way to write the iterated integral. $\square$

3. Compute the integral $\int_0^2 \int_0^x e^{x^2} \, \mathrm{d}y \, \mathrm{d}x$.

   Solution: the region of integration is given by

$$0 \leq y \leq x, \quad 0 \leq x \leq 2.$$

   As such, it is the triangle with vertices $(0, 0)$, $(2, 2)$ and $(2, 0)$ (we're not drawing it but you probably should). Thus,

$$\int_0^2 \int_0^x e^{x^2} \, \mathrm{d}y \, \mathrm{d}x = \int_0^2 \left[y e^{x^2} \right]_0^x \mathrm{d}x = \int_0^2 x e^{x^2} \, \mathrm{d}x = \left[\frac{1}{2} e^{x^2} \right]_0^2 = \frac{1}{2}(e^4 - 1). \qquad \square$$

4. Compute $\int_0^3 \int_{y^2}^9 y \sin(x^2) \, \mathrm{d}x \, \mathrm{d}y$.

Solution: the region of integration is

$$y^2 \leq x \leq 9, \quad 0 \leq y \leq 3.$$

Since it is difficult (read: impossible) to find an anti-derivative of $\sin(x^2)$ with respect to x, we change the order of integration. To do so cleanly, it suffices to notice that the region can be written as

$$0 \leq y \leq \sqrt{x}, \quad 0 \leq x \leq 9.$$

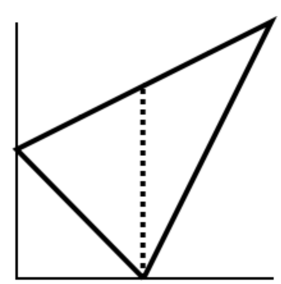
Thus,

$$\begin{aligned}\int_0^3 \int_{y^2}^9 y\sin(x^2)\,\mathrm{d}x\,\mathrm{d}y &= \int_0^9 \int_0^{\sqrt{x}} y\sin(x^2)\,\mathrm{d}y\,\mathrm{d}x = \int_0^9 \left[\frac{y^2}{2}\sin(x^2)\right]_0^{\sqrt{x}} \mathrm{d}x = \int_0^9 \frac{x}{2}\sin(x^2)\,\mathrm{d}x \\ &= \left[-\frac{1}{4}\cos(x^2)\right]_0^9 = \frac{1}{4}(1-\cos 81). \qquad \square\end{aligned}$$

5. What is the volume of the solid bounded by the planes $z = x + 2y + 4$ and $z = 2x + y$, above the triangle in the xy plane with vertices $A(1,0,0)$, $B(2,1,0)$ and $C(0,1,0)$?

Solution: in the $xy-$plane, the equations of the boundary of ΔABC are

$$\begin{aligned} AC: &\quad y = -x + 1 \leftrightsquigarrow x = -y + 1 \\ BC: &\quad y = 1 \\ AB: &\quad y = x - 1 \leftrightsquigarrow x = y + 1 \end{aligned}$$



The region of integration R can be written as

$$0 \leq y \leq 1, \; -y+1 \leq x \leq y+1,$$

and the volume of interest is

$$\begin{aligned} V &= \int_R |(x+2y+4) - (2x+y)|\,\mathrm{d}A = \int_0^1 \int_{-y+1}^{y+1} (y - x + 4)\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \left[yx - \frac{x^2}{2} + 4x\right]_{-y+1}^{y+1} \mathrm{d}y \\ &= \int_0^1 \left[\left(y(y+1) - \frac{(y+1)^2}{2} + 4(y+1)\right) - \left(y(-y+1) - \frac{(-y+1)^2}{2} + 4(-y+1)\right)\right] \mathrm{d}y \\ &= \int_0^1 (2y^2 + 6y)\,\mathrm{d}y = \left[\frac{2y^3}{3} + 3y\right]_0^1 = \frac{11}{3}. \qquad \square \end{aligned}$$

6. Compute $\int_W h \; \mathrm{d}V$, where $h(x, y, z) = ax + by + cz$ and

$$W = \{(x, y, z) \mid 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 2\}.$$

Solution: the region of integration is rectangular, so there are no hardships:

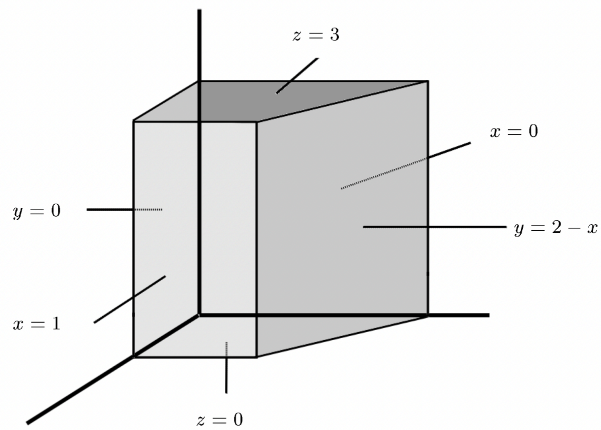
$$\begin{aligned}\int_W h \,\mathrm{d}V &= \int_0^1 \int_0^1 \int_0^2 (ax + by + cz)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \int_0^1 \left[axz + byz + c\frac{z^2}{2}\right]_0^2 \mathrm{d}x\,\mathrm{d}y \\ &= \int_0^1 \int_0^1 (2ax + 2by + 2c)\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \left[ax^2 + 2bxy + 2cx\right]_0^1 \mathrm{d}y \\ &= \int_0^1 (a + 2by + 2c)\,\mathrm{d}y = \left[ay + by^2 + 2cy\right]_0^1 = a + b + 2c. \qquad \square\end{aligned}$$

7. Sketch the region of integration W of the triple integral $\int_0^1 \int_0^{2-x} \int_0^3 f(x, y, z)\,\mathrm{d}z\,\mathrm{d}y\;\mathrm{d}x$.

Solution: the region is defined by

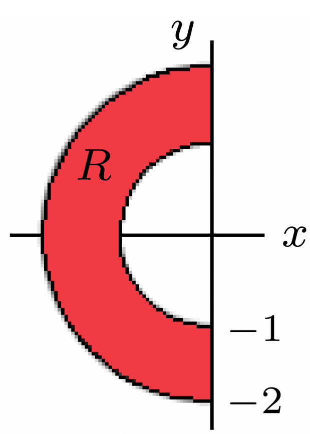
$$0 \le z \le 3, \quad 0 \le y \le 2 - x, \quad 0 \le x \le 1.$$

Thus, it is a box bounded by 6 planes: $z = 0$, $z = 3$, $y = 0$, $y = 2 - x$, $x = 0$, $x = 1$.



$\square$

8. Let $f : R \to \mathbb{R}$ be defined as below. Write $\int_R f \,\mathrm{d}A$ as an iterated integral.



Solution: in polar coordinates, the region becomes

$$1 \leq r \leq 2, \quad \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}.$$

Thus,

$$\int_R f(x,y)\,\mathrm{d}A = \int_1^2 \int_{\pi/2}^{3\pi/2} f(r\cos\theta, r\sin\theta) r\,\mathrm{d}\theta\,dr. \qquad \square$$

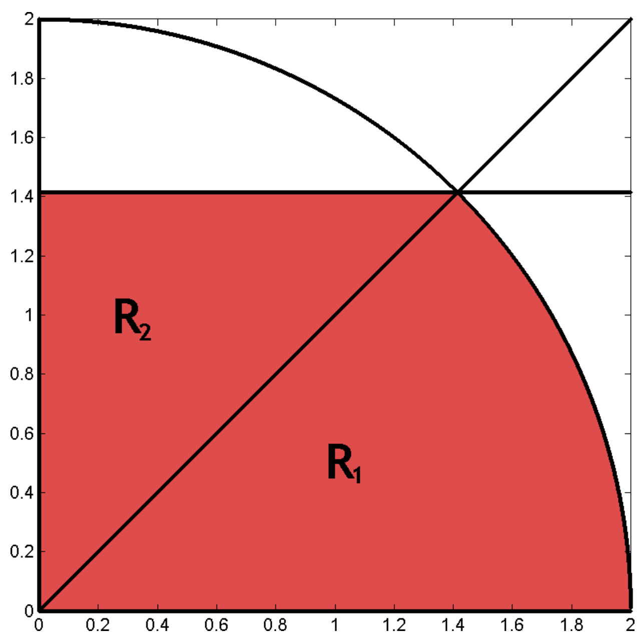
9. Compute $\int_0^{\sqrt{2}} \int_0^{\sqrt{4-y^2}} xy\,\mathrm{d}x\,\mathrm{d}y$.

