Chapter 3

Limits and Continuity

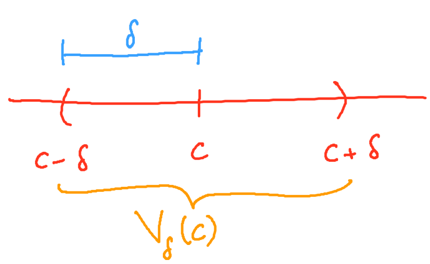
The main objects of study in analysis are **functions**. In this chapter, we introduce the $\varepsilon-\delta$ **definition of the limit** of a function, provide results that help to compute such limits, identify two types of **continuity**, and present some of the theorems that form the basis of analytical endeavours.

3.1 Limit of a Function

The objects we have studied thus far are functions of $\mathbb{N}$ into $\mathbb{R}$. However, most of calculus deals with functions of $\mathbb{R}$ into $\mathbb{R}$. How do we generalize the concepts and results we have derived for sequences to functions?

Let $A \subseteq \mathbb{R}$ and $c \in \mathbb{R}$. The **neighbourhood** $V_\delta(c)$, where $\delta > 0$, is the interval

$$V_\delta(c) = \{x \in \mathbb{R} : |x - c| < \delta\} = (c - \delta, c + \delta).$$



The point $c \in \mathbb{R}$ is a **limit point** (or **cluster point**) of A if every neighbourhood $V_\delta(c)$ contains at least one point $x \in A$ other than c.

> **Example:** consider the set $A \subseteq \mathbb{R}$ drawn below.
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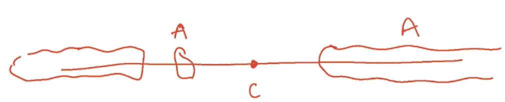
> 
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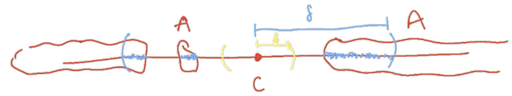
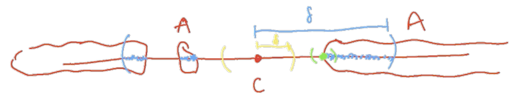
The $V_\delta(c)-$neighbourhood in blue contains points in A other than c, but c is not a limit point of A since the $V_\delta(c)-$neighbourhood in yellow does not contain points of A.



The point at the centre of the green interval is a limit point of A, however.



The **set of all limit points** of A is denoted by $\overline{A}$; a limit point of A does not have to be in A.

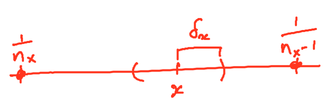
Example: what are the limit points of $A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$?

Solution: let $n \in \mathbb{N}$. The distance between a point $\frac{1}{n}$ and its immediate successor/predecessor $\frac{1}{n\pm1}$ is

$$\frac{1}{n} - \frac{1}{n\pm1} = \frac{1}{n(n\pm1)} > \frac{1}{3n^2}.$$

Let $\delta = \frac{1}{3n^2}$. Then $V_\delta(\frac{1}{n}) = (\frac{1}{n} - \frac{1}{3n^2}, \frac{1}{n} + \frac{1}{3n^2}) \subseteq (\frac{1}{n-1}, \frac{1}{n+1})$, so the only point of A in $V_\delta(\frac{1}{n})$ is $\frac{1}{n}$. Thus $\frac{1}{n} \notin \overline{A}$. No negative real number is a limit point of A; indeed, if $x < 0$, set $\delta = \frac{|x|}{2}$. Then $V_\delta(x) \subseteq (-\infty, 0)$ and so contains no point of A. Similarly, no real number strictly greater than 1 is a limit point of A. Hence $\overline{A} \subseteq [0,1] \setminus A$.

Let $x \in (0,1] \setminus A$. By the Archimedean property, $\exists n_x \in \mathbb{N}$ s.t. $n_x > \frac{1}{x} > n_x - 1$, so $\frac{1}{n_x} < x < \frac{1}{n_x-1}$. Set $\delta_x = \frac{1}{2}\min\{|x - \frac{1}{n_x}|, |x - \frac{1}{n_x-1}|\}$. Then $V_{\delta_x}(x)$ contains none of the points of A.



The only remaining possibility is $x = 0$. Let $\delta > 0$. By the Archimedean property, $\exists N_\delta$ such that $\frac{1}{N_\delta} < \delta$. But $0 \neq \frac{1}{N_\delta} \in A$, Thus

$$\varnothing \neq \left\{\frac{1}{N_\delta}\right\} \subseteq V_\delta \cap A = (-\delta, \delta) \cap A,$$

so $x = 0$ is the only limit point of A: $\overline{A} = \{0\}$. □

Directly determining the limit points of a set is a time-intensive endeavour. Thankfully, there is a link between limit points and convergent sequences.

Theorem 24
A point $c \in \mathbb{R}$ is a limit point of A if and only if there is a sequence $(a_n) \subseteq A$, with $a_n \neq c$ for $n \in \mathbb{N}$, such that $a_n \to c$.

Proof: suppose c is a limit point of A. By definition, the neighbourhood $V_{\frac{1}{n}}(c)$ must contain a point $a_n \neq c \in A$, for all $n \in \mathbb{N}$. Let $\varepsilon > 0$. By the Archimedean property, $\exists N_\varepsilon > \frac{1}{\varepsilon}$ s.t. $\frac{1}{N_\varepsilon} < \varepsilon$. Thus

$$n > N_\varepsilon \Longrightarrow 0 < |a_n - c| < \frac{1}{n} < \frac{1}{N_\varepsilon} < \varepsilon, \quad \text{i.e. } a_n \to c.$$

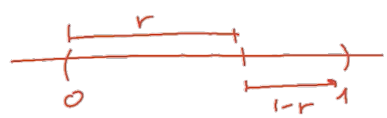
Conversely, suppose that there is a sequence $(a_n) \subseteq A$, with $a_n \neq c$ for all $n \in \mathbb{N}$, such that $a_n \to c$. Let $\delta > 0$. By definition, $\exists N_\delta \in \mathbb{N}$, such that $0 < |a_n - c| < \delta$ for all $n > N_\delta$. Then $a_n \in V_\delta(c)$ and $a_n \neq c$ for all $n > N_\delta$. Thus any neighbourhood of c contains at least one $a_n \neq c$, so $c \in \overline{A}$. ■

Any limit point of A is in fact the limit of a sequence in A, and *vice-versa*.

Example: let $A = [0, 1] \cap \mathbb{Q}$. What are the limit points of A?

Solution: any convergent sequence $(a_n) \subseteq A$ is such that $0 \leq a_n \leq 1$ for all $n \in \mathbb{N}$, so its limit must also lie in $[0, 1]$, according to Theorem 15. On the other hand, Theorem 24 tells us that any limit point of A is the limit of a sequence of rationals in $[0, 1]$. The sequences $(\frac{1}{n})$ and $(1 - \frac{1}{n})$ lie in A. Since $\frac{1}{n} \to 0$ and $1 - \frac{1}{n} \to 1$, then $0, 1 \in \overline{A}$.

Now, let $r \in (0, 1)$. Set $\eta = \min\{r, 1 - r\}$.



Then $\eta > 0$ and $\frac{1}{\eta} > 0$. By the Archimedean property, $\exists M \in \mathbb{N}$ s.t. $M > \frac{1}{\eta}$. Then

$$0 \leq r - \eta < r - \frac{1}{M} > r + \frac{1}{M} < r + \eta \leq 1,$$

since $\eta = r$ if $r \leq 1/2$ and $\eta = 1 - r$ if $r \geq 1/2$. So

$$n > M \Longrightarrow 0 < r - \frac{1}{n} < r + \frac{1}{n} < 1.$$

But the density theorem states that for all $n > M$, $\exists a_n \neq r \in \mathbb{Q}$ such that

$$r - \frac{1}{n} < a_n < r + \frac{1}{n}.$$

The sequence (a_n) thus constructed converges to r. Indeed, let $\varepsilon > 0$. According to the Archimedean property, $\exists N \in \mathbb{N}$ such that $N > \frac{1}{\varepsilon}$.

Set $N_\varepsilon = \max\{M, N\}$. Then

$$n > N_\varepsilon \Longrightarrow 0 < |a_n - r| < \frac{1}{n} < \frac{1}{N_\varepsilon} < \varepsilon,$$

and so $a_n \to r$ and $r \in \overline{A}$. Consequently, $\overline{A} = [0, 1]$. ■

Intuitively, a limit of a function f at c is a value L towards which $f(x)$ "approaches" as x gets closer to c, if it exists. But what does that actually mean? What would need to happen for the value not to exist?

Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, and $c \in \overline{A}$: $L \in \mathbb{R}$ is the **limit of f at c** if

$$\forall \varepsilon > 0,\ \exists \delta_\varepsilon > 0 \text{ such that } 0 < |x - c| < \delta_\varepsilon \text{ and } x \in A \Longrightarrow |f(x) - L| < \varepsilon,$$

which we denote by

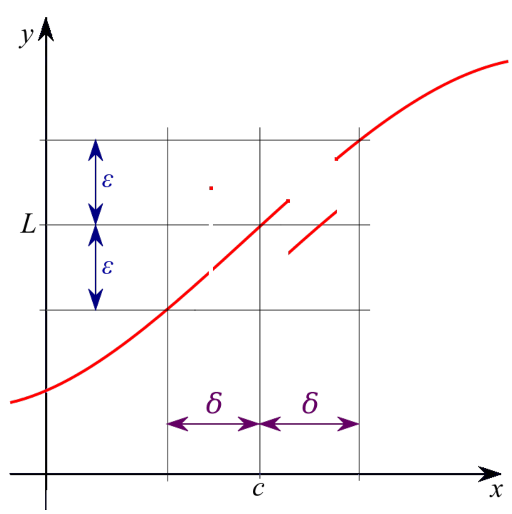
$$\lim_{x\to c} f(x) = L \quad \text{or by} \quad f(x) \to L, \text{ when } x \to c.$$

The limit of f at c is **not** $L \in \mathbb{R}$ if

$$\exists \varepsilon_0 > 0,\ \forall \delta > 0,\ \exists x_\delta \in A \text{ such that } 0 < |x_\delta - c| < \delta_\varepsilon \text{ and } |f(x_\delta) - L| \geq \varepsilon_0,$$

which we denote by

$$\lim_{x\to c} f(x) \neq L \quad \text{or by} \quad f(x) \not\to L, \text{ when } x \to c.$$



The underlying principle is the same as that of the limit of a sequence: given $\varepsilon > 0$, we need to find a $\delta_\varepsilon > 0$ which satisfies the definition. Graphically, this is equivalent to putting a horizontal strip of width 2ε around the line $y = L$, and showing that there is a neighbourhood $V_{\delta_\varepsilon}(c)$ such that $f(x)$ is in the strip for any $x \in V_{\delta_\varepsilon}$.

Examples

1. Let $f : [0, 1) \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} 2, & x \in (0,1) \\ 3, & x = 0 \end{cases}$$

Show $\lim_{x\to 0} f(x) = 2$.

Proof: let $\varepsilon > 0$. Set $\delta_\varepsilon = 1$. Then

$$x \in [0,1) \text{ and } 0 < |x - c| < \delta_\varepsilon \implies |f(x) = 2| = 0 < 0 \cdot \delta < \varepsilon,$$

which completes the proof. ■

2. Let $f : [0, \infty) \to \mathbb{R}$ be defined by $f(x) = \frac{x^2+2x+2}{x+1}$. Show $\lim_{x\to 2} f(x) = \frac{10}{3}$.

Proof: let $\varepsilon > 0$. Set $\delta_\varepsilon = \varepsilon$. Then

$$\left|\frac{x^2+2x+2}{x+1} - \frac{10}{3}\right| = \left|\frac{3(x^2+2x+2) - 10(x+1)}{x+1}\right| = \left|\frac{3x^2-4x-4}{3x+3}\right|$$
$$= \underbrace{\left|\frac{3x+2}{3x+3}\right|}_{<1} |x-2| < |x-2| < \delta_\varepsilon = \varepsilon$$

when $x \geq 0$ and $0 < |x - 2| < \delta_\varepsilon$. ■

3. Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = x^2 \cos(1/x)$. Show that $\lim_{x\to 0} f(x) = 0$.