   Solution: The region of integration R is defined by

$$0 \leq x \leq \sqrt{4-y^2}, \quad 0 \leq y \leq \sqrt{2}.$$

   We separate this region into two subregions R_1 and R_2 with the line $y = x$. Thus,

$$\int_R xy\,\mathrm{d}A = \int_{R_1} xy\,\mathrm{d}A + \int_{R_2} xy\,\mathrm{d}A.$$



   The regions' geometry indicates that polar coordinates have to be used in the first region, while cartesian coordinates will be appropriate in the second region.
   In polar coordinates, R_1 is

$$0 \leq r \leq 2, \quad 0 \leq \theta \leq \frac{\pi}{4},$$

   whence

$$\begin{aligned}\int_{R_1} xy\,\mathrm{d}A &= \int_0^2 \int_0^{\pi/4} (r\cos\theta)(r\sin\theta) r\,\mathrm{d}\theta\,dr = \int_0^2 \int_0^{\pi/4} r^3 \cos\theta \sin\theta\,\mathrm{d}\theta\,dr \\ &= \int_0^2 \int_0^{\pi/4} \frac{r^3}{2} \sin 2\theta\,\mathrm{d}\theta\,dr = \int_0^2 \left[-\frac{r^3}{4}\cos 2\theta\right]_0^{\pi/4} \mathrm{d}\theta\,dr = \int_0^2 \frac{r^3}{4}\,dr = \left[\frac{r^4}{16}\right]_0^2 = 1.\end{aligned}$$

In cartesian coordinates, R_2 is

$$0 \le x \le \sqrt{2}, \quad x \le y \le \sqrt{2},$$

whence

$$\begin{aligned}\int_{R_2} xy \,\mathrm{d}A &= \int_0^{\sqrt{2}} \int_x^{\sqrt{2}} xy \,\mathrm{d}y \,\mathrm{d}x = \int_0^{\sqrt{2}} \left[\frac{xy^2}{2}\right]_0^{\sqrt{2}} \mathrm{d}x = \int_0^{\sqrt{2}} \frac{x(2-x^2)}{2} \,\mathrm{d}x \\ &= \left[\frac{x^2}{2} - \frac{x^4}{8}\right]_0^{\sqrt{2}} = \frac{1}{2}.\end{aligned}$$

Thus, $\int_R xy \,\mathrm{d}A = \int_{R_1} xy \,\mathrm{d}A + \int_{R_2} xy \,\mathrm{d}A = 1 + \frac{1}{2} = \frac{3}{2}$. □

10. Compute $\int_W \sin(x^2+y^2)\,\mathrm{d}V$, where W is the cylinder centered about the z axis from $z=-1$ to $z=3$ and with radius 1.

    Solution: in cylindrical coordinates, W is

    $$0 \le r \le 1, \quad 0 \le \theta \le 2\pi, \quad -3 \le z \le 1.$$

    Thus,

    $$\int_W \sin(x^2+y^2)\,\mathrm{d}V = \int_{-3}^1 \int_0^{2\pi} \int_0^1 \sin(r^2) r \,dr \,\mathrm{d}\theta \,\mathrm{d}z = 4\pi(1-\cos 1). \quad \square$$

11. Using spherical coordinates, compute the triple integral of $f(\rho,\theta,\varphi) = \sin\varphi$ on the region defined by $0 \le \theta \le 2\pi$, $0 \le \varphi \le \frac{\pi}{4}$, $1 \le \rho \le 2$.

    Solution: in spherical coordinates, the region is

    $$0 \le \theta \le 2\pi, \quad 0 \le \varphi \le \frac{\pi}{4}, \quad 1 \le \rho \le 2.$$

    Thus, the integral is

    $$\begin{aligned} I &= \int_0^{2\pi} \int_0^{\pi/4} \int_1^2 \sin\varphi \rho^2 \sin\varphi \,\mathrm{d}\rho \,\mathrm{d}\varphi \,\mathrm{d}\theta = \int_0^{2\pi} \int_0^{\pi/4} \int_1^2 \rho^2 \sin^2\varphi \,\mathrm{d}\rho \,\mathrm{d}\varphi \,\mathrm{d}\theta \\ &= \int_0^{2\pi} \int_0^{\pi/4} \left[\frac{\rho^3}{3} \sin^2\varphi\right]_1^2 \mathrm{d}\varphi \,\mathrm{d}\theta = \int_0^{2\pi} \int_0^{\pi/4} \frac{7}{3} \sin^2\varphi \,\mathrm{d}\varphi \,\mathrm{d}\theta \\ &= \int_0^{2\pi} \frac{7}{6} [\varphi - \sin\varphi\cos\varphi]_0^{\pi/4} \,\mathrm{d}\theta = \int_0^{2\pi} \frac{7}{12}\left(\frac{\pi}{2} - 1\right) \mathrm{d}\theta = \frac{14\pi}{12}\left(\frac{\pi}{2}-1\right) = \frac{7\pi}{6}(\pi - 1). \quad \square \end{aligned}$$

12. Compute

    $$\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-z^2}}^{\sqrt{1-x^2-z^2}} (x^2+y^2+z^2)^{-1/2} \,\mathrm{d}y \,\mathrm{d}z \,\mathrm{d}x.$$

    Solution: in spherical coordinates, the region of integration is

    $$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \quad 0 \le \varphi \le \pi, \quad 0 \le \rho \le 1.$$

Thus, the integral is

$$\begin{aligned} I &= \int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-z^2}}^{\sqrt{1-x^2-z^2}} (x^2+y^2+z^2)^{-1/2}\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x = \int_{-\pi/2}^{\pi/2}\int_0^{\pi}\int_0^1 \frac{1}{\sqrt{\rho^2}}\rho^2\sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}\theta \\ &= \int_{-\pi/2}^{\pi/2}\int_0^{\pi}\int_0^1 \rho\sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}\theta = \int_{-\pi/2}^{\pi/2}\int_0^{\pi}\left[\frac{\rho^2}{2}\sin\varphi\right]_0^1 \mathrm{d}\varphi\,\mathrm{d}\theta = \int_{-\pi/2}^{\pi/2}\int_0^{\pi}\frac{\sin\varphi}{2}\,\mathrm{d}\varphi\,\mathrm{d}\theta \\ &= \int_{-\pi/2}^{\pi/2}\left[-\frac{\cos\varphi}{2}\right]_0^{\pi}\mathrm{d}\theta = \int_{-\pi/2}^{\pi/2}\mathrm{d}\theta = \pi. \qquad \square \end{aligned}$$

13. Compute

$$\int_0^1\int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x^2+y^2)^{-1/2}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z.$$

Solution: in cylindrical coordinates, the region of integration is

$$0\le r\le 1,\quad 0\le\theta\le 2\pi,\quad 0\le z\le 1.$$

In that case, the integral of interest is

$$\begin{aligned} I &= \int_0^1\int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x^2+y^2)^{-1/2}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z = \int_0^1\int_0^{2\pi}\int_0^1 \frac{1}{\sqrt{r^2}} r\,dr\,\mathrm{d}\theta\,\mathrm{d}z \\ &= \int_0^1\int_0^{2\pi}\int_0^1 dr\,\mathrm{d}\theta\,\mathrm{d}z = 2\pi. \qquad \square \end{aligned}$$

14. Compute $\int_0^1\int_{\sqrt{x}}^1 e^{y^3}\,\mathrm{d}y\,\mathrm{d}x$.

Solution: the region of integration is given by

$$0\le x\le y^2,\quad 0\le y\le 1.$$

Thus, the integral of interest is

$$\int_0^1\int_{\sqrt{x}}^1 e^{y^3}\,\mathrm{d}y\,\mathrm{d}x = \int_0^1\int_0^{y^2} e^{y^3}\,\mathrm{d}x\,\mathrm{d}y = \int_0^1\left[xe^{y^3}\right]_{x=0}^{x=y^2}\mathrm{d}y = \int_0^1 y^2e^{y^3}\,\mathrm{d}y = \left[\frac{e^{y^3}}{3}\right]_0^1 = \frac{e-1}{3}. \qquad \square$$

15. Sketch the solid bounded by the the surfaces $z=0$, $y=0$, $z=a-x+y$ and $y=a-\frac{1}{a}x^2$, where a is a positive constant. What is the volume of that solid?

Solution: the solid's base is the parabolic region in the $xy-$plane bounded by the line $y=0$ and the parabola $y=a-\frac{1}{a}x^2$. The volume of this solid is thus

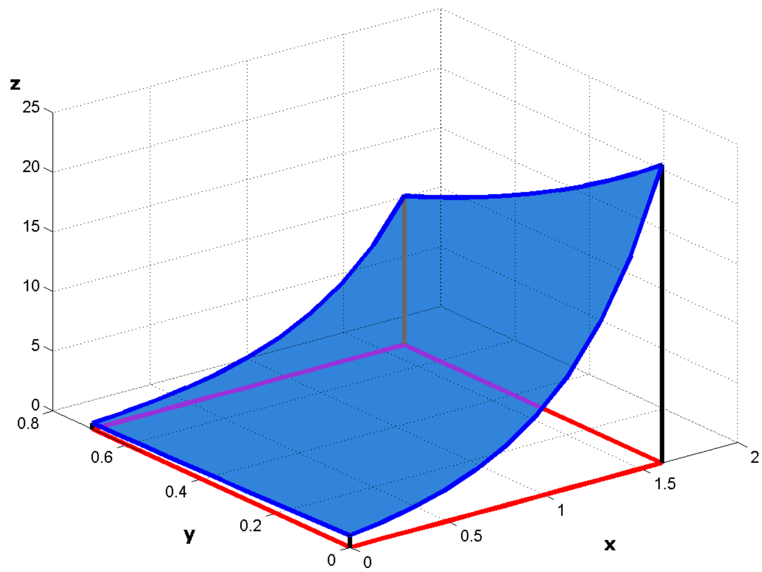
$$V = \iint_D (a-x+y)\,\mathrm{d}A = \iint_D (a+y)\,\mathrm{d}A,$$

(why can we eliminate the x in the integral?) so that

$$\begin{aligned} V &= \int_{-a}^a\int_0^{a-\frac{1}{a}x^2}(a+y)\,\mathrm{d}y\,\mathrm{d}x = \int_{-a}^a\left[ay+\frac{y^2}{2}\right]_{y=0}^{y=a-\frac{1}{a}x^2}\mathrm{d}x \\ &= 2\int_0^a\left(\frac{3}{2}a^2-2x^2+\frac{x^4}{2a^2}\right)\mathrm{d}x = \left[3a^2x-\frac{4}{3}x^3+\frac{1}{5a^2}x^5\right]_0^a = 3a^3-\frac{4}{3}a^3+\frac{1}{5}a^3 = \frac{28}{15}. \qquad \square \end{aligned}$$

16. Evaluate $\int_0^{\ln 2}\int_0^{\ln 5} e^{2x-y}\,\mathrm{d}x\,\mathrm{d}y$.

Solution: the region of integration appears in red R, while the surface $z = e^{2x-y}$ shows up in blue.

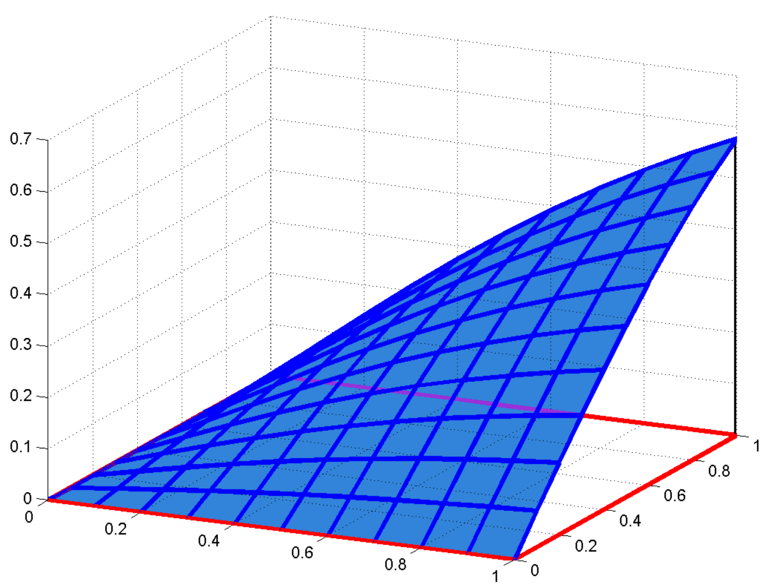


Since R is a rectangle, we can proceed directly:

$$\int_0^{\ln 2}\int_0^{\ln 5} e^{2x-y}\,\mathrm{d}x\,\mathrm{d}y = \int_0^{\ln 2}\left[\frac{1}{2}e^{2x-y}\right]_{x=0}^{x=\ln 5}\mathrm{d}y = \int_0^{\ln 2} 12e^{-y}\,\mathrm{d}y = \left[-12e^{-y}\right]_{y=0}^{y=\ln 2} = 6. \qquad \square$$

17. Evaluate $\int_0^1\int_0^1 \frac{xy}{\sqrt{x^2+y^2+1}}\,\mathrm{d}x\,\mathrm{d}y$.

Solution: the region of integration appears in red R, while the surface $z = \frac{xy}{\sqrt{x^2+y^2+1}}$ shows up in blue.

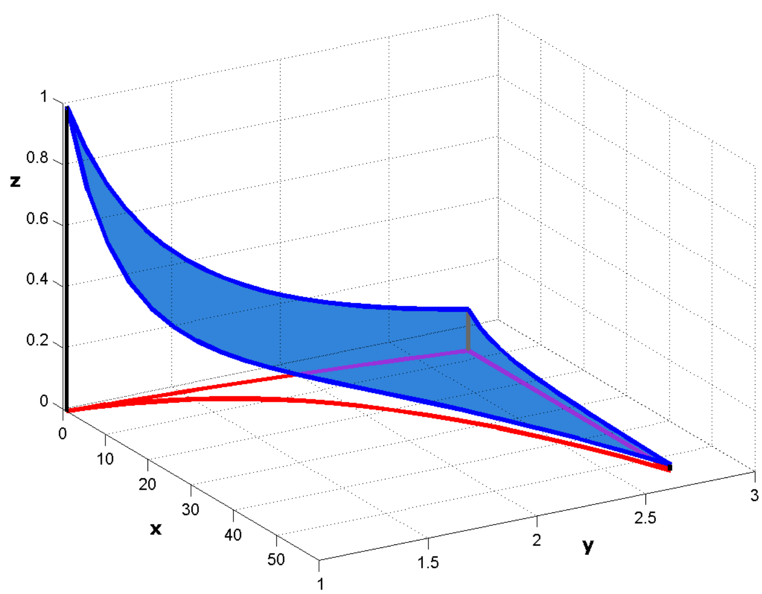


Since R is a rectangle, we can proceed directly:

$$\begin{aligned}
I &= \int_0^1 \int_0^1 \frac{xy}{\sqrt{x^2+y^2+1}}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^1 \left[y\sqrt{x^2+y^2+1}\right]_{x=0}^{x=1} \mathrm{d}y = \int_0^1 y\left[\sqrt{y^2+2}-\sqrt{y^2+1}\right]\mathrm{d}y \\
&= \int_0^1 y\sqrt{y^2+2}\,\mathrm{d}y - \int_0^1 y\sqrt{y^2+1}\,\mathrm{d}y = \left[\frac{1}{3}(y^2+2)^{3/2}\right]_{y=0}^{y=1} - \left[\frac{1}{3}(y^2+1)^{3/2}\right]_{y=0}^{y=1} \\
&= \sqrt{3} - \frac{4}{3}\sqrt{2} + \frac{1}{3}. \qquad \square
\end{aligned}$$

18. Let $D = \{(x,y) \mid 1 \leq y \leq e, y^2 \leq x \leq y^4\}$. Compute $\iint_D \frac{1}{x}\,\mathrm{d}A$.

Solution: the region of integration appears in red R, while the surface $z = \frac{1}{x}$ shows up in blue.

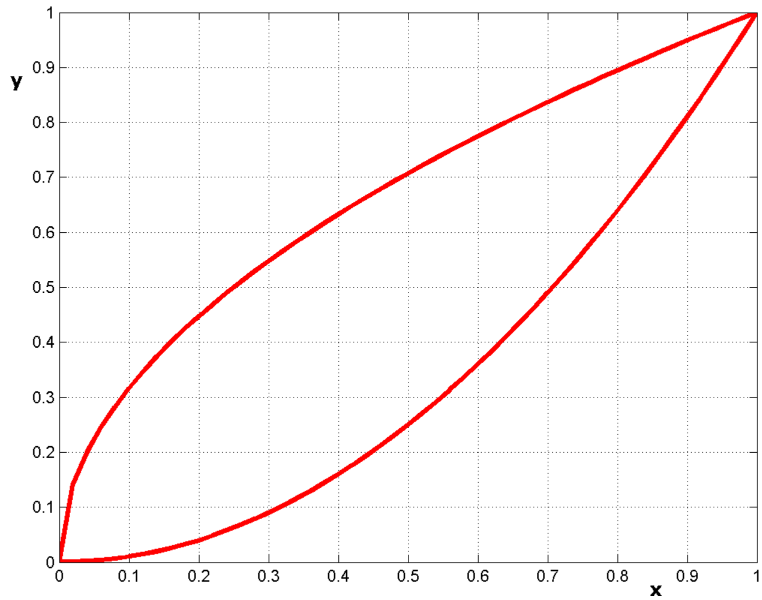


The double integral can be expressed as an iterated integral:

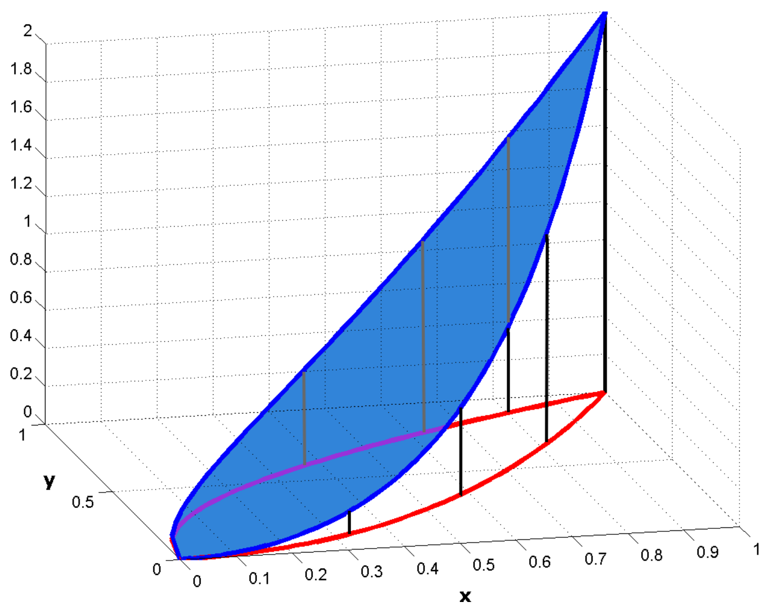
$$\begin{aligned}
\iint_D \frac{1}{x}\,\mathrm{d}A &= \int_1^e \int_{y^2}^{y^4} \frac{1}{x}\,\mathrm{d}x\,\mathrm{d}y = \int_1^e [\ln|x|]_{y^2}^{y^4}\,\mathrm{d}y = \int_1^e \left[\ln|y^4| - \ln|y^2|\right]\,\mathrm{d}y \\
&= \int_1^e \left[\ln|y^2|\right]\,\mathrm{d}y = \int_1^e \left[\ln y^2\right]\,\mathrm{d}y = 2\int_1^e \ln y\,\mathrm{d}y = 2\left[y\ln y - y\right]_1^e = 2. \qquad \square
\end{aligned}$$

19. What is the volume of the solid lying under the paraboloid $z = x^2 + y^2$ and above the domain bounded by $y = x^2$ and $x = y^2$?

Solution: the domain D is shown below:



Thus, $D = \{(x, y) \mid 0 \leq x \leq 1, x^2 \leq y \leq \sqrt{x}\}$ and the solid of interest is shown in the following figure:



Its volume is thus

$$V = \iint_D (x^2 + y^2)\,\mathrm{d}A = \int_0^1 \int_{x^2}^{\sqrt{x}} (x^2 + y^2)\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_{x^2}^{\sqrt{x}} \mathrm{d}x$$

$$= \int_0^1 \left[x^{5/2} - x^4 + \frac{x^{3/2}}{3} - \frac{x^6}{3}\right] \mathrm{d}x = \left[\frac{2}{7}x^{7/2} - \frac{x^5}{5} + \frac{2}{15}x^{5/2} - \frac{x^7}{21}\right]_0^1 = \frac{6}{35}. \qquad \square$$

20. Let R be the disk of radius 5, centered at the origin. Evaluate $\iint_R x\,\mathrm{d}A$.

Solution: in polar coordinates, R rewrites as

$$R_{(r,\theta)} = \{(r, \theta) \mid 0 \leq r \leq 5, 0 \leq \theta \leq 2\pi\}.$$

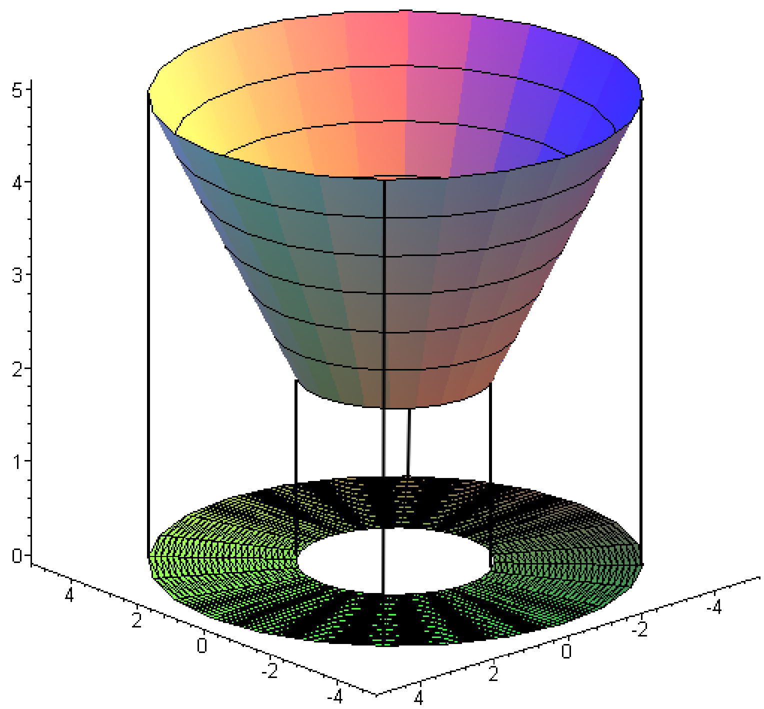
Since $x=r\cos\theta$, the change of variables formula yields

$$\iint_R x\,\mathrm{d}A=\int_0^5\int_0^{2\pi} r\cos\theta\cdot r\,\mathrm{d}\theta\,dr=\int_0^5\int_0^{2\pi} r^2\cos\theta\,\mathrm{d}\theta\,dr=\int_0^5\left[r^2\sin\theta\right]_0^{2\pi}\,dr=0.$$

Are you suprised by this result? You should not be. □

21. What is the volume of the solid lying under the cone $z=\sqrt{x^2+y^2}$ and above the ring $4\leq x^2+y^2\leq 25$ located in the $xy-$plane?

Solution: the solid of interest is shown here:



If $R=\{(x,y)\mid 4\leq x^2+y^2\leq 25\}$, we wish to evaluate $\iint_R\sqrt{x^2+y^2}\,\mathrm{d}A$. In polar coordinates, we have

$$R_{(r,\theta)}=\{(r,\theta)\mid 2\leq r\leq 5, 0\leq\theta\leq 2\pi\}$$

and $\sqrt{x^2+y^2}=\sqrt{r^2}=r$, whence

$$\iint_R\sqrt{x^2+y^2}\,\mathrm{d}A=\int_2^5\int_0^{2\pi} r\cdot r\,\mathrm{d}\theta\,dr=\int_2^5\int_0^{2\pi} r^2\,\mathrm{d}\theta\,dr=\int_2^5 2\pi r^2\,dr=78\pi. \qquad \square$$

22. Compute $\int_0^2\int_0^{\sqrt{2x-x^2}}\sqrt{x^2+y^2}\,\mathrm{d}y\,\mathrm{d}x$.

Solution: the region of integration is shown below:

In polar coordinates, this regions rewrites as

$$R_{(r,\theta)} = \{(r,\theta) : 0 \le \theta \le \pi/2, 0 \le r \le 2\cos\theta\},$$

whence the integral of interest is

$$I = \int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,\mathrm{d}y\,\mathrm{d}x = \int_0^{\pi/2}\int_0^{2\cos\theta} \sqrt{r^2}\cdot r\,dr\,\mathrm{d}\theta = \int_0^{\pi/2} \left[\frac{r^3}{3}\right]_{r=0}^{r=2\cos\theta} \mathrm{d}\theta$$
$$= \int_0^{\pi/2} \left(\frac{8}{3}\cos^3\theta\right) \mathrm{d}\theta = \left[\frac{8}{9}\cos^2\theta\sin\theta + \frac{16}{9}\sin\theta\right]_0^{\pi/2} = \frac{16}{9}. \qquad \square$$

23. Find the mass and the centre of mass of the metal plate occupying the domain

$$D = \{(x,y) \mid 0 \le x \le 2,\ 0 \le y \le 3\},$$

if the density function of the plate is $\rho(x,y) = y$.

Solution: the total mass of the plate is $m = \iint_D \rho(x,y)\,\mathrm{d}A$, while the coordinates of the centre of mass $(\overline{x},\overline{y})$ are given by

$$\overline{x} = \frac{1}{m}\iint_D x\rho(x,y)\,\mathrm{d}A \quad \text{and} \quad \overline{y} = \frac{1}{m}\iint_D y\rho(x,y)\,\mathrm{d}A.$$

Thus,

$$m = \int_0^2\int_0^3 y\,\mathrm{d}y\,\mathrm{d}x = \int_0^2 \left[\frac{y^2}{2}\right]_0^3 \mathrm{d}x = \int_0^2 \frac{9}{2}\,\mathrm{d}x = 9$$
$$\overline{x} = \frac{1}{9}\int_0^2\int_0^3 xy\,\mathrm{d}y\,\mathrm{d}x = \frac{1}{9}\int_0^2 \left[x\frac{y^2}{2}\right]_0^3 \mathrm{d}x = \frac{1}{9}\int_0^2 \frac{9}{2}x\,\mathrm{d}x = 1$$
$$\overline{y} = \frac{1}{9}\int_0^2\int_0^3 y^2\,\mathrm{d}y\,\mathrm{d}x = \frac{1}{9}\int_0^2 \left[\frac{y^3}{3}\right]_0^3 \mathrm{d}x = \frac{1}{9}\int_0^2 9\,\mathrm{d}x = 2. \qquad \square$$

24. Evaluate $\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^x yz\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x$.