Proof: note that $c \in A = \mathbb{R} \setminus \{0\}$. We can only use the definition of the limit if $c \in \overline{A}$. That it does so is a given, as $(\frac{1}{n}) \subseteq A$ and $\frac{1}{n} \to 0$, with $\frac{1}{n} \neq 0$ for all $n \in \mathbb{N}$, according to Theorem 24.

Let $\varepsilon > 0$ and set $\delta_\varepsilon = \sqrt{\varepsilon}$. Then

$$\left|x^2 \cos(1/x) - 0\right| = |x|^2 \underbrace{|\cos(1/x)|}_{\leq 1} \leq |x|^2 = |x - 0|^2 < \delta_\varepsilon^2 < \varepsilon,$$

whenever $x \in \mathbb{R} \setminus \{0\}$ and $0 < |x - 0| < \delta_\varepsilon$. ■

As is the case with sequences, a function has at most one limit at any of its limit points c.

Theorem 25
Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ and c a limit point of A. Then f has at most one limit at c.

Proof: suppose that

$$\lim_{x\to c} f(x) = L' \quad \text{and} \quad \lim_{x\to c} f(x) = L'', \quad \text{where } L' < L''.$$

Let $\varepsilon = \frac{L''-L'}{3} > 0$. By definition, $\exists \delta'_\varepsilon, \delta''_\varepsilon$ s.t. $|f(x) - L'| < \varepsilon$ and $|f(x) - L''| < \varepsilon$ whenever $x \in A$ and $0 < |x - c| < \delta'_\varepsilon$, $0 < |x - c| < \delta''_\varepsilon$.

Set $\delta_\varepsilon = \min\{\delta'_\varepsilon, \delta''_\varepsilon\}$. Then, whenever $x \in A$ and $0 < |x - c| < \delta_\varepsilon$,

$$\begin{aligned} f(x) < L'' + \varepsilon = L' + \frac{L''-L'}{3} &= \frac{2L'+L''}{3} = \frac{L'+L''}{3} + \frac{L'}{3} \\ < \frac{L'+L''}{3} + \frac{L''}{3} < \frac{2L''+L'}{3} &= L'' - \frac{L''-L'}{3} = L'' - \varepsilon < f(x), \end{aligned}$$

which is a contradiction, hence $L' \not< L''$. The proof that $L'' \not< L'$ is identical. ■

As is the case with sequences, the definition is useless if we do not have a candidate for L beforehand. The next result allows us to get such a candidate before using the definition.

Theorem 26 (SEQUENTIAL CRITERION)
Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ and c a limit point of A. Then

$$\lim_{x\to c} f(x) = L \quad \textit{if and only if} \quad \lim_{n\to\infty} f(x_n) = L$$

for any sequence $(x_n) \subseteq A$ such that $x_n \to c$, with $x_n \neq c$ for all $n \in \mathbb{N}$.

Proof: assume $\lim_{x\to c} f(x) = L$. Let $\varepsilon > 0$. Then $\exists \delta_\varepsilon > 0$ such that

$$x \in A \text{ and } 0 < |x - c| < \delta_\varepsilon \implies |f(x) - L| < \varepsilon.$$

Suppose $(x_n) \subseteq A$ is such that $x_n \neq c$ for all $n \in \mathbb{N}$ and $x_n \to c$. Then $\exists M_{\delta_\varepsilon} > 0$ such that $0 < |x_n - c| < \delta_\varepsilon$ whenever $n > M_{\delta_\varepsilon}$.

Let $N_\varepsilon = M_{\delta_\varepsilon}$. Then

$$x_n \neq c \in A \text{ and } n > N_\varepsilon \implies 0 < |x_n - c| < \delta_\varepsilon \implies |f(x_n) - L| < \varepsilon,$$

which is to say $f(x_n) \to L$.

Conversely, if $\lim_{x\to c} f(x) \neq L$, then $\exists \varepsilon_0 > 0$ s.t. $\forall \delta > 0$, $\exists x_\delta \in A$ with $0 < |x_\delta - c| < \delta$ but $|f(x) - L| \geq \varepsilon_0$. Thus, for $n \in \mathbb{N}$ and $\delta = \frac{1}{n}$, $\exists x_n = x_\delta$ as above.

The sequence $(x_n) \subseteq A$ is such that $0 < |x_n - c| < \frac{1}{n}$ and $|f(x_n) - L| \geq \varepsilon_0$. According to the squeeze theorem, $x_n \to c$, with $|f(x_n) - L| \geq \varepsilon_0$ for all $n \in \mathbb{N}$. Thus $f(x_n) \not\to L$. ■

Let us take a look at a few examples.

Examples

1. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 3x^3 + x + 1$. Compute $\lim_{x\to 7} f(x)$.

   Solution: let $(x_n) \subseteq \mathbb{R} \setminus \{7\}$ with $x_n \to 7$. Then

$$\begin{aligned}\lim_{n\to\infty} f(x_n) &= \lim_{n\to\infty} (3x_n^2 + x_n + 1) = 3\left(\lim_{n\to\infty} x_n\right)^2 + \lim_{n\to\infty} x_n + 1 \\ &= 3 \cdot 7^3 + 7 + 1 = 1037.\end{aligned}$$

   Thus $f(x) \to 1037$ when $x \to 7$, according to Theorem 26. □

2. Let $f : (2, \infty) \to \mathbb{R}$, $f(x) = \frac{(x-1)(x-2)}{(x-2)}$. Compute $\lim_{x\to 2} f(x)$.

   Solution: let $(x_n) \subseteq \mathbb{R} \setminus \{2\}$ with $x_n \to 2$. Then

$$\begin{aligned}\lim_{n\to\infty} f(x_n) &= \lim_{n\to\infty} \frac{(x_n - 1)(x_n - 2)}{(x_n - 2)} = \lim_{n\to\infty} (x_n - 1) = \lim_{n\to\infty} x_n - 1 \\ &= 2 - 1 = 1.\end{aligned}$$

   Since (x_n) was arbitrary, $f(x) \to 1$ when $x \to 2$, according to Theorem 26. □

3. Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = x^2 \cos(1/x)$. Show that $\lim_{x\to 0} f(x) = 0$.

   Proof: let $(x_n) \subseteq \mathbb{R} \setminus \{0\}$ be any sequence converging to 0. Then

$$0 \leq |x_n^2 \cos(1/x_n)| \leq |x_n^2| = |x_n|^2.$$

   However, since $x_n \to 0$, then both $|x_n| \to 0$ and $|x_n|^2 \to 0$, which is to say that

$$\lim_{n\to 0} |x_n^2 \cos(1/x_n)| = 0$$

   according to the squeeze theorem. Thus $x_n^2 \cos(1/x_n) \to 0$. Since (x_n) was arbitrary, $f(x) \to 0$ when $x \to 0$, according to the sequential criterion. ■

4. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} 0, & x \in \mathbb{Q} \\ 1, & x \notin \mathbb{Q} \end{cases}$$

Show that $\lim_{x \to 0} f(x)$ does not exist.

Proof: define $(x_n), (y_n)$ by $x_n = \frac{1}{n}$, $y_n = \frac{\sqrt{2}}{n}$ for all $n \in \mathbb{N}$. Then $(x_n) \subseteq \mathbb{Q}$ and $(y_n) \subseteq \mathbb{R} \setminus \mathbb{Q}$. Furthermore, $x_n, y_n \to 0$, with $x_n, y_n \neq 0$ for all $n \in \mathbb{N}$. But $f(x_n) = 0$ and $f(y_n) = 1$ for all $n \in \mathbb{N}$, so

$$\lim_{n \to \infty} f(x_n) = 0 \neq 1 = \lim_{n \to \infty} f(y_n),$$

thus $\lim_{x \to 0} f(x)$ does not exist. ■

5. Let sgn $: \mathbb{R} \to \mathbb{R}$ be the function defined by

$$\text{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

Show that $\lim_{x \to 0}(x + \text{sgn}(x))$ does not exist.

Proof: define $(x_n), (y_n)$ by $x_n = \frac{1}{n}$, $y_n = -\frac{1}{n}$ for all $n \in \mathbb{N}$. Then $x_n, y_n \to 0$, with $x_n, y_n \neq 0$ for all $n \in \mathbb{N}$.

But $f(x_n) = \frac{1}{n} + \text{sgn}\left(\frac{1}{n}\right) = \frac{1}{n} + 1$, and $f(y_n) = -\frac{1}{n} + \text{sgn}\left(-\frac{1}{n}\right) = -\frac{1}{n} - 1$ for all $n \in \mathbb{N}$, so

$$\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} \left(\frac{1}{n} + 1\right) \neq -1 = \lim_{n \to \infty} \left(\frac{1}{n} + 1\right) = \lim_{n \to \infty} f(x_n),$$

thus $\lim_{x \to 0} f(x)$ does not exist. ■

To show that the limit does not exist, it is enough to find two specific sequences $(x_n), (y_n) \subseteq A$, with $x_n, y_n \neq c$ for all $n \in \mathbb{N}$ and $x_n, y_n \to c$, such that $f(x_n) \to L_1$, $f(y_n) \to L_2$, $L_1 \neq L_2$.

But we cannot show that the limit L exists by finding two sequences $(x_n), (y_n) \subseteq A$ with $x_n, y_n \neq c$ for all $n \in \mathbb{N}$, $x_n, y_n \to c$, and $f(x_n), f(y_n) \to L$.

Note that at no point have we needed to use the graph of a function to compute a limit or prove its existence.

3.2 Properties of Limits

Limits behave quite nicely with respect to the usual operations.

Theorem 27 (Operations on Limits)
Let $A \subseteq \mathbb{R}$, $f, g : A \to \mathbb{R}$, and c a limit point of A. Suppose $f(x) \to L$ and $g(x) \to M$ when $x \to c$. Then

1. $\lim_{x\to c} |f(x)| = |L|$;
2. $\lim_{x\to c}(f(x) + g(x)) = L + M$;
3. $\lim_{x\to c} f(x)g(x) = LM$;
4. $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}$, *if $g(x) \neq 0$ for all $x \in A$ and if $M \neq 0$.*

Proof: this result is an easy consequence of Theorems 14 and 26. Let $(x_n) \subseteq A$ with $x_n \neq c$ and $x_n \to c$ for all $n \in \mathbb{N}$. Then $f(x_n) \to L$ and $g(x_n) \to M$.

1. $\lim_{x\to c} |f(x)| = \lim_{n\to\infty} |f(x_n)| = \left|\lim_{n\to\infty} f(x_n)\right| = L.$
2. $\lim_{x\to c}\left[f(x) + g(x)\right] = \lim_{n\to\infty}\left[f(x_n) + g(x_n)\right] = \lim_{n\to\infty} f(x_n) + \lim_{n\to\infty} g(x_n) = L + M.$
3. $\lim_{x\to c}\left[f(x)g(x)\right] = \lim_{n\to\infty}\left[f(x_n)g(x_n)\right] = \lim_{n\to\infty} f(x_n) \cdot \lim_{n\to\infty} g(x_n) = LM.$
4. $\lim_{x\to c}\left[\frac{f(x)}{g(x)}\right] = \lim_{n\to\infty}\left[\frac{f(x_n)}{g(x_n)}\right] = \frac{\lim_{n\to\infty} f(x_n)}{\lim_{n\to\infty} g(x_n)} = \frac{L}{M}$, if $g(x) \neq 0$ for $x \in A$ and if $M = 0$. ■

There is also a squeeze theorem for functions, but it is not nearly as useful as the corresponding result for sequences.

Theorem 28 (squeeze theorem for Functions)
Let $A \subseteq \mathbb{R}$, $f, g, h : A \to \mathbb{R}$, and c a limit point of A. If $f(x) \leq g(x) \leq h(x)$ for all $x \in A$ and if $f(x), h(x) \to L$ when $x \to c$, then $g(x) \to L$ when $x \to c$.