Solution: this can be done directly:

$$I = \int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^x yz\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x = \int_0^3 \int_0^{\sqrt{9-x^2}} \left[\frac{y^2 z}{2}\right]_0^x \mathrm{d}z\,\mathrm{d}x = \int_0^3 \int_0^{\sqrt{9-x^2}} \frac{x^2 z}{2}\,\mathrm{d}z\,\mathrm{d}x$$
$$= \int_0^3 \left[\frac{x^2 z^2}{4}\right]_0^{\sqrt{9-x^2}} \mathrm{d}x = \int_0^3 \frac{x^2(9-x^2)}{4}\,\mathrm{d}x = \left[-\frac{x^5}{20} + \frac{3}{4}x^3\right]_0^3 = \frac{81}{10}. \qquad \square$$

25. Compute $\iiint_E e^x\,\mathrm{d}V$, where

$$E = \{(x,y,z) : 0 \le y \le 1, 0 \le x \le y, 0 \le z \le x+y\}.$$

Solution: again, this can be done directly, with the help of an iterated integral.

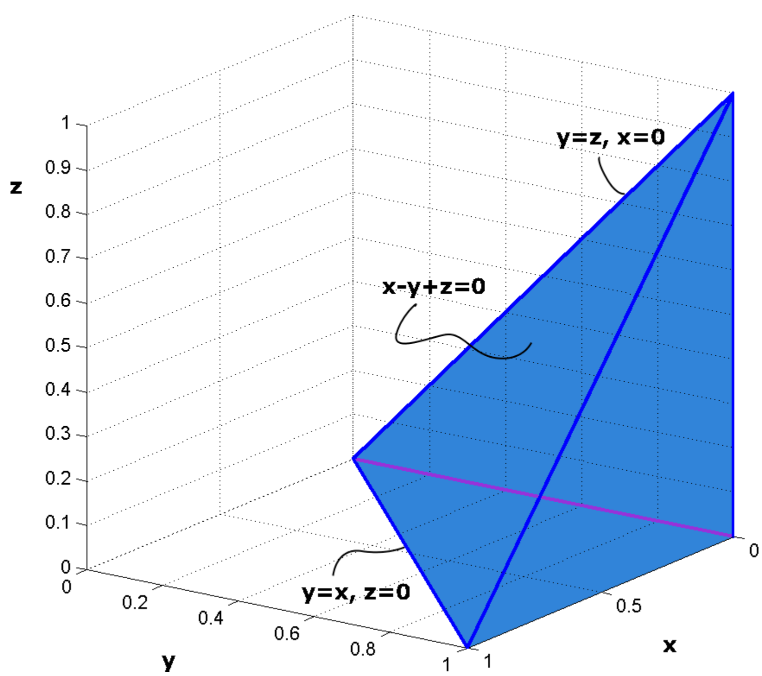
$$I = \int_0^1 \int_0^y \int_0^{x+y} e^x\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \int_0^y [e^x z]_{z=0}^{z=x+y}\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \int_0^y e^x(x+y)\,\mathrm{d}x\,\mathrm{d}y$$
$$= \int_0^1 [e^x(x+y-1)]_{x=0}^{x=y}\,\mathrm{d}y = \int_0^1 (e^y-1)(y-1)\,\mathrm{d}y = \left[2ye^y - 3e^y + y - \frac{y^2}{2}\right]_0^1 = \frac{7}{2} - e. \qquad \square$$

26. Compute $\iiint_E xz\,\mathrm{d}V$, where E is the pyramid with vertices $(0,0,0)$, $(0,1,0)$, $(1,1,0)$ and $(0,1,1)$.

Solution: we can define E by

$$E = \{(x,y,z) \mid 0 \le y \le 1, 0 \le z \le y, 0 \le x \le y - z\},$$

as can be seen on the figure below.



Thus,

$$\iiint_E xz\,\mathrm{d}V = \int_0^1 \int_0^y \int_0^{y-z} xz\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y = \int_0^1 \int_0^y \frac{1}{2}(y-z)^2 z\,\mathrm{d}z\,\mathrm{d}y$$
$$= \frac{1}{2}\int_0^1 \left[\frac{1}{2}y^2z^2 - \frac{2}{3}yz^3 + \frac{1}{4}z^4\right]_{z=0}^{z=y} \mathrm{d}y = \frac{1}{24}\int_0^1 y^4\,\mathrm{d}y = \frac{1}{24}\left[\frac{1}{25}\right]_0^1 = \frac{1}{120}. \qquad \square$$

27. Let W be a three-dimensional solid. Its volume can be computed by the following iterated integral:

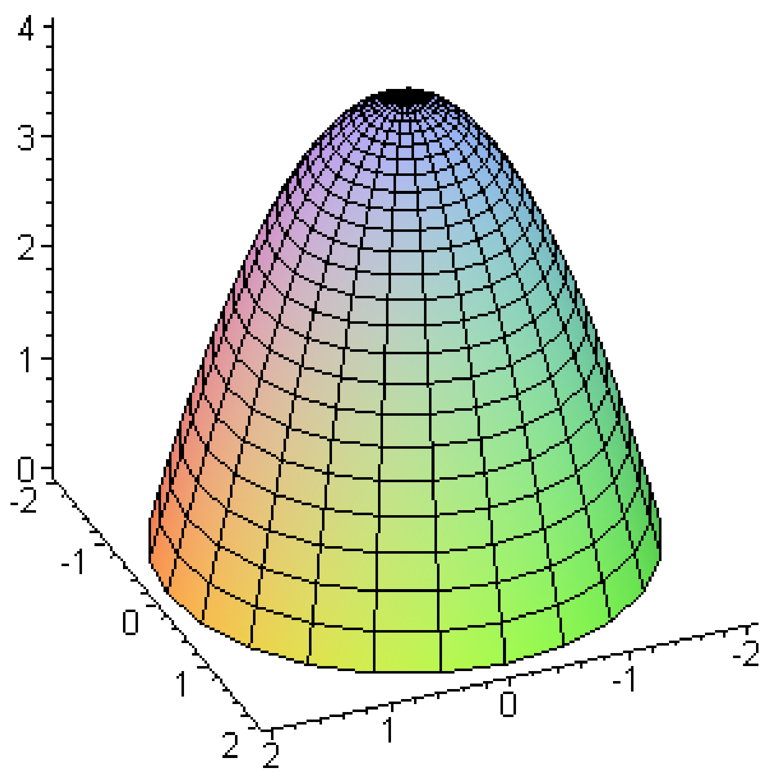
$$V(W) = \int_0^{2\pi} \int_0^2 \int_0^{4-r^2} r\,\mathrm{d}z\,dr\,\mathrm{d}\theta.$$

Find W and $V(W)$.

Solution: in cartesian coordinates, $V(W) = \iiint_W \mathrm{d}V$. The volume integral is given in cylindrical coordinates, from which we can conclude that

$$W_{(r,\theta,z)} = \{(r,\theta,z) \mid 0 \le \theta \le 2\pi, 0 \le r \le 2, 0 \le z \le 4 - r^2\}.$$

In cartesian coordinates, the solid of interest lies under the paraboloid $z = 4 - x^2 - y^2$ and above the disk in the $xy-$plane of radius 2 centered at the origin.



Thus,

$$\begin{aligned}
V(W) &= \int_{-2}^{-2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{4-x^2-y^2} \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = \int_0^{2\pi}\int_0^2\int_0^{4-r^2} r\,\mathrm{d}z\,dr\,\mathrm{d}\theta \\
&= \int_0^{2\pi}\int_0^2 [rz]_{z=0}^{z=4-r^2}\,dr\,\mathrm{d}\theta = \int_0^{2\pi}\int_0^2 r(4-r^2)\,dr\,\mathrm{d}\theta \\
&= \int_0^{2\pi}\left[-\frac{r^4}{4}+2r^2\right]_0^2 \mathrm{d}\theta = 4\int_0^{2\pi}\mathrm{d}\theta = 8\pi. \qquad \square
\end{aligned}$$

28. Let W be a three-dimensional solid. Its volume can be computed by the following iterated integral:

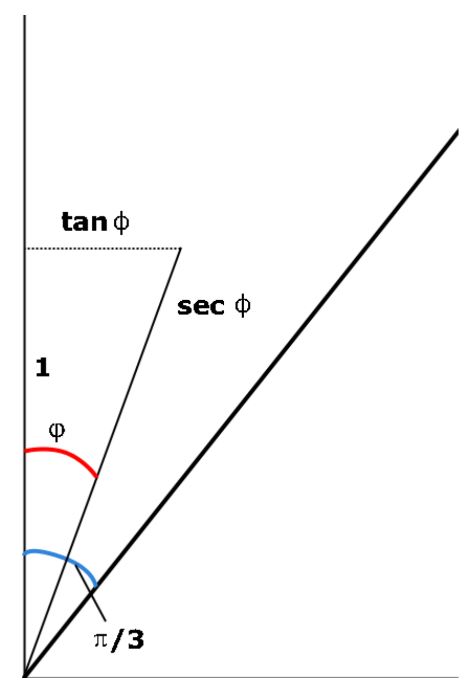
$$\int_0^{\pi/3}\int_0^{2\pi}\int_0^{\sec\varphi} \rho^2 \sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi.$$

Find W and $V(W)$.

Solution: in cartesian coordinates, $V(W) = \iiint_W \mathrm{d}V$. The volume integral is given in spherical coordinates, from which we can conclude that

$$W_{(\rho,\theta,\varphi)} = \{(\rho,\theta,\varphi) \mid 0 \leq \rho \leq \pi/3, 0 \leq \theta \leq 2\pi, 0 \leq \rho \leq \sec\varphi\}.$$

Using the first two sets of inequalities, we see that the solid is part of the cone whose surface is $z = \frac{1}{\sqrt{3}}\sqrt{x^2+y^2}$ (in cartesian coordinates): when the radius is $\rho = \sec\varphi$, the height of the of the point in cartesian coordinates is automatically 1, as can be seen when we provide a transverse slice of the cone:



Thus, the volume of the cone is

$$V(W) = \int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} \int_{\frac{1}{\sqrt{3}}\sqrt{x^2+y^2}}^{1} \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = \int_0^{\pi/3}\int_0^{2\pi}\int_0^{\sec\varphi} \rho^2 \sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi$$
$$= \int_0^{\pi/3}\int_0^{2\pi} \left[\frac{1}{3}\rho^3 \sin\varphi\right]_{\rho=0}^{\rho=\sec(\varphi)} \mathrm{d}\theta\,\mathrm{d}\varphi = \int_0^{\pi/3} \left[\frac{1}{3}\sec^3\varphi \sin\varphi\theta\right]_{\theta=0}^{\theta=2\pi} \mathrm{d}\varphi$$
$$= \int_0^{\pi/3} \frac{2\pi}{3} \sec^3\varphi \sin\varphi\,\mathrm{d}\varphi = \left[\frac{\pi}{3}\sec^2\varphi\right]_0^{\pi/3} = \pi,$$

However, you do know how to compute the volume of a cone when the height and the radius are known: $V = \frac{1}{3}\pi r^2 h$. How does that compare to your answer? □

29. Compute $\iiint_B (x^2+y^2+z^2)\,\mathrm{d}V$, where B is the unit ball $x^2+y^2+z^2 \leq 1$.

Solution: in spherical coordinates, the region can be written as

$$B_{(\rho,\theta,\varphi)} = \{(\rho,\theta,\varphi) \mid 0 \leq \rho \leq 1, 0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq \pi\},$$

with $\rho^2 = x^2+y^2+z^2$, whence

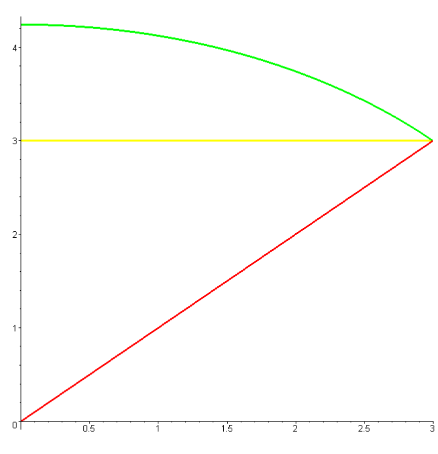
$$I = \iiint_B (x^2+y^2+z^2)\,\mathrm{d}V = \int_0^1\int_0^{2\pi}\int_0^{\pi} \rho^2 \cdot \rho^2 \sin\varphi\,\mathrm{d}\varphi\,\mathrm{d}\theta\,\mathrm{d}\rho$$
$$= \int_0^1\int_0^{2\pi}\int_0^{\pi} \rho^4 \sin\varphi\,\mathrm{d}\varphi\,\mathrm{d}\theta\,\mathrm{d}\rho = \int_0^1\int_0^{2\pi} \left[-\rho^4\cos\varphi\right]_{\varphi=0}^{\varphi=\pi/3} \mathrm{d}\theta\,\mathrm{d}\rho$$
$$= \int_0^1\int_0^{2\pi} \frac{\rho^4}{2}\,\mathrm{d}\theta\,\mathrm{d}\rho = \int_0^1 \pi\rho^4\,\mathrm{d}\rho = \pi\left[\frac{\rho^5}{5}\right]_0^1 = \frac{\pi}{5}. \quad \square$$

30. Evaluate

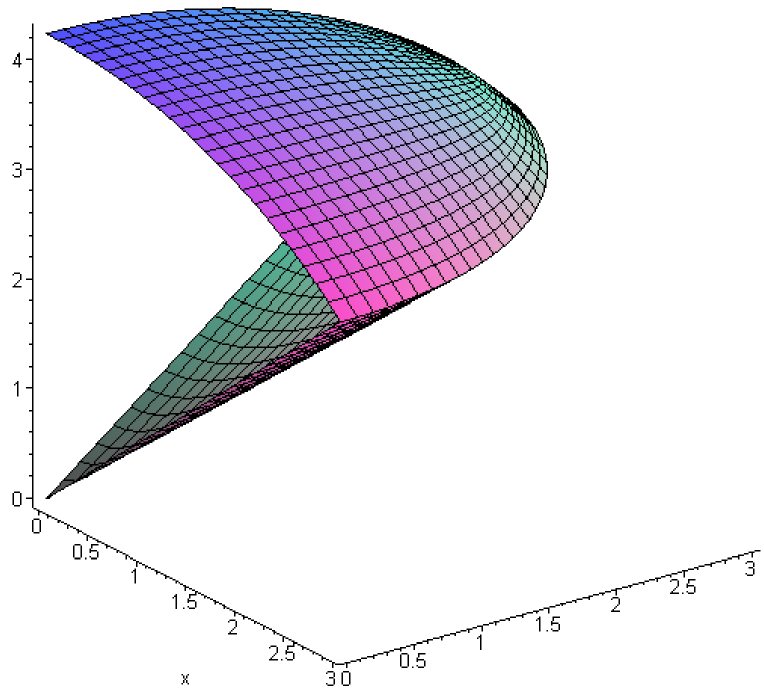
$$\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} (x^2+y^2+z^2)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y.$$

Solution: □

31. **Solution:** the volume of integration is defined by the solid lying above the the disk of radius 3 in the first quadrant of the $xy-$plane and bounded by the cone $z^2 = x^2+y^2$ and the sphere $x^2+y^2+z^2 = 18$; as such, it is the solid of revolution of the following curve



around the z axis, under a rotation of $\frac{\pi}{2}$ radians:



In spherical coordinates, the region becomes

$$\left\{(\rho,\theta,\varphi) \mid 0\leq \rho\leq \sqrt{18}, 0\leq\theta\leq\frac{\pi}{2}, 0\leq\varphi\leq\frac{\pi}{4}\right\}$$

with $\rho^2=x^2+y^2+z^2$, whence

$$\begin{aligned}
I &= \int_0^3\int_0^{\sqrt{9-y^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}}(x^2+y^2+z^2)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y=\int_0^{\sqrt{18}}\int_0^{\pi/2}\int_0^{\pi/4}\rho^2\cdot\rho^2\sin\varphi\,\mathrm{d}\varphi\,\mathrm{d}\theta\,\mathrm{d}\rho \\
&= \int_0^{\sqrt{18}}\int_0^{\pi/2}\int_0^{\pi/4}\rho^4\sin\varphi\,\mathrm{d}\varphi\,\mathrm{d}\theta\,\mathrm{d}\rho=\int_0^{\sqrt{18}}\int_0^{\pi/2}\left[-\rho^4\cos\varphi\right]_{\varphi=0}^{\varphi=\pi/4}\,\mathrm{d}\theta\,\mathrm{d}\rho \\
&= \int_0^{\sqrt{18}}\int_0^{\pi/2}\left[1-\cos(\pi/4)\right]\rho^4\,\mathrm{d}\theta\,\mathrm{d}\rho=\int_0^{\sqrt{18}}\frac{\pi}{2}(1-\cos(\pi/4))\rho^4\,\mathrm{d}\rho \\
&= \left[\frac{\pi}{2}(1-\cos(\pi/4))\frac{\rho^5}{5}\right]_0^{\sqrt{18}}=\frac{\pi}{2}(1-\cos(\pi/4))\frac{\sqrt{18}^5}{5}. \qquad \square
\end{aligned}$$

32. Compute the volume of the solid bounded by the cone $z=\sqrt{x^2+y^2}$ and the sphere of radius $a>0$ whose center is located at the origin.