Proof: let $(x_n) \subseteq A$, with $x_n \neq c$ for all $n \in \mathbb{N}$ and $x_n \to c$. According to the sequential criterion,
$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} h(x_n) = L.$$
Since $f(x_n) \leq g(x_n) \leq h(x_n)$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} g(x_n) = L$, by the squeeze theorem (for sequences). Since (x_n) was arbitrary, we conclude that $g(x) \to L$, again by the sequential criterion. ■

Let's take a look at some examples.

Examples

1. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = k$, $k \in \mathbb{R}$. Show that $\lim_{x\to c} f(x) = k$ for all $c \in \mathbb{R}$.

   Proof: let $\varepsilon > 0$. Set $\delta_\varepsilon = \varepsilon$. Then $|f(x) - k| = |k - k| = 0 < \varepsilon$, when $0 < |x - c| < \delta_\varepsilon = \varepsilon$. ■

2. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x$. Show that $\lim_{x\to c} f(x) = f(c)$ for all $c \in \mathbb{R}$.

   Proof: let $\varepsilon > 0$. Set $\delta_\varepsilon = \varepsilon$. Then $|f(x) - c| = |x - c| < \delta_\varepsilon = \varepsilon$, when $0 < |x - c| < \delta_\varepsilon = \varepsilon$. ■

3. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{x^3+2x-4}{x^2+1}$. Compute $\lim_{x\to 3} f(x)$.

   Solution: according to Theorem 27, and the preceding examples,

$$\lim_{x\to 3}(x^3 + 2x + 4) = \left(\lim_{x\to 3} x\right)^3 + 2\left(\lim_{x\to 3} x\right) + \lim_{x\to 3} 4 = 3^2 + 2(3) + 3 = 37$$
$$\lim_{x\to 3}(x^2 + 1) = \left(\lim_{x\to 3} x\right)^2 + 1 = 3^2 + 1 = 10,$$

   and so $\lim_{x\to 3} \dfrac{x^3 + 2x - 4}{x^2 + 1} = \frac{10}{3}$, **because** $x^2 + 1 \neq 0$ for all $x \in \mathbb{R}$. □

4. Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = x^2 \cos(1/x)$. Show that $\lim_{x\to 0} f(x) = 0$.

   Proof: we cannot use the multiplication component of Theorem 27 to compute the limit since $\lim_{x\to 0} \cos(1/x)$ does not exist.

   Indeed, let $(x_n), (y_n) \subseteq \mathbb{R} \setminus \{0\}$ be such that $x_n = \frac{1}{(2n-1)\pi}$, and $y_n = \frac{1}{2n\pi}$ for all $n \in \mathbb{N}$. Then $x_n, y_n \to 0$. But

$$\cos\left(\frac{1}{x_n}\right) = \cos((2n-1)\pi) = -1 \quad \text{and} \quad \cos\left(\frac{1}{y_n}\right) = \cos(2n\pi) = 1$$

   for all $n \in \mathbb{N}$. Then

$$\cos(1/x_n) \to -1 \neq 1 \leftarrow \cos(1/y_n).$$

   This does not mean that

$$\lim_{x\to 0} x^2 \cos\left(\frac{1}{x}\right)$$

   does not exist, only that we cannot use Theorem 27 to compute it.

   In fact, the squeeze theorem for functions does the trick, with $-x^2 \leq f(x) \leq x^2$. ■

Other sequence concepts have analogous definitions in the world of functions. Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ and $c \in \overline{A}$. The function f is **bounded on some neighbourhood of** c if $\exists \delta > 0$ and $M > 0$ are such that $|f(x)| \leq M$ for all $x \in A \cap V_\delta(c)$.

Theorem 29
If $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $c \in \overline{A}$, and $\lim_{x \to c} f(x) = L$ for some $L \in \mathbb{R}$, then f is bounded on some neighbourhood of c.

Proof: Let $\varepsilon = 1$. By definition, $\exists \delta_1 > 0$ such that $|f(x) - L| < 1$ whenever $x \in A$ and $0 < |x - c| < \delta_1$. Since

$$|f(x)| - |L| < |f(x) - L|,$$

then $|f(x)| - |L| \leq 1$ whenever $x \in A$ and $0 < |x - c| < \delta_1$.

If $c \notin A$, set $M = |L| + 1$. If $c \in A$, set $M = \max\{|f(c)|, |L| + 1\}$. In either case, $|f(x)| \leq M$ whenever $x \in A$ and $0 < |x - c| < \delta_1$. ■

3.3 Continuous Functions

Functions like polynomials, or trigonometric functions, are continuous, which is a fundamental notion of calculus.

Intuitively, a function is continuous at a point if the graph of the function at that point can be traced without lifting the pen. The notion of "continuity" is fundamental is calculus.

But we emphasized earlier that limits could be computed/shown to exist without referring to the graph of a function. What does that mean for continuity?

Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, and $c \in A$; f **is continuous at** c if

$$\forall \varepsilon > 0, \exists \delta_\varepsilon > 0 \text{ such that } |x - c| < \delta_\varepsilon \text{ and } x \in A \Longrightarrow |f(x) - f(c)| < \varepsilon.$$

When computing the limit of f at c, we are interested in the behaviour of the function near c, but not at c. When we are dealing with continuity, we also **include the behaviour at** c. When c is a **limit point of** A, this definition actually means that

$$\lim_{x \to c} f(x) = f(c).$$

If $c \notin \overline{A}$, the expression $\lim_{x \to c} f(x)$ is meaningless.[1] In that case, f is automatically continuous at c. Indeed, there will then be a $\delta > 0$ such that $V_\delta(c)$ contains no point of A but c. Then for $\varepsilon > 0$, whenever $x \in A$ and $|x - c| < \delta$ (i.e., whenever $x = c$), we have

$$|f(x) - f(c)| = |f(c) - f(c)| = 0 < \varepsilon.$$

[1]Since there are no sequence $(x_n) \subseteq A$ with $x_n \neq c$ for all $n \in \mathbb{N}$ and $x_n \to c$.

The definition contains 3 statements: a function f is continuous at c if

1. $f(c)$ is defined;
2. $\lim_{x \to c} f(x)$ exists, and
3. $\lim_{x \to c} f(x) = f(c)$.

Let $B \subseteq A$. If f is continuous for all $c \in B$, then we say that f **is continuous on** B.

Examples

- Let $f : [0, \infty) \to \mathbb{R}$, $f(x) = \frac{x^2+2x+2}{x+1}$. Is f continuous at $c = 2$?

  Solution: since 2 is a limit point of $[0, \infty)$, we need only verify if $\lim_{x \to 2} f(x) = f(2)$. But we have already seen that $f(x) \to \frac{10}{3} = f(2)$ when $x \to 2$, so f is continuous at $c = 2$. $\square$

- Let $f : [0, 1) \to \mathbb{R}$,
  $$f(x) = \begin{cases} 2, & x \in (0, 1) \\ 3, & x = 0 \end{cases}$$
  Is f continuous at $c = 0$?

  Solution: since 0 is a limit point of $[0, 1)$, we need only verify if $\lim_{x \to 0} f(x) = f(0)$. But we have already seen that $f(x) \to 2 \neq 3 = f(0)$ when $x \to 0$, so f is not continuous at $c = 0$. $\square$

- Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 3x^3 + x + 1$. Is f continuous at $c = 7$?

  Solution: since 7 is a limit point of $\mathbb{R}$, we need only verify if $\lim_{x \to 7} f(x) = f(7)$. But we have already seen that $f(x) \to 1037 = f(7)$ when $x \to 7$, so f is continuous at $c = 7$. $\square$

- Let $f : \mathbb{R} \to \mathbb{R}$,
  $$f(x) = \begin{cases} 0, & x \in \mathbb{Q} \\ 1, & x \notin \mathbb{Q} \end{cases}$$
  Is f continuous at $c = 0$?

  Solution: as $f(0) = 0$, we only need to verify if $\lim_{x \to 0} f(x) = f(0)$. But we have already seen that $\lim_{x \to 0} f(x)$ does not exist, so f is not continuous at $c = 0$. $\blacksquare$

- Let $f:(2,\infty)\to\mathbb{R}$, $f(x)=\frac{(x-1)(x-2)}{(x-2)}$. Is f continuous at $c=2$?

  Solution: since f is not defined at $c=2$ and since $2\notin A$, f is not continuous at $c=2$. $\square$

- Let $f:\mathbb{R}\to\mathbb{R}$, $f(x)=k$, $k\in\mathbb{R}$. Is f continuous on $\mathbb{R}$?

  Solution: since all $c\in\mathbb{R}$ are limit points of $\mathbb{R}$, we need only verify if $\lim_{x\to c} f(x)=f(c)$. But we have already seen that $f(x)\to k=f(c)$ for all $c\in\mathbb{R}$, so f is continuous on $\mathbb{R}$. $\square$

- Let $f:[0,\infty)\to\mathbb{R}$, $f(x)=\sqrt{x}$. Is f continuous on $[0,\infty)$?

  Solution: let $\varepsilon>0$. If $c=0$, set $\delta_\varepsilon=\varepsilon$. Then

$$x\geq 0 \text{ and } |x-0|<\delta_\varepsilon \implies f(x)-f(0)|=\sqrt{x}=\sqrt{|x-0|}<\sqrt{\delta_\varepsilon=\varepsilon},$$

  so f is continuous at $c=0$. If $c>0$, set $\delta_\varepsilon=\sqrt{c}\varepsilon$. Then

$$|f(x)-f(c)|=|\sqrt{x}-\sqrt{c}|=\frac{|x-c|}{\sqrt{x}+\sqrt{c}}<\frac{|x-c|}{\sqrt{c}}<\frac{\delta_\varepsilon}{\sqrt{c}}=\varepsilon$$

  whenever $x\geq 0$ and $|x-c|<\delta_\varepsilon$. Hence f is continuous at any $c>0$. $\square$

- Let $f:\mathbb{R}\to\mathbb{R}$,

$$f(x)=\begin{cases} x, & x\in\mathbb{Q} \\ 0, & x\notin\mathbb{Q}\end{cases}$$

  Is f continuous at $c=0$? At $c\neq 0$?

  Solution: since $f(0)=0$, we need to see if $\lim_{x\to 0} f(x)=0$. Let $\varepsilon>0$ and set $\delta_\varepsilon>0$. Then $|x-0|<\delta_\varepsilon \implies |f(x)-f(0)|=|f(x)|\leq |x|=|x-0|<\delta_\varepsilon=\varepsilon$, so f is continuous at $c=0$. Now let $n\in\mathbb{N}$. According to the density theorem, $\exists x_n\in\mathbb{Q}$, $y_n\notin\mathbb{Q}$ such that

$$c<x_n+c+\frac{1}{n} \quad \text{and} \quad c<y_n<c+\frac{1}{n}.$$

  According to the sequence squeeze theorem, $x_n,y_n\to c$. But $f(x_n)=x_n$ and $f(y_n)=0$ for all $n\in\mathbb{N}$, so

$$\lim_{n\to\infty} f(x_n)=\lim_{n\to\infty} x_n=c \quad \text{and} \quad \lim_{n\to\infty} f(y_n)=\lim_{n\to\infty} 0=0.$$

  Since $c\neq 0$, these limits are different, and so $\lim_{x\to c} f(x)$ does not exist, according to the sequential criterion. $\square$

- Let $A = \{x \in \mathbb{R} \mid x > 0\}$. Consider the function $f : A \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \in \mathbb{Q}, \text{ with } \gcd(m,n) = 1 \end{cases}$$

Where is f is continuous?

Solution: we consider two types of limit points of A: $a \in \mathbb{Q}$ and $b \notin \mathbb{Q}$. If $0 < a \in \mathbb{Q}$, let $(x_n) \subseteq A \cap \mathbb{Q}^\complement$ be such that $x_n \to a$. Then $f(x_n) \to 0$. But $f(a) > 0$, so $f(x) \not\to f(a)$ when $x \to a$, according to the sequential criterion.