Solution: let

$$A=\overline{B}(0,a)\cap\text{Cone}=\{(x,y,z)\mid x^2+y^2+z^2\leq a^2 \text{ and } z\geq\sqrt{x^2+y^2}\}$$

If $(x,y,z)\in A$, then

$$x^2+y^2\leq z^2\leq a^2-(x^2+y^2),$$

whence $x^2 + y^2 \leq \frac{a^2}{2}$. Denote

$$C = \left\{ (x, y) \mid x^2 + y^2 \leq \frac{a^2}{2} \right\}.$$

We then have

$$A = \{(x, y, z) : (x, y) \in C, \sqrt{x^2 + y^2} \leq z \leq \sqrt{a^2 - (x^2 + y^2)}\}$$

and so

$$\begin{aligned} \text{Vol}(A) &= \iiint_A \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint_C \left(\sqrt{a^2 - (x^2 + y^2)} - \sqrt{x^2 + y^2} \right) \mathrm{d}x\,\mathrm{d}y \\ &= \int_{[0, a/\sqrt{2}]} \int_{[-\pi, \pi]} \left(\sqrt{a^2 - r^2} - r \right) r\,\mathrm{d}\theta\, dr = \cdots = \frac{2\pi a^3}{3} \left(1 - \frac{1}{\sqrt{2}} \right). \qquad \square \end{aligned}$$

33. Compute the volume of the solid bounded by the paraboloïds $z = 10 - x^2 - y^2$ and $z = 2(x^2 + y^2 - 1)$.

Solution: let

$$A = \{(x, y, z) \mid 2(x^2 + y^2 - 1) \leq z \leq 10 - x^2 - y^2\}$$

If $(x, y, z) \in A$, then $x^2 + y^2 \leq 4$ (why?). Denote

$$B = \{(x, y) : x^2 + y^2 \leq 4\}.$$

We then have

$$A = \{(x, y, z) \mid (x, y) \in B, 2(x^2 + y^2 - 1) \leq z \leq 10 - x^2 - y^2\}$$

and so

$$\begin{aligned} \text{Vol}(A) &= \iiint_A \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint_B \left((10 - x^2 - y^2) - 2(x^2 + y^2 - 1) \right) \mathrm{d}x\,\mathrm{d}y \\ &= 3 \iint_B \left(4 - (x^2 + y^2) \right) \mathrm{d}x\,\mathrm{d}y = 3 \int_{[0,2]} \int_{[-\pi, \pi]} (4 - r^2) r\,\mathrm{d}\theta\, dr = \cdots = 24\pi. \qquad \square \end{aligned}$$

34. Let T be the triangle with vertices $(0, 0)$, $(0, 1)$ and $(1, 0)$. Compute $\iint_T \exp\left(\frac{y-x}{y+x}\right) \mathrm{d}x\,\mathrm{d}y$ using

   a) polar coordinates;
   b) the change of variables $u = y - x$, $v = y + x$.

Solution:

a) Let $x = r\cos\theta$, $y = r\sin\theta$. Then

$$\begin{aligned} I &= \iint_T \exp\left(\frac{y-x}{y+x}\right)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{[0,\pi/2]}\int_{[0,(\sin\theta+\cos\theta)^{-1}]} \exp\left(\frac{\sin\theta-\cos\theta}{\sin\theta+\cos\theta}\right) r\,dr\,\mathrm{d}\theta \\ &= \int_{[0,\pi/2]} \exp\left(\frac{\sin\theta-\cos\theta}{\sin\theta+\cos\theta}\right)\left(\int_{[0,(\sin\theta+\cos\theta)^{-1}]} r\,dr\right)\mathrm{d}\theta \\ &= \frac{1}{2}\int_{[0,\pi/2]} \exp\left(\frac{\sin\theta-\cos\theta}{\sin\theta+\cos\theta}\right)\left(\frac{1}{\sin\theta+\cos\theta}\right)^2 \mathrm{d}\theta \end{aligned}$$

Set $t = \frac{\sin\theta-\cos\theta}{\sin\theta+\cos\theta}$. Then $dt = \frac{2}{(\sin\theta+\cos\theta)^2}\,\mathrm{d}\theta$ so that

$$I = \frac{1}{4}\int_{[-1,1]} \exp(t)\,dt = \frac{e-e^{-1}}{4}.$$

b) Let $y = \frac{1}{2}(u+v)$, $x = \frac{1}{2}(v-u)$. Then

$$I = \frac{1}{2}\iint_{T'} \exp\left(\frac{u}{v}\right)\,\mathrm{d}u\,\mathrm{d}V$$

where T' is the triangle in the $uv-$plane bounded by the points $(0,0)$, $(-1,1)$ and $(1,1)$. Then

$$I = \frac{1}{2}\int_{[0,1]}\int_{[-v,v]} \exp\left(\frac{u}{v}\right)\,\mathrm{d}u\,\mathrm{d}V = \cdots = \frac{e-e^{-1}}{4}. \qquad \square$$

35. Compute the area of the planar region bounded by $y = x^2$, $y = 2x^2$, $x = y^2$ and $x = 3y^2$.

Solution: denote the region in question by D and set $u = \frac{y}{x^2}$ and $v = \frac{x}{y^2}$. Then $(x,y) \in D$ if and only if $(u,v) \in R$, where R is the rectangle defined by $1 \leq u \leq 2$ and $1 \leq v \leq 3$. Let $\varphi : D \to R$ be defined by $\varphi(x,y) = (u,v) = (\frac{y}{x^2}, \frac{x}{y^2})$. Then we have

$$J_\varphi(x,y) = \det D\varphi(x,y) = \frac{3}{x^2y^2} = 3u^2v^2$$

and

$$\left|J_{\varphi^{-1}}(u,v)\right| = \frac{1}{|J_\varphi(x,y)|} = \frac{1}{3u^2v^2}.$$

Consequently,

$$\text{Area}(D) = \iint_D \mathrm{d}x\,\mathrm{d}y = \iint R\frac{1}{3u^2v^2}\,\mathrm{d}u\,\mathrm{d}V = \frac{1}{3}\int_{[1,2]}\int_{[1,3]} \frac{1}{v^2u^2}\,\mathrm{d}V\,\mathrm{d}u = \cdots = \frac{1}{9}. \qquad \square$$

36. For what values of $k \in \mathbb{R}$ does the integral

$$\iint_{x^2+y^2\leq 1} \frac{\mathrm{d}x\,\mathrm{d}y}{(x^2+y^2)^k}$$

converge? For each such k, find the value to which it converges.

Solution: first, note that

$$\iint_{x^2+y^2\le 1} \frac{\mathrm{d}x\,\mathrm{d}y}{(x^2+y^2)^k} = \lim_{\varepsilon\to 0} \iint_{\varepsilon^2\le x^2+y^2\le 1} \frac{\mathrm{d}x\,\mathrm{d}y}{(x^2+y^2)^k}.$$

In polar coordinates, we have

$$\iint_{\varepsilon^2\le x^2+y^2\le 1} \frac{\mathrm{d}x\,\mathrm{d}y}{(x^2+y^2)^k} = \int_{[\varepsilon,1]}\int_{[0,2\pi]} \frac{1}{r^{2k-1}}\,\mathrm{d}\theta\,dr = 2\pi\int_{[\varepsilon,1]} \frac{dr}{r^{2k-1}}.$$

Then,

$$\lim_{\varepsilon\to 0}\int_{[\varepsilon,1]} \frac{dr}{r^{2k-1}}$$

if and only if $2k-1<1$, i.e. $k<1$. Furthermore,

$$\int_{[\varepsilon,1]} \frac{dr}{r^{2k-1}} = \frac{1}{2(1-k)} - \frac{\varepsilon^{2(1-k)}}{2(1-k)},$$

and so

$$\iint_{x^2+y^2\le 1} \frac{\mathrm{d}x\,\mathrm{d}y}{(x^2+y^2)^k} = \frac{\pi}{1-k}$$

when $k<1$. □

37. Find the volume of the solid bounded by the interior of the sphere $x^2+y^2+z^2=a^2$ and the interior of the cylinder $x^2+y^2=a^2$, $a>0$.

Solution: let V be the volume sought. Set

$$B = \{(x,y) \mid x^2+y^2\le a^2\}.$$

We have

$$\begin{aligned} V &= 2\iint_B \sqrt{2a^2-(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y = 2\int_{[0,a]}\int_{[0,2\pi]} \sqrt{2a^2-r^2}\,\mathrm{d}\theta\,dr \\ &= 4\pi\int_{[0,a]} \sqrt{2a^2-r^2}\,r\,dr = \cdots = \frac{4\pi}{3}\left(2^{3/2}-1\right)a^3. \qquad \square \end{aligned}$$

38. Find the volume of the solid bounded by the interior of the cone $z^2=x^2+y^2$ lying above the paraboloïd $z=6-x^2-y^2$.