If $0 < b \notin \mathbb{Q}$, let $\varepsilon > 0$. By the Archimedean property, there exists an integer $N_0 > \frac{1}{\varepsilon}$. There can only be a finite set of rationals with denominator $< N_0$ in the interval $(b-1, b+1)$. Indeed, if $n < N_0$ and $\frac{m}{n} \in (b-1, b+1)$ then whenever $|k| > 2n$, we have:

$$\left|\frac{m+k}{n} - \frac{m}{n}\right| = \frac{|k|}{n} > 2 \Longrightarrow \frac{m+k}{n} \notin (b-1, b+1).$$

Consequently, $\exists \delta > 0$ such that there are no rational number $\frac{m}{n}$ with denominator $< N_0$ in $(b-\delta, b+\delta)$, which is to say that for all $x \in (b-\delta, b+\delta)$, either $f(x) = 0$ (when x is irrational) or $f(x) = \frac{1}{n} \leq \frac{1}{N_0}$ (when x is rational).

Thus, if $|x - b| < \delta$ and $x \in A$, we have

$$|f(x) - f(b)| = |f(x) - 0| = |f(x)| \leq \frac{1}{N_0} < \varepsilon,$$

so $f(x) \to f(b)$ when $x \to b$, i.e., f is only continuous on $A \cap (\mathbb{R} \setminus \mathbb{Q})$. □

Continuity behaves very nicely with respect to elementary operations on functions.

Theorem 30 (Operations on Continuous Functions)
Let $A \subseteq \mathbb{R}$, $f, g : A \to \mathbb{R}$, and $c \in A$. If f, g are continuous at c, then

1. *$|f|$ is continuous at c;*
2. *$f + g$ is continuous at c;*
3. *fg is continuous at c;*
4. *$\frac{f}{g}$ is continuous at c if $g \neq 0$ on A.*

Proof: if $c \notin \overline{A}$, there is nothing to prove. If $c \in \overline{A}$, then

$$\lim_{x\to c} f(x) = f(c) \quad \text{and} \quad \lim_{x\to c} g(x) = g(c).$$

We can then apply Theorem 27 directly with $L = f(c)$ and $M = g(c)$. ■

Since constants and the identity function are continuous on $\mathbb{R}$ (as we saw in the preceding examples), so are polynomial functions. Furthermore, rational functions are continuous on their domain.

The **composition** of the functions $f : A \to B$ and $g : B \to C$ is the function $g \circ f : A \to C$, with $(g \circ f)(x) = g(f(x))$ for all $x \in A$.

Theorem 31 (Composition of Continuous Functions)
Let $A, B \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $g : B \to \mathbb{R}$, $c \in A$. If f is continuous at c, g is continuous at $f(c)$, and $f(A) \subseteq B$, then $g \circ f : A \to B$ is continuous at c.

Proof: let $\varepsilon > 0$. As g is continuous at $f(c)$, $\exists \delta_\varepsilon > 0$ such that

$$y \in B \text{ and } |y - f(c)| < \delta_\varepsilon \implies |g(y) - g(f(c))| < \varepsilon.$$

Since f is continuous at c, $\exists \eta_{\delta_\varepsilon} = \eta_\varepsilon > 0$ such that

$$x \in A \text{ and } |x - c| < \eta_{\delta_\varepsilon} \implies |f(x) - f(c)| < \delta_\varepsilon \implies$$

$$x \in A \text{ and } |x - c| < \eta_\varepsilon \implies |(g \circ f)(x) - (g \circ f)(c)| = |g(f(x)) - g(f(c))| < \varepsilon,$$

which completes the proof. ■

It is not too difficult to see that Theorems 30 and 31 remain valid if we replace "continuous at c" with "continuous at A".

Example: let $f : [0, \infty) \to \mathbb{R}$, defined by $f(x) = \sqrt{3x^3 + x + 1}$. Show that f is continuous on $[0, \infty)$.

Proof: we can write $f = g \circ h$, where $g : [0, \infty) \to \mathbb{R}$, $g(y) = \sqrt{y}$ and $h : \mathbb{R} \to \mathbb{R}$, $h(x) = 3x^2 + x + 1$. Since g and h are both continuous on their domains and $h(\mathbb{R}) \subseteq [0, \infty)$, g is continuous on $[0, \infty)$, according to Theorem 31. ■

An **algebraic** function is a function obtained via the (possibly repeated) composition of rational functions and root functions. The class of algebraic functions is continuous on its domain. The same goes for trigonometric, exponential, and logarithmic functions, *via* their power series definition.

3.4 Max/Min Theorem

We begin our study of the classical theorems of calculus. Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$. The function $f : A \to \mathbb{R}$ is **bounded** on A if $\exists M > 0$ such that $|f(x)| < M$ for all $x \in A$.

Examples

1. $f : [0, 1] \to \mathbb{R}$, $f(x) = x^2$, is bounded on $[0, 1]$ as $|f(x)| < 2, \forall x \in [0, 1]$.
2. $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^2$, is not bounded on $\mathbb{R}$ Indeed, suppose $\exists M > 0$ such that $|f(x)| < M$ for all $x \in \mathbb{R}$. Then $|x^2| = |x|^2 < M$ for all $x \in \mathbb{R}$, i.e. $|x| < \sqrt{M}$ for all $x \in \mathbb{R} \implies M$ is an upper bound of $\mathbb{R}$. But there is no such bound, $\therefore g$ is not bounded on $\mathbb{R}$.
3. $f : (0, 1) \to \mathbb{R}$, $f(x) = \frac{1}{x}$, is not bounded on $(0, 1]$, but it is bounded on $[a, 1]$ for all $a \in (0, 1]$. □

There is a link between continuity and boundedness.

Theorem 32
If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, then f is bounded on $[a, b]$.

Proof: suppose f is not bounded on $[a, b]$. Hence, for all $n \in \mathbb{N}$, $\exists x_n \in [a, b]$ such that $|f(x_n)| > n$. However, $(x_n) \subseteq [a, b]$ so that (x_n) is bounded.

According to Bolzano-Weierstrass, $\exists (x_{n_k}) \subseteq (x_n)$ such that $x_{n_k} \to \hat{x} \in [a, b]$, since

$$a \leq x_{n_k} \leq b \quad \text{for all } k.$$

Since f is continuous, we have

$$f(\hat{x}) = \lim_{x \to \hat{x}} f(x) = \lim_{k \to \infty} f(x_{n_k}),$$

so $(f(x_{n_k}))$ is bounded, being a convergent sequence. But this contradicts the assumption that $|f(x_{n_k})| > n_k \geq k$ for all k. Hence f is bounded on $[a, b]$. ■

Continuous functions on closed, bounded sets have a useful property. Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$. We say that f **reaches a global maximum on** A if $\exists x^* \in A$ such that $f(x^*) \geq f(x)$ for all $x \in A$. Similarly, f **reaches a global minimum on** A if $\exists x_* \in A$ such that $f(x_*) \leq f(x)$ for all $x \in A$.

Theorem 33 (Max/Min Theorem)
If $f : [a, b] \to \mathbb{R}$ is continuous, then f reaches a global maximum and a global minimum of $[a, b]$.

Proof: let $f([a,b]) = \{f(x) \mid x \in [a,b]\}$. According to Theorem 32, $f([a,b])$ is bounded as f is continous, and so, by completeness of $\mathbb{R}$,

$$s^* = \sup\{f(x) \mid x \in [a,b]\} \quad \text{and} \quad s_* = \inf\{f(x) \mid x \in [a,b]\}$$

both exist. We need only show $\exists x^*, x_* \in [a,b]$ such that $f(x^*) = s^*$ and $f(x_*) = s_*$.

Since $s^* - \frac{1}{n}$ is not an upper bound of $f([a,b])$ for every $n \in \mathbb{N}$, $\exists x_n \in [a,b]$ with

$$s^* - \frac{1}{n} < f(x_n) \leq s^*, \quad \text{for all } n \in \mathbb{N}.$$

According to the squeeze theorem, we must have $f(x_n) \to s^*$ (this says nothing about whether x_n converges or not, however).

But $(x_n) \subseteq [a,b]$ is bounded, so applying the Bolzano-Weierstrass theorem, we find that $\exists (x_{n_k}) \subseteq (x_n)$ such that $x_{n_k} \to x^* \in [a,b]$. As f is continuous,

$$s^* = \lim_{k\to\infty} f(x_{n_k}) = f\Big(\lim_{k\to\infty} x_{n_k}\Big) = f(x^*).$$

The existence of $x_* \in [a,b]$ such that $f(x_*) = s_*$ is shown similarly. ■

Let's take a look at some examples.

Examples

1. The function $f : [0,1] \to \mathbb{R}$, $f(x) = x^2$, reaches its maximum and minimum on $[0,1]$ since f is continuous, being a polynomial.

2. Let $f : [0,1) \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} 2, & x \in (0,1) \\ 3, & x = 0 \end{cases}$$

   The function f is not continuous on $[0,1)$, and $[0,1)$ is not closed and bounded, so we cannot use the max/min theorem to conclude that f reaches its global max/min on $[0,1)$... even though it does: 3 at $x^* = 0$ and 2 at any $x_* \in (0,1)$.[2]

3. The function $f : [a,1] \to \mathbb{R}$, $a \in (0,1]$, defined by $f(x) = \frac{1}{x}$ reaches its global max/global min on $[a,1]$ as f is continuous on $[a,1]$, being rational there.

4. The function $f : (0,1] \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is continuous on $(0,1]$, but we cannot use the max/min theorem as $(0,1]$ is not closed. In this case, f has no global maximum, but it does have a global minimum at $x = 1$. □

3.5 Intermediate Value Theorem

The following result has many applications; notably it can help locate the roots of a function.

Theorem 34
Let $f : [a, b] \to \mathbb{R}$ be continuous. If $\exists \alpha, \beta \in [a, b]$ such that $f(\alpha)f(\beta) < 0$, then $\exists \gamma \in (a, b)$ such that $f(\gamma) = 0$.

Proof: we prove the result for $f(\alpha) < 0 < f(\beta)$; the other case is similar. Write $\alpha_1 = \alpha$, $\beta_1 = \beta$, $I_1 = [\alpha_1, \beta_1]$, and $\gamma_1 = \frac{\alpha_1+\beta_1}{2}$. There are 3 possibilities:

i. if $f(\gamma_1) = 0$, set $\gamma = \gamma_1$; then $\gamma \in (\alpha_1, \beta_1)$ and the theorem is proven;
ii. if $f(\gamma_1) > 0$, set $\alpha_2 = \alpha_1$, $\beta_2 = \gamma_1$;
iii. if $f(\gamma_1) < 0$, set $\alpha_2 = \gamma_1$, $\beta_2 = \beta_1$.

In the last two cases, set $I_2 = [\alpha_2, \beta_2]$. Then $I_1 \supseteq I_2$, length$(I_1) = \frac{\beta_1-\alpha_1}{2^0}$ and

$$f(\alpha_2) < 0 < f(\beta_2).$$

This is the base case $n = 1$ of an induction process, which can be extended for all $n \in \mathbb{N}$. Either one of two things can occur:

1. $\exists n \in \mathbb{N}$ such that $f(\gamma_n) = 0$, with $\gamma_n \in (\alpha_n, \beta_n) \subseteq (\alpha, \beta)$, in which case the theorem is proven, or

2. there is a chain of nested intervals

$$I_1 \supseteq I_2 \supseteq \cdots I_k \supseteq I_{k+1} \supseteq \cdots$$

where $I_n = [\alpha_n, \beta_n]$, length$(I_n) = \frac{\beta_n-\alpha_n}{2^{n-1}}$, $f(\alpha_n) < 0 < f(\beta_n)\ \forall n \in \mathbb{N}$.

According to the nested intervals theorem, since

$$\inf_{n\in\mathbb{N}}\{\text{length}(I_n)\} = \lim_{n\to\infty} \frac{\beta_n - \alpha_n}{2^{n-1}} = 0,$$

$\exists c \in [\alpha, \beta] \subseteq [a, b]$ such that $\bigcap_{n\in\mathbb{N}} I_n = \{c\}$.

It remains to show that $f(c) = 0$. Note that the sequences $(\alpha_n), (\beta_n)$ both converge to c. Indeed, let $\varepsilon > 0$. By the Archimedean property, $\exists N_\varepsilon \in \mathbb{N}$ such that $N_\varepsilon > \log_2(\frac{\beta-\alpha}{\varepsilon}) + 1$.

Since $c \in I_n$ for all $n \in \mathbb{N}$, then $|\alpha_n - c| <$ length$(I_n) = \frac{\beta-\alpha}{2^{n-1}} < \varepsilon$ whenever $n > N_\varepsilon$. The proof that $\beta_n \to c$ is identical.

Since f is continuous on $[a,b]$, it is also continuous at c. Thus,

$$\lim_{n\to\infty} f(\alpha_n) = \lim_{n\to\infty} f(\beta_n) = f(c).$$

But $f(\alpha_n) < 0$ for all n, so, Theorem 15:

$$f(c) = \lim_{n\to\infty} f(\alpha_n) \leq 0.$$

Using the same Theorem, we have $f(c) \geq 0$. Then $f(c) = 0$. Lastly, note that $c \neq \alpha, \beta$; otherwise, $f(\alpha)f(\beta) = 0$.

This concludes the proof, with $\gamma = c$. ■

We can use the result to revisit a corollary from Chapter 1.

Example: Show that $\exists x \in \mathbb{R}^+$ such that $x^2 = 2$.

Proof: the function $f : [0,2] \to \mathbb{R}$ defined by $f(x) = x^2 - 2$ is continuous on $[0,2]$. As $f(0) = 0^2 - 2 = -2 < 0$ and $f(2) = 2^2 - 2 = 2 > 0$, $\exists \gamma \in (0,2)$ such that $\gamma^2 - 2 = 0$, so $\gamma^2 = 2$, according to Theorem 34. ■

This result easily generalizes to the following.

Theorem 35 (INTERMEDIATE VALUE THEOREM)
Let $f : [a,b] \to \mathbb{R}$ be continuous. If $\exists \alpha < \beta \in [a,b]$ s.t. $f(\alpha) < k < f(\beta)$ or $f(\alpha) > k > f(\beta)$, then $\exists \gamma \in (a,b)$ such that $f(\gamma) = k$.

Proof: assume that $f(\alpha) < k < f(\beta)$; the proof for the other case is similar. Consider the function $g : [a,b] \to \mathbb{R}$ defined by $g(x) = f(x) - k$. Theorem 30 shows that g is continuous on $[a,b]$. Furthermore,

$$g(\alpha) = f(\alpha) - k < k - k = 0 < f(\beta) - k = g(\beta).$$

By Theorem 34, $\exists \gamma \in (\alpha, \beta)$ such that $g(\gamma) = f(\gamma) - k = 0$. Thus $f(\gamma) = k$. ■

The following result combines the max/min and the intermediate value theorems.

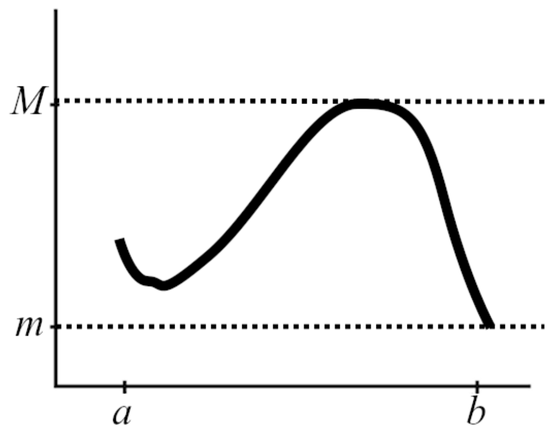
Theorem 36
If $f : [a,b] \to \mathbb{R}$ is continuous, then $f([a,b])$ is a closed and bounded interval.

Proof: Let $m = \inf\{f[a,b]\}$ and $M = \sup\{f[a,b]\}$. According to the max/min theorem, $\exists \alpha, \beta \in [a,b]$ such that $f(\alpha) = m$ and $f(\beta) = M$.

If $m = M$, then f is constant and $f([a,b]) = [m,m] = [M,M]$. If $m < M$, then $\alpha \neq \beta$. Furthermore, $m \leq f(x) \leq M$ for all $x \in [a,b]$, so that $f([a,b]) \subseteq [m,M]$.

Now, let $k \in [m,M]$. According to the intermediate value theorem, $\exists \gamma$ between α and β such that $f(\gamma) = k$. Hence $k \in f([a,b])$ and so $[m,M] \subseteq f([a,b])$. Consequently, $f([a,b]) = [m,M]$. ■

The image of any interval by a continuous function is always an interval, but the only time that we know for a fact that image is of the same type as the original is when the original is closed and bounded.



Examples

1. Let $f : [0,1] \to \mathbb{R}$, $f(x) = 2x - 1$. Then $f([0,1])$ is closed and bounded (in fact, $f([0,1]) = [-1,1]$, but the endpoints of $f([-1,1])$ are not provided by Theorem 36).

2. The function $f : (0,2\pi) \to \mathbb{R}$ defined by $f(x) = \sin x$ is continuous and $f((0,2\pi)) = [-1,1]$, but Theorem 36 does not apply.

3.6 Uniform Continuity

If $f : A \to \mathbb{R}$ is continuous (on A), then for $\varepsilon > 0$ and $c \in A$, the $\delta_\varepsilon > 0$ that is used to show continuity of f at c generally depends on ε **and** on c. But there might be instances when δ_ε depends only on ε.

The function f is **uniformly continuous** on A if

$$x, y \in A \text{ and } |x-y| < \delta_\varepsilon \Longrightarrow |f(x) - f(y)| < \varepsilon.$$

The notion of uniform continuity is more restrictive than that of (simple) continuity.

Theorem 37
If $f : A \to \mathbb{R}$ is uniformly continuous on A, then f is continuous on A.

Proof: let $c \in A$ and $\varepsilon > 0$. As f is uniformly continuous on A, $\exists \delta_\varepsilon > 0$ such that

$$|f(x) - f(y)| < \varepsilon \quad \text{whenever } |x - y| < \delta_\varepsilon \text{ and } x, y \in A.$$

In particular, if $y = c$ then

$$|f(x) - f(c)| < \varepsilon \quad \text{whenever } |x - c| < \delta_\varepsilon \text{ and } x \in A.$$

As c is arbitrary, f is continuous on A. ■

The converse of Theorem 37 is false, as the following example shows.

Example: show that $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is continuous on $(0, \infty)$ but not uniformly continuous on $(0, \infty)$.

Proof: that f is continuous on $(0, \infty)$ is immediate, as it is a rational function. Let $(x_n) = (\frac{1}{n}) \subseteq (0, \infty)$. Clearly, (x_n) is a Cauchy sequence as it is a convergent sequence. But $f(x_n) = \frac{1}{1/n} = n$ for all $n \in \mathbb{N}$, so $(f(x_n))$ is not a Cauchy sequence in $\mathbb{R}$ (as it is not bounded, and thus divergent).

According to a lemma that we will prove next, f cannot be uniformly continuous on $(0, \infty)$. ■

In a sense, continuity only requires that there be no "holes" in the function; uniform continuity requires that the combination of domain and rule plays "nicely".

Lemma: if f is uniformly continuous on A and $(x_n) \subseteq A$ is a Cauchy sequence, then $f(x_n)$ is a Cauchy sequence.

Proof: if $(x_n) \subseteq A$ is a Cauchy sequence and $\delta > 0$, $\exists N_\delta \in \mathbb{N}$ such that $|x_m - x_n| < \delta$ whenever $m, n > N_\delta$.

But f is uniformly continuous on A, so that $\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$ such that

$$x, y \in A \text{ and } |x - y| < \delta_\varepsilon \Longrightarrow |f(x) - f(y)| < \varepsilon.$$

Combining these two statements, with $N_\varepsilon = M_{\delta_\varepsilon}$, yields

$$m, n > N_\varepsilon \Longrightarrow |x_m - x_n| < \delta_\varepsilon \Longrightarrow |f(x_m) - f(x_n)| < \varepsilon,$$

and so $(f(x_n))$ is a Cauchy sequence. ■

While continuous functions are not generally uniformly continuous, there is a specific class of functions for which continuity is equivalent to uniform continuity.

Theorem 38
Let $f : [a,b] \to \mathbb{R}$. Then f is uniformly continuous on $[a,b]$ if it is continuous on $[a,b]$.

Proof: this is the converse of Theorem 37. Assume f is continuous on $[a,b]$. If f is not uniformly continuous, then $\exists \varepsilon_0 > 0$ such that $\forall \delta > 0$, $\exists x_\delta, y_\delta \in [a,b]$ with

$$|f(x_\delta) - f(y_\delta)| \geq \varepsilon_0 \text{ and } |x_\delta - y_\delta| < \delta.$$

For $n \in \mathbb{N}$, let $\delta_n = \frac{1}{n}$. The corresponding sequences $(x_{\delta_n}), (y_{\delta_n})$ lie in $[a,b]$, with

$$|x_{\delta_n} - y_{\delta_n}| < \delta_n = \frac{1}{n} \quad \text{and} \quad |f(x_{\delta_n}) - f(y_{\delta_n})| \geq \varepsilon_0, \quad \forall n \in \mathbb{N}.$$

As (x_{δ_n}) is bounded, $\exists (x_{\delta_{n_k}}) \subseteq (x_{\delta_n})$ such that $x_{\delta_{n_k}} \to z$ with $k \to \infty$, according to the Bolazano-Weierstrass theorem.

Furthermore, $z \in [a,b]$ according to Theorem 15. The corresponding sequence $(y_{\delta_{n_k}})$ also converges to z since

$$0 \leq |y_{\delta_{n_k}} - z| \leq |y_{\delta_{n_k}} - x_{\delta_{n_k}}| + |x_{\delta_{n_k}} - z| < \frac{1}{n_k} + |x_{\delta_{n_k}} - z|$$

according to the squeeze theorem, as both $\frac{1}{n_k}, |x_{\delta_{n_k}} - z| \to 0$ with $k \to \infty$. But f is continuous, so both $(f(x_{\delta_{n_k}})), (f(y_{\delta_{n_k}})) \to f(z)$, which is impossible as we have $|f(x_{\delta_n}) - f(y_{\delta_n})| \geq \varepsilon_0, \forall n \in \mathbb{N}$. Thus f must be uniformly continuous. ■

There is something "special" about the interval $[a,b]$ that allows for all sorts of interesting results when combined with continuous functions; as we shall see in Chapters 8, 9, 16-17.

Example: show $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{1+x^2}$ is uniformly continuous on $(0,1)$.