Solution: let V be the volume sought. Set

$$B = \{(x,y) \mid x^2+y^2\le 4\}.$$

We have

$$\begin{aligned} V &= 2\iint_B \left(6-(x^2+y^2)-\sqrt{x^2+y^2}\right)\,\mathrm{d}x\,\mathrm{d}y = \int_{[0,2]}\int_{[0,2\pi]} (6-r^2-r)r\,\mathrm{d}\theta\,dr \\ &= 2\pi\int_{[0,2]} (6-r^2-r)r\,\mathrm{d}\theta\,dr = \cdots = \frac{32\pi}{3}. \qquad \square \end{aligned}$$

39. Find the volume of the solid bounded by the plane $z = 3x+4y$ lying below the paraboloïd $z = x^2 + y^2$.

    Solution: the intersection of the paraboloïd and the plane is $\{(x, y, z) \mid 3x + 4y = z = x^2 + y^2\}$. The set

    $$D = \{(x, y) \mid 3x + 4y = x^2 + y^2\}$$

    is the circle of radius $\frac{5}{2}$ centered at $(\frac{3}{2}, 2)$. For every $(x, y) \in D$, $x^2 + y^2 \leq 3x + 4y$. Let V be the volume sought. Set

    $$B = \{(x, y) \mid x^2 + y^2 \leq 4\}.$$

    We have

    $$V = \iint_B \left(3x + 4y - (x^2 + y^2)\right) \, \mathrm{d}x \, \mathrm{d}y.$$

    Using the change of variable

    $$x = \frac{3}{2} + r \cos\theta, \quad y = 2 + r \sin\theta,$$

    we obtain $V = \frac{1875\pi}{64}$. □

23.9 Exercises

1. Prepare a 2-page summary of this chapter, with important definitions and results.
2. Let $\mathfrak{S}$ be a $\sigma-$algebra. Show that
    a) $A_1, A_2, \ldots, A_n, \ldots \in \mathfrak{S} \implies \bigcap_{n\geq 1} A_n \in \mathfrak{S}$;
    b) $A, B \in \mathfrak{S} \implies A \cap B^c \in \mathfrak{S}$, and
    c) $\varnothing, \mathbb{R}^n \in \mathfrak{S}$.
3. Complete the proof of Lemma 291.1.
4. Compute $\iint s_1(x, y) \, \mathrm{d}x \, \mathrm{d}y$ and $\iint s_2(x, y) \, \mathrm{d}x \, \mathrm{d}y$ in the example of Section 21.3.
5. In the example of Section 21.3, show that:
    a) for $1 \leq i \leq 2^n$, we have Area$(A_i^n) = \frac{1}{4^n}\left(i - \frac{1}{2}\right)$;
    b) for $2^n + 1 \leq i \leq 2^{n+1}$, we have Area$(A_i^n) = \frac{1}{4^n}\left(2^{n+1} - i - \frac{1}{2}\right)$.
6. Complete the proof of Corollary 294.
7. Is the converse of the third solved problem (Borel-Lebesgue integration on $\mathbb{R}^n$) true?
8. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a Borel function and $d \in \mathbb{R}$. Show that $\{z \in \mathbb{R}^2 \mid f(z) < d\} \in \mathcal{B}(\mathbb{R}^2)$.
9. Complete the proof of Proposition 289 for $f + g$ and fg.

10. Show that if $g : \mathbb{R}^2 \to \mathbb{R}$ and
$$\{z \in \mathbb{R}^2 \mid g(z) < d\} \in \mathcal{B}(\mathbb{R}^2)$$
for all $d \in \mathbb{R}$, then g is a Borel function.

11. Show that $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$ but that Area$(\mathbb{Q}^2) = 0$.

12. Show that $\mathcal{V}_n = \{f : \mathbb{R}^n \to \mathbb{R} \mid f$ finite, Borel, integrable$\}$ is a vector space and that the Borel-Lebesgue integral is a linear functional over $\mathcal{V}_n$.

13. Complete the proof of Theorem 301.

14. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(z) = \exp(-\|z\|^2)$. Find a sequence of simple functions
$$0 \leq s_1 \leq s_2 \leq \ldots \leq s_n \leq f$$
for which $s_n(z) \to f(z)$ for all $z \in \mathbb{R}^2$. Can you use the sequence to compute $\int f \, \mathrm{d}m$? If so, do so.

15. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by
$$f(z) = \begin{cases} x^2 + y^2 & \text{if } (x, y) \in [0,1] \times [0,1] \\ 0 & \text{otherwise} \end{cases}$$
Find a sequence of simple functions
$$0 \leq s_1 \leq s_2 \leq \ldots \leq s_n \leq f$$
for which $s_n(z) \to f(z)$ for all $z \in \mathbb{R}^2$. Can you use the sequence to compute $\int f \, \mathrm{d}m$? If so, do so.

16. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by
$$f(z) = \begin{cases} x^2 + y^2 & \text{if } x^2 + y^2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$
Find a sequence of simple functions
$$0 \leq s_1 \leq s_2 \leq \ldots \leq s_n \leq f$$
for which $s_n(z) \to f(z)$ for all $z \in \mathbb{R}^2$. Can you use the sequence to compute $\int f \, \mathrm{d}m$? If so, do so.

17. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by
$$f(z) = \begin{cases} x + y + z & \text{if } (x, y, z) \in [0,1] \times [0,1] \times [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Find a sequence of simple functions

$$0 \leq s_1 \leq s_2 \leq \ldots \leq s_n \leq f$$

for which $s_n(z) \to f(z)$ for all $z \in \mathbb{R}^3$. Can you use the sequence to compute $\int f \, \mathrm{d}m$? If so, do so.

18. Give a proof of the Lebesgue monotone convergence theorem.

19. Prove Theorem 300.

20. Show that $\Psi(r, \theta) = (x, y)$ is a diffeomorphism between U and V for polar coordinates.

21. Show that $|J_\Psi(r, \varphi, \theta)| = r^2 \sin \varphi$ for spherical coordinates.

22. What is the volume of the solid defined by the intersection of the two cylinders $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$?

23. What is the volume of the solid Q directly above the region bounded by $0 \leq x \leq 1$, $1 \leq y \leq 2$ in the $xy-$plane and below the plane $z = 4 - x - y$?

24. Evaluate the integral $\iint_D x^2 y \, \mathrm{d}x \, \mathrm{d}y$ where D is the region bounded by the curves $y = x^2$ and $x = y^2$ in the first quadrant.

25. Let $f, f_1 : I \to \mathbb{R}$ be two continuous functions for which $f_1 \leq f$. If

$$A = \{(x, y) \in \mathbb{R}^2 \mid f_1(x) \leq y \leq f(x)\},$$

show that

$$\iint \chi_A(x, y) \, \mathrm{d}x \, \mathrm{d}y = \int_I (f_1(x) - f(x)) \, \mathrm{d}x.$$

Can you use this result to show that

$$\text{Graph}(f) = \{(x, f(x)) \mid x \in I\}$$

has $2D$ measure 0?

26. The Gamma and Beta functions are defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt, \quad \text{for } x > 0$$
$$B(x, y) = \int_0^1 t^{x-1} (1 - t)^{y-1} \, dt, \quad \text{for } x > 0, y > 0$$

Show that the following properties hold:

a) $\Gamma(x + 1) = x\Gamma(x), \quad (x > 0)$;
b) $\Gamma(n + 1) = n!, \quad (n = 0, 1, 2, \ldots)$;

c) $\Gamma(x) = 2\int_0^\infty s^{2x-1}e^{-s^2}\,ds, \quad (x>0)$;

d) $\Gamma(\frac{1}{2}) = \sqrt{\pi}, \quad \Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$;

e) $B(x,y) = 2\int_0^{\pi/2} \cos^{2x-1}\theta \sin^{2y-1}\theta\,\mathrm{d}\theta, \quad (x>0, y>0)$;

f) $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad (x>0, y>0)$.

27. Find the volume of the solid bounded by the interior of each of the cylinders $x^2+y^2=a^2$, $x^2+z^2=a^2$ and $y^2+z^2=a^2$, $a>0$.

28. Let S be the sphere of radius $a>0$ centered at $(0,0,a)$. Show that $\iiint_S z^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{8}{5}\pi a^5$.

29. Compute $\iiint e^{-(x^2+y^2+z^2)}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$.

30. Show that $S^1 = \{(x,y) \mid x^2+y^2=1\}$ has $2D$ measure 0.

31. Show that every countable subset of $\mathbb{R}^2$ has $2D$ measure 0.