Proof: let $\varepsilon > 0$. Set $\delta_\varepsilon = \varepsilon$. Note that $\forall z \in \mathbb{R}$, $0 \leq (|z|-1)^2 = z^2 - 2|z| + 1 \Longrightarrow 2|z| \leq 1 + z^2 \Longrightarrow |\frac{z}{1+z^2}| \leq 1/2$. Then whenever $|x-y| < \delta_\varepsilon$, we have:

$$\begin{aligned} |f(x) - f(y)| &= \left|\tfrac{1}{1+x^2} - \tfrac{1}{1+y^2}\right| = \left|\tfrac{y^2-x^2}{(1+x^2)(1+y^2)}\right| = \left|\tfrac{x+y}{(1+x^2)(1+y^2)}\right| |x-y| \\ &\leq \left(\left|\tfrac{y}{1+y^2}\right| \cdot \underbrace{\tfrac{1}{1+x^2}}_{\leq 1} + \left|\tfrac{x}{1+x^2}\right| \cdot \underbrace{\tfrac{1}{1+y^2}}_{\leq 1}\right)|x-y| \\ &\leq \left(\underbrace{\left|\tfrac{y}{1+y^2}\right|}_{\leq 1/2} + \underbrace{\left|\tfrac{x}{1+x^2}\right|}_{\leq 1/2}\right)|x-y| \leq |x-y| < \delta_\varepsilon = \varepsilon, \end{aligned}$$ ■

3.7 Solved Problems

1. Show $\lim_{x\to c} x^3 = c^3$ for any $c \in \mathbb{R}$.

   Proof: if $|x-c|<1$, then $|x|<|c|+1$. Let $\varepsilon>0$ and set $\delta_\varepsilon = \min\{1, \frac{\varepsilon}{3|c|^2+3|c|+1}\}$. Then

$$\begin{aligned}|x^3-c^3| &= |x-c||x^2+cx+c^2| \leq |x-c|\left(|x|^2+|c||x|+|c|^2\right)\\ &< |x-c|\left((|c|+1)^2+|c|(|c|+1)+|c|^2\right)\\ &= |x-c|\left(3|c|^2+3|c|+1\right)\\ &< \delta_\varepsilon\cdot\left(3|c|^2+3|c|+1\right) \leq \frac{\varepsilon}{3|c|^2+3|c|+1}\cdot\left(3|c|^2+3|c|+1\right)=\varepsilon,\end{aligned}$$

   whenever $0<|x-c|<\delta_\varepsilon$ and $x\in\mathbb{R}$. ■

2. Let $f:\mathbb{R}\to\mathbb{R}$ and let $c\in\mathbb{R}$. Show that $\lim_{x\to c} f(x)=L$ if and only if $\lim_{x\to 0} f(x+c)=L$.

   Proof: we have

$$\lim_{x\to c} f(x) = L$$
$$\Updownarrow$$
$$\forall\varepsilon>0, \exists\delta_\varepsilon>0 \text{ s.t. } |f(x)-L|<\varepsilon \text{ when } 0<|x-c|<\delta_\varepsilon$$
$$\Updownarrow$$
$$\text{Set } x=y+c:\ \forall\varepsilon>0, \exists\delta_\varepsilon>0 \text{ s.t. } |f(y+c)-L|<\varepsilon \text{ when } 0<|y|<\delta_\varepsilon$$
$$\Updownarrow$$
$$\forall\varepsilon>0, \exists\delta_\varepsilon>0 \text{ s.t. } |f(y+c)-L|<\varepsilon \text{ when } 0<|y-0|<\delta_\varepsilon$$
$$\Updownarrow$$
$$\lim_{y\to 0} f(y+c) = L,$$

   which completes the proof. ■

3. Use either the $\varepsilon-\delta$ definition of the limit or the sequential criterion for limits to establish the following limits:

   a) $\lim_{x\to 2}\frac{1}{1-x}=-1$;

   b) $\lim_{x\to 1}\frac{x}{1+x}=\frac{1}{2}$;

   c) $\lim_{x\to 0}\frac{x^2}{|x|}=0$, and

   d) $\lim_{x\to 1}\frac{x^2-x+1}{x+1}=\frac{1}{2}$

Proof:

a) Let $\varepsilon > 0$ and set $\delta_\varepsilon = \min\{\frac{1}{2}, \frac{\varepsilon}{2}\}$. Then

$$0 < |x-2| < \delta_\varepsilon \Longrightarrow |x-2| < \frac{1}{2} \Longleftrightarrow \frac{3}{2} < x < \frac{5}{2}$$
$$\Longleftrightarrow \frac{1}{2} < x-1 < \frac{3}{2} \Longleftrightarrow \frac{1}{x-1} < 2.$$

Thus

$$\left|\frac{1}{1-x} - (-1)\right| = \frac{1}{|x-1|}|x-2| = \frac{1}{x-1}|x-2| < 2\delta_\varepsilon < \varepsilon$$

whenever $0 < |x-2| < \delta_\varepsilon$ and $x \in \mathbb{R}$. (Note that if $0 < |x-2| < \delta_\varepsilon$, we've seen that $x > \frac{3}{2}$ and so that $|x-1| = x-1$. This explains why we have gotten rid of the absolute values above.)

b) Let $\varepsilon > 0$ and set $\delta_\varepsilon = \min\{\frac{1}{2}, 3\varepsilon\}$. Then

$$0 < |x-1| < \delta_\varepsilon \Longrightarrow |x-1| < \frac{1}{2} \Longleftrightarrow \frac{1}{2} < x < \frac{3}{2}$$
$$\Longleftrightarrow 3 < 2(x+1) < 5 \Longleftrightarrow \frac{1}{2(x+1)} < \frac{1}{3}.$$

Thus

$$\left|\frac{x}{1+x} - \frac{1}{2}\right| = \frac{1}{2|x+1|}|x-1| = \frac{1}{2(x+1)}|x-1| < \frac{1}{3}\delta_\varepsilon < \varepsilon$$

whenever $0 < |x-1| < \delta_\varepsilon$ and $x \in \mathbb{R}$. (Note that if $0 < |x-1| < \delta_\varepsilon$, we've seen that $2(x+1) > 3$ and so that $2|x+1| = 2(x+1)$. This explains why we have gotten rid of the absolute values above.)

c) Let $(x_n) \subseteq \mathbb{R}$ be a sequence s.t. $x_n \to 0$ and $x_n \neq 0$ for all n. Then

$$\frac{x_n^2}{|x_n|} = \frac{|x_n|^2}{|x_n|} = |x_n| \to 0,$$

by theorem 14. By the sequence squeeze theorem, the limit must be thus 0.

d) Let $\varepsilon > 0$ and set $\delta_\varepsilon = \min\{\frac{1}{2}, \frac{3}{2}\varepsilon\}$. Then

$$0 < |x-1| < \delta_\varepsilon \Longrightarrow |2x-1| < 2 \text{ and } \left|\frac{1}{2(x+1)}\right| < \frac{1}{3}.$$

Thus ,whenever $0 < |x-1| < \delta_\varepsilon$ and $x \in \mathbb{R}$, we have

$$\left|\frac{x^2-x+1}{x+1} - \frac{1}{2}\right| = \left|\frac{2x-1}{2(x+1)}\right||x-1| < \frac{2}{3}|x-1| < \frac{2}{3}\delta_\varepsilon < \varepsilon.$$

This completes the exercise. ■

4. Show that the following limits do not exist:

a) $\lim\limits_{x\to 0} \frac{1}{x^2}$, with $x > 0$;

b) $\lim_{x\to 0} \frac{1}{\sqrt{x}}$, with $x > 0$;

c) $\lim_{x\to 0}(x + \text{sgn}(x))$, and

d) $\lim_{x\to 0} \sin(1/x^2)$, with $x > 0$.

Solution: in each instance, we only give some sequence(s) for which Theorem 26 shows the limit does not exist.

a) $x_n = \frac{1}{n} \to 0$, but $f(x_n) = \frac{1}{1/n^2} = n^2 \to \infty$.

b) $x_n = \frac{1}{n} \to 0$, but $f(x_n) = \frac{1}{1/\sqrt{n}} = \sqrt{n} \to \infty$.

c) $x_n = \frac{1}{n}, y_n = -\frac{1}{n} \to 0$, but $f(x_n) = \frac{1}{n} + 1 \to 1$, $f(y_n) = -\frac{1}{n} - 1 \to -1$.

d) $x_n = \sqrt{\frac{2}{(4n+1)\pi}}, y_n = \sqrt{\frac{2}{(4n+3)\pi}} \to 0$ but

$$f(x_n) = \sin\left(\frac{4n+1}{2}\pi\right) \to 1,\ f(y_n) = \sin\left(\frac{4n+3}{2}\pi\right) \to -1.$$

This completes the exercise. □

5. Let $c \in \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be such that $\lim_{x\to c} (f(x))^2 = L$. Show that if $L = 0$, then $\lim_{x\to c} f(x) = 0$. Show that if $L \neq 0$, then f may not have a limit at c.

Proof: if $\lim_{x\to c} (f(x))^2 = 0$ then $\forall \eta > 0, \exists \delta_\eta > 0$ such that

$$|f(x)|^2 = \left|(f(x))^2 - 0\right| < \eta$$

whenever $0 < |x - c| < \delta_\eta$. Let $\varepsilon > 0$.

By definition of the real numbers, $\exists \eta_\varepsilon > 0$ such that $\varepsilon = \sqrt{\eta_\varepsilon}$. Set $\delta_\varepsilon = \delta_{\eta_\varepsilon}$. Then

$$|f(x) - 0| = |f(x)| = \sqrt{|f(x)|^2} < \sqrt{\eta_\varepsilon} = \varepsilon$$

whenever $0 < |x - c| < \delta_\varepsilon$.

Now, consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}.$$

Then $(f(x))^2 \equiv 1$ and

$$\lim_{x\to 0} (f(x))^2 = \lim_{x\to 0} 1 = 1.$$

But $\lim_{x\to 0} f(x)$ does not exist since $(x_n) = (\frac{1}{n}), (y_n) = (-\frac{1}{n})$ are sequences such that $x_n, y_n \to 0$, $x_n, y_n \neq 0$ for all n and

$$f(x_n) = -1 \to -1 \neq 1 \leftarrow 1 = f(y_n).$$

This completes the proof. ■

6. Let $f : \mathbb{R} \to \mathbb{R}$, let J be a closed interval in $\mathbb{R}$ and let $c \in J$. If f_2 is the restriction of f to J, show that if f has a limit at c then f_2 has a limit at c. Show the converse is not necessarily true.

Proof: suppose $\lim_{x\to c} f(x) = L$ exists. Then, $\forall \varepsilon > 0, \exists \delta_\varepsilon > 0$ s.t. $|f(x) - L| < \varepsilon$ whenever $0 < |x - c| < \delta_\varepsilon$. But $f_2(x) = f(x)$ for all $x \in J \subseteq \mathbb{R}$, so $\forall \varepsilon > 0, \exists \delta_\varepsilon > 0$ (exactly as above) s.t. $|f_2(x) - L| = |f(x) - L| < \varepsilon$ whenever $0 < |x - c| < \delta_\varepsilon$ and $x \in J$, and so $\lim_{x\to c} f_2(x) = L$.

Now consider $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in (-\infty, 0) \cup (1, \infty) \\ 1 & \text{if } x \in [0, 1] \end{cases},$$

with $J = [0, 1]$ and $f_2 = f|_J$. Then $\lim_{x\to 1} f_2(x) = 1$ but $\lim_{x\to 1} f(x)$ does not exist. ■

7. Determine the following limits and state which theorems are used in each case.

a) $\lim_{x\to 2} \sqrt{\frac{2x+1}{x+3}}$, $(x > 0)$;

b) $\lim_{x\to 2} \frac{x^2-4}{x-2}$, $(x > 0)$;

c) $\lim_{x\to 0} \sqrt{\frac{(x+1)^2-1}{x}}$, $(x > 0)$, and

d) $\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1}$, $(x > 0)$.

Solution: We will do c) in its entirety and only give the answers to the others.

Consider the sequence $(x_n) = (\frac{1}{n})$. Then $x_n \to 0$, $x_n \neq 0\ \forall n \in \mathbb{N}$, and

$$\frac{(x_n+1)^2-1}{x_n} = \frac{\left(\frac{1}{n}+1\right)^2-1}{\frac{1}{n}} = \frac{1}{n} + 2 \to 2.$$

Hence, if $\lim_{x\to 0} \frac{(x+1)^2-1}{x}$ exists, its value must be 2, by Theorem 26.

Let $\varepsilon > 0$. Set $\delta_\varepsilon = \varepsilon$. Then when $0 < |x - 0| < \delta_\varepsilon$ and $x > 0$, we have

$$\left|\frac{(x+1)^2-1}{x} - 2\right| = \left|\frac{x^2+2x+1-1-2x}{x}\right| = \left|\frac{x^2}{x}\right| = |x| = |x-0| < \delta_\varepsilon = \varepsilon.$$

a) 1 b) 4 d) $\frac{1}{2}$ □

8. Give examples of functions f and g such that f and g do not have limits at point c, but both $f+g$ and fg have limits at c.

Solution: Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

and $g(x) = -f(x)$ for all $x \in \mathbb{R}$. Then $f(x) + g(x) \equiv 0$ and $f(x)g(x) \equiv -1$. As a result,

$$\lim_{x\to 0}(f+g)(x) = 0 \quad \text{and} \lim_{x\to 0}(fg)(x) = -1,$$

but the limits of f and g don't exist at 0 (see solved problem 5). □

9. Determine whether the following limits exist in $\mathbb{R}$:

a) $\lim\limits_{x\to 0} \sin\left(\frac{1}{x^2}\right)$, with $x \neq 0$;

b) $\lim\limits_{x\to 0} x\sin\left(\frac{1}{x^2}\right)$, with $x \neq 0$;

c) $\lim\limits_{x\to 0} \text{sgn}\sin\left(\frac{1}{x}\right)$, with $x \neq 0$, and

d) $\lim\limits_{x\to 0} \sqrt{x}\sin\left(\frac{1}{x^2}\right)$, with $x > 0$.

Solution:

a) Let $(x_n) = (\frac{1}{\sqrt{n\pi}})$ and $(y_n) = (\sqrt{\frac{2}{(4n+1)\pi}})$ for all $n \in \mathbb{N}$. Then $x_n, y_n \to 0$ and $x_n, y_n \neq 0$ for all $n \in \mathbb{N}$. But

$$\sin\left(\frac{1}{x_n^2}\right) = \sin(n\pi) = 0 \quad \text{and} \quad \sin\left(\frac{1}{y_n^2}\right) = \sin\left(\frac{(4n+1)\pi}{2}\right) = 1$$

for all $n \in \mathbb{N}$.

Then $\sin(1/x_n^2) \to 0$ and $\sin(1/y_n^2) \to 1$. As $0 \neq 1$, $\lim\limits_{x\to 0} \sin\left(\frac{1}{x^2}\right)$ doesn't exist.

b) Consider the sequence $(x_n) = (\frac{1}{\sqrt{n\pi}})$. Then $x_n \to 0$ and $x_n \neq 0$ for all $n \in \mathbb{N}$. Furthermore,

$$x_n \sin\left(\frac{1}{x_n^2}\right) = \frac{1}{\sqrt{n\pi}}\sin(n\pi) = \frac{1}{\sqrt{n\pi}} \cdot 0 \to 0.$$

As a result, if $\lim\limits_{x\to 0} x\sin\left(\frac{1}{x^2}\right)$ exists, it must take the value 0. Let $\varepsilon > 0$. Set $\delta_\varepsilon = \varepsilon$. Then

$$\left| x\sin\left(\frac{1}{x^2}\right) - 0 \right| = |x|\left|\sin\left(\frac{1}{x^2}\right)\right| \leq |x| = |x-0| < \delta_\varepsilon = \varepsilon$$

whenever $0 < |x-0| < \delta_\varepsilon$ and $x > 0$. Hence $\lim\limits_{x\to 0} x\sin\left(\frac{1}{x^2}\right) = 0$.

c) Let $(x_n) = \left(\frac{2}{(2n+1)\pi}\right)$. Then $x_n \to 0$, $x_n \neq 0$ for all $n \in \mathbb{N}$ and

$$\text{sgn}\left(\sin\left(\frac{1}{x_n}\right)\right) = \text{sgn}\left((-1)^n\right) = (-1)^n,$$

which does not converge. Hence $\lim_{x\to 0} \text{sgn}\left(\sin\left(\frac{1}{x}\right)\right)$ does not exist.

d) $\lim_{x\to 0} \sqrt{x}\sin\left(\frac{1}{x^2}\right) = 0$, with the same proof as b), save for $\delta_\varepsilon = \varepsilon^2$. □

10. Let $f : \mathbb{R} \to \mathbb{R}$ be s.t. $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Assume $\lim_{x\to 0} f(x) = L$ exists. Prove that $L = 0$ and that f has a limit at every point $c \in \mathbb{R}$.

    Proof: as f is additive, we have $f(2x) = f(x+x) = f(x) + f(x) = 2f(x)$, so that

    $$L = \lim_{y\to 0} f(y) = \lim_{2x\to 0} f(2x) = \lim_{x\to 0} f(2x) = \lim_{x\to 0} 2f(x) = 2\lim_{x\to 0} f(x) = 2L;$$

    hence $L = 2L$ and $L = 0$, i.e., $\lim_{x\to 0} f(x) = 0$.

    Now, let $c \in \mathbb{R}$. Then

    $$\begin{aligned}\lim_{x\to c} f(x) &= \lim_{x\to c}\left(f(x-c) + f(c)\right) = \lim_{x\to c} f(x-c) + \lim_{x\to c} f(c)\\ &= \lim_{y\to 0} f(y) + f(c) = 0 + f(c) = f(c).\end{aligned}$$

    As f is defined on all of $\mathbb{R}$, $f(c)$ exists for all $c \in \mathbb{R}$, and so $\lim_{x\to c} f(x) = f(c)$ exists for all $c \in \mathbb{R}$. ■

11. Let $K > 0$ and let $f : \mathbb{R} \to \mathbb{R}$ satisfy the condition

    $$|f(x) - f(y)| \leq K|x-y|$$

    for all $x, y \in \mathbb{R}$. Show that f is continuous on $\mathbb{R}$.

    Proof: let $c \in \mathbb{R}$ and $\varepsilon > 0$. Set $\delta_\varepsilon = \frac{\varepsilon}{K}$. Then

    $$|f(x) - f(c)| \leq K|x-c| < K\delta_\varepsilon < K\frac{\varepsilon}{K} = \varepsilon$$

    whenever $|x-c| < \delta_\varepsilon$. ■

12. Let $f : (0,1) \to \mathbb{R}$ be bounded and s.t. $\lim_{x\to 0} f(x)$ does not exist. Show that there are two convergent sequences $(x_n), (y_n) \subseteq (0,1)$ with $x_n, y_n \to 0$ and $f(x_n) \to \xi$, $f(y_n) \to \zeta$, but $\xi \neq \zeta$.

    Proof: for $n \in \mathbb{N}$, let $I_n = (0, 1/n)$ and set

    $$s_n = \sup f(I_n) \quad \text{and} \quad t_n = \inf f(I_n).$$

    These are well-defined as $f(I_n)$ is bounded.

By construction, (s_n) is decreasing and (t_n) is increasing. Since

$$s_1 \geq s_n = \sup f(I_n) \geq \inf f(I_n) = t_n \geq t_1,$$

(s_n) is bounded below by t_1 and (t_n) is bounded above by s_1. Hence $s_n \to s$ and $t_n \to t$ exist, by the bounded monotone convergence theorem.

For $n \in \mathbb{N}$, let $x_n, y_n \in I_n$ be s.t.

$$|f(x_n) - s_n| < \frac{1}{n} \quad \text{and} \quad |f(y_n) - t_n| < \frac{1}{n}.$$

This can always be done as $s_n - \frac{1}{n}$ and $t_n + \frac{1}{n}$ are not the supremum and the infimum, respectively, of $f(I_n)$. Then, $x_n, y_n \to 0$ and $x_n, y_n \neq 0$ for all $n \in \mathbb{N}$.

Furthermore, $f(x_n) \to s$ and $f(y_n) \to t$ according to the sequence squeeze theorem; indeed, $s_n - \frac{1}{n} < f(x_n) \leq s_n$, $t_n \leq f(y_n) < t_n + \frac{1}{n}$, $s_n \to s$, and $t_n \to t$, and the statement follows.

Now, suppose that $s = t = \ell$. Then $s_n, t_n \to \ell$. Let $\varepsilon > 0$. $\exists N_1, N_2 \in \mathbb{N}$ s.t. $|s_n - \ell| < \varepsilon$ whenever $n > N_1$ and $|t_n - \ell| < \varepsilon$ whenever $n > N_2$. Set $N_\varepsilon = \max\{N_1, N_2\}$. Then

$$\ell - \varepsilon < t_n \leq s_n < \ell - \varepsilon$$

whenever $n > N_\varepsilon$. Set $\delta_\varepsilon = \frac{1}{N_\varepsilon}$. Then

$$\ell - \varepsilon < t_{N_\varepsilon} = \inf f(I_{N_\varepsilon}) \leq f(x) \leq \sup f(I_{N_\varepsilon}) \leq s_{N_\varepsilon} < \ell + \varepsilon,$$

that is, $|f(x) - \ell| < \varepsilon$ whenever $0 < |x - 0| < \frac{1}{N_\varepsilon} = \delta_\varepsilon$. Hence $\lim_{x\to 0} f(x) = \ell$, which contradicts the hypothesis that the limit does not exist. Consequently, $s \neq t$, which completes the proof. ■

13. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ and let $P = \{x \in \mathbb{R} : f(x) > 0\}$. If $c \in P$, show that there exists a neighbourhood $V_\delta(c) \subseteq P$.

    Proof: let $c \in P$. Then $f(c) > 0$ and $\exists \varepsilon_0 > 0$ s.t. $f(c) - \varepsilon_0 > 0$. By the continuity of f, $\exists \delta_{\varepsilon_0}$ s.t. $|f(x) - f(c)| < \varepsilon_0$ whenever $|x - c| < \delta_{\varepsilon_0}$.

    Thus, $0 < f(c) - \varepsilon_0 < f(x)$ for all $x \in V_{\delta_{\varepsilon_0}}$, i.e. $V_{\delta_{\varepsilon_0}} \subseteq P$. ■

14. Prove that if an additive function is continuous at some point $c \in \mathbb{R}$, it is continuous on $\mathbb{R}$.

    Proof: in the light of a previous question on the topic, it is sufficient to show that if $\lim_{x\to c} f(x) = f(c)$ for some $c \in \mathbb{R}$, then $\lim_{x\to 0} f(x) = 0$. Let f be continuous at c. Then

$$\begin{aligned} f(c) = \lim_{x\to c} f(x) &= \lim_{x\to c} (f(x-c) + f(c)) \\ &= \lim_{x\to c} f(x-c) + \lim_{x\to c} f(c) = \lim_{y\to 0} f(y) + f(c), \end{aligned}$$

    hence $\lim_{y\to 0} f(y) = 0$, which completes the proof. ■

15. If f is a continuous additive function on $\mathbb{R}$, show that $f(x) = cx$ for all $x \in \mathbb{R}$, where $c = f(1)$.

Proof: let $n \in \mathbb{N}$. Then

$$f(1) = f\left(\frac{n}{n}\right) = f\left(\frac{1}{n} + \cdots + \frac{1}{n}\right) = f\left(\frac{1}{n}\right) + \cdots + f\left(\frac{1}{n}\right) = nf\left(\frac{1}{n}\right),$$

hence $\frac{1}{n}f(1) = f\left(\frac{1}{n}\right)$.

Set $c = f(1)$. Let $y \in \mathbb{Q}$. Then $y = \frac{m}{n}$, where $m \in \mathbb{Z}$ and $n \in \mathbb{N}^{\times}$, and

$$f(y) = f\left(\frac{m}{n}\right) = mf\left(\frac{1}{n}\right) = m\frac{1}{n}f(1) = yc.$$

Let $x \in \mathbb{R}$. Since x is a limit point of $\mathbb{Q}$, $\exists (x_n) \subseteq \mathbb{Q}$ s.t. $x_n \to x$, with $x_n \neq x$ for all $n \in \mathbb{N}$. But $f(x_n) \to f(x)$, by continuity, so $f(x_n) = cx_n \to cx$, by the above discussion. Hence, $f(x) = cx$. ■

16. Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be a continuous function on I s.t. $\forall x \in I$, $\exists y \in I$ s.t. $|f(y)| \leq \frac{1}{2}|f(x)|$. Show $\exists c \in I$ s.t. $f(c) = 0$.

Proof: let $x_1 \in I$. By hypothesis, $\exists x_2 \in I$ s.t.

$$|f(x_2)| \leq \frac{1}{2}|f(x_1)|.$$

Since $x_2 \in I$, $\exists x_3 \in I$ s.t.

$$|f(x_3)| \leq \frac{1}{2}|f(x_2)| \leq \frac{1}{2}\left(\frac{1}{2}|f(x_1)|\right) = \frac{1}{2^2}|f(x_1)|,$$

and so on. The sequence $(x_n) \subseteq I$ thusly built satistfies

$$0 \leq |f(x_n)| \leq \frac{1}{2^{n-1}}|f(x_1)|,$$

by induction (can you show this?).

Then $\lim_{n\to\infty} |f(x_n)| = 0$, by the squeeze theorem, and so $f(x_n) \to 0$. As (x_n) is bounded, it has a convergent subsequence (x_{n_k}) (according to the Bolzano-Weierstrass theorem) whose limit c is in I (because $a \leq x_n \leq b$ for all n).

Since $(f(x_{n_k}))$ is a subsequence of $(f(x_n))$, then

$$\lim_{k\to\infty} f(x_{n_k}) = 0.$$

However,

$$\lim_{k\to\infty} f(x_{n_k}) = f\left(\lim_{k\to\infty} x_{n_k}\right) = f(c),$$

as f is continuous. Hence $f(c) = 0$. ■

17. Show that every polynomial with odd degree has at least one real root.

Proof: let
$$f(x) = a_{2n+1}x^{2n+1} + a_{2n}x^{2n} + \cdots + a_1x + a_0,$$
where $a_i \in \mathbb{R}$ for $i = 0, \ldots, 2n+1$. Assume that $a_{2n} \neq 0$.[3] Let
$$M = \max\left\{(2n+1)\frac{|a_{2n}|}{|a_{2n+1}|}, \left(\frac{|a_{2n-k}|}{|a_{2n}|}\right)^{1/k}; k = 1, \ldots, 2n\right\}.$$
Then, whenever $|x| \geq M$,

- $|a_{2n}||x^{2n}| \geq |a_{2n}||x^{2n}|$;
- $|a_{2n}||x^{2n}| \geq |a_{2n-1}||x^{2n-1}|$;
- $\cdots$;
- $|a_{2n}||x^{2n}| \geq |a_1||x|$, and
- $|a_{2n}||x^{2n}| \geq |a_0|$,

and so
$$\begin{aligned}|a_{2n}x^{2n} + \cdots a_0| &\leq |a_{2n}||x^{2n}| + \cdots + |a_0| \leq |a_{2n}||x^{2n}| + \cdots + |a_{2n}||x^{2n}| \\ &= (2n+1)|a_{2n}||x^{2n}| \leq |a_{2n+1}||x^{2n+1}| = |a_{2n+1}x^{2n+1}|,\end{aligned}$$
from which we concude that $f(M+1)f(-M-1) < 0$.

As f is continuous on $[-M-1, M+1]$, $\exists c \in [-M-1, M+1]$ s.t. $f(c) = 0$, by the intermediate value theorem. ■

18. Let $f : [0,1] \to \mathbb{R}$ be continuous, with $f(0) = f(1)$. Show $\exists c \in [0, \frac{1}{2}]$ s.t. $f(c) = f(c + \frac{1}{2})$.

Proof: let $g : [0, \frac{1}{2}] \to \mathbb{R}$ be defined by $g(x) = f(x) - f(x + \frac{1}{2})$. By construction, g is continuous on $[0, \frac{1}{2}]$. If $g(0) = g(1/2) = 0$, there is nothing else to show. Otherwise,
$$g(0) = f(0) - f(1/2) \quad \text{and} \quad g(1/2) = f(1/2) - f(1) = f(1/2) - f(0);$$
hence $g(0)g(\frac{1}{2}) < 0$. By the intermediate value theorem, $\exists c \in [0, \frac{1}{2}]$ s.t. $g(c) = 0$, that is $f(c) - f(c + \frac{1}{2}) = 0$. This completes the proof. ■

19. Show that $f(x) = \frac{1}{x^2}$ is uniformly continuous on $A = [1, \infty)$, but not on $B = (0, \infty)$.

Proof: if $x, y \in A$, then $x, y \geq 1$. In particular, $|x| = x$ and $|y| = y$, and $\frac{1}{x^2y}, \frac{1}{xy^2} \leq 1$. Let $\varepsilon > 0$ and set $\delta_\varepsilon = \frac{\varepsilon}{2}$. Then
$$\begin{aligned}|f(x) - f(y)| &= \left|\frac{1}{x^2} - \frac{1}{y^2}\right| = \left|\frac{y^2 - x^2}{x^2y^2}\right| = \frac{|y+x||y-x|}{x^2y^2} \\ &= |y-x|\left(\frac{y}{x^2y^2} + \frac{x}{x^2y^2}\right) = |x-y|\left(\frac{1}{x^2y} + \frac{1}{xy^2}\right) \leq 2|x-y| < 2\delta_\varepsilon = \varepsilon\end{aligned}$$
whenever $|x - y| < \delta_\varepsilon$ and $x, y \in A$.

[3] If that is not the case, the proof will proceed in a similar fashion, but a_{2n} will be replaced by the first a_i that is non-zero, starting with a_{2n-1}; if all coefficients are 0, then the real root is 0.

We show that the negation of the definition of uniform continuity holds on B. Let $\varepsilon = 1$ and $\delta > 0$. Then, $\exists N \in \mathbb{N}$ s.t. $\frac{1}{N^2} < \delta$. Set $x_N = \frac{1}{N}$ and $y_N = \frac{1}{N+1}$. Clearly, $x_N, y_N \in B$ and

$$|x_N - y_N| = \left|\frac{1}{N} - \frac{1}{N+1}\right| = \frac{1}{N(N+1)} \leq \frac{1}{N^2} < \delta.$$

However,

$$|f(x_N) - f(y_N)| = |N^2 - (N+1)^2| = 2N + 1 > \varepsilon,$$

that is, f is not uniformly continuous on B. ■

20. If $f(x) = x$ and $g(x) = \sin x$, show that f and g are both uniformly continuous on $\mathbb{R}$ but that their product is not uniformly continuous on $\mathbb{R}$.

Proof: let $\varepsilon > 0$ and set $\delta_\varepsilon = \varepsilon$. Then

$$|f(x) - f(y)| = |x - y| < \delta_\varepsilon = \varepsilon$$

and

$$\begin{aligned} |g(x) - g(y)| = |\sin x - \sin y| &= 2\left|\sin\left(\frac{1}{2}(x-y)\right)\cos\left(\frac{1}{2}(x+y)\right)\right| \\ &\leq 2\frac{1}{2}|x-y| \cdot 1 = |x-y| < \delta_\varepsilon = \varepsilon \end{aligned}$$

(the second-last inequality can be obtained using Taylor's theorem on the sin function, see Chapter 4), whenever $|x - y| < \delta_\varepsilon$ and $x, y \in \mathbb{R}$. Hence f and g are both uniformly continuous.

Set $h(x) = x \sin x$. Let $\varepsilon = 1$ and $\delta > 0$. Them $\exists N \in \mathbb{N}$ s.t. $\frac{1}{N} < \delta$ and $K \in \mathbb{N}$ s.t.

$$K > \frac{1}{4}\left(1 - \cos\frac{1}{N}\right)^{-1} + 3.$$

Define

$$x_K = \frac{4K-3}{2}\pi \quad \text{and} \quad y_K = \frac{4K-3}{2}\pi - \frac{1}{N}.$$

Then $|x_K - y_K| = \frac{1}{N} < \delta$ and

$$|h(x_K) - h(y_K)| \geq \frac{4K-3}{2}\pi\left(1 - \cos\frac{1}{N}\right) > \frac{\pi}{2} > 1 = \varepsilon,$$

and so h is not uniformly continuous. ■

21. Let $A \subseteq \mathbb{R}$ and suppose that f has the following property: $\forall \varepsilon > 0$, $\exists g_\varepsilon : A \to \mathbb{R}$ s.t. g_ε is uniformly continuous on A with $|f(x) - g_\varepsilon(x)| < \varepsilon$ for all $x \in A$. Show f is uniformly continuous on A.

Proof: let $\varepsilon > 0$. Then $\frac{\varepsilon}{3} > 0$ and there exists $g_{\varepsilon/3}$ as in the hypothesis: hence $\exists \eta_{\varepsilon/3} > 0$ s.t. $|g_{\varepsilon/3}(x) - g_{\varepsilon/3}(y)| < \frac{\varepsilon}{3}$ whenever $|x-y| < \eta_{\varepsilon/3}$ and $x, y \in A$. Set $\delta_\varepsilon = \eta_{\varepsilon/3}$. Then

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - g_{\varepsilon/3}(x) + g_{\varepsilon/3}(x) - g_{\varepsilon/3}(y) + g_{\varepsilon/3}(y) - f(y)| \\ &\leq |f(x) - g_{\varepsilon/3}(x)| + |g_{\varepsilon/3}(x) - g_{\varepsilon/3}(y)| + |g_{\varepsilon/3}(y) - f(y)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

whenever $|x-y| < \delta_\varepsilon$ and $x, y \in A$. Hence, f is uniformly continuous on A. ■

22. Is a continuous $p-$periodic fonction on $\mathbb{R}$ bounded and uniformly continuous on $\mathbb{R}$?

Proof: since f is continuous, then $|f|$ is also continuous, being the composition of two continuous functions. As f is $p-$periodic, $\exists c \in [0, p]$ s.t.

$$\sup_{x\in\mathbb{R}} |f(x)| = \sup_{x\in[0,p]} |f(x)| = |f(c)|,$$

by the max/min theorem. Hence f is bounded by $|f(c)|$ on $\mathbb{R}$.

Let $\varepsilon > 0$. By hypothesis, f is continuous on the closed interval $[-1, p+1]$, which implies that that f is uniformly continuous on $[-1, p+1]$ (according to Theorem 38). Then, $\exists \delta_\varepsilon > 0$ s.t. $|f(x) - f(y)| < \varepsilon$ whenever $|x-y| < \delta_\varepsilon$ and $x, y \in [-1, p+1]$.

Without loss of generality, we can assume that $\delta_\varepsilon < 1$. Let $x, y \in \mathbb{R}$ s.t. $|x-y| < \delta_\varepsilon$. Then $\exists k \in \mathbb{Z}$ and $\alpha, \beta \in [-1, p+1]$ s.t. $x = \alpha + kp$ and $y = \beta + kp$.

Thus $|\alpha - \beta| = |x-y| < \delta_\varepsilon$ and $|f(x) - f(y)| = |f(\alpha) - f(\beta)| < \varepsilon$, since f is uniformly continuous on $[-1, p+1]$; consequently, f is uniformly continuous. ■

23. Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} \frac{(-1)^n}{n} & \text{if } x = 1/n \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise} \end{cases}$$

Prove that g is continuous at 0.

Proof: let $\varepsilon > 0$. Set $\delta_\varepsilon = \varepsilon$. Then,

$$\left|\frac{1}{n} - 0\right| < \delta \Longrightarrow \left|g\left(\frac{1}{n}\right) - g(0)\right| = \left|\frac{1}{n}\right| = \left|\frac{1}{n} - 0\right| < \delta_\varepsilon = \varepsilon$$

whenever $|1/n - 0| < \delta_\varepsilon$, so g is continuous at 0. ■

3.8 Exercises

1. Prepare a 2-page summary of this chapter, with important definitions and results.

2. Let $A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Show that no real number strictly greater than 1 can be a limit point of A.

3. Prove the "min" part of Theorem 33.

4. Complete the solution of solved problem 7.

5. Let $f : \mathbb{R} \to \mathbb{R}$. The **pre-image** of a subset $B \subseteq \mathbb{R}$ under f is

$$f^{-1}(B) = \{a \in A \mid f(a) \in B\}.$$

   Prove that f is continuous if and only if the pre-image of every open subset of $\mathbb{R}$ is an open subset of $\mathbb{R}$.

6. A function $f : A \to \mathbb{R}$ is said to be **Lipschitz** if there is a positive number M such that

$$|f(x) - f(y)| \leq M|x - y| \quad \forall\, x, y \in A.$$

   Show that a Lipschitz function must be uniformly continuous, but that uniformly continuous functions do not have to be Lipschitz.