Chapter 4

Differential and Integral Calculus

We have spent a fair amount of time and energy on concepts like the limit, continuity, and uniform continuity, with the goal of making differential and integral calculus sound. In this chapter, we introduce the concepts of **differentiability** and **Riemann-integrability** for functions, and prove a number of useful calculus results.

4.1 Differentiation

Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$, and $c \in I$. The real number L is the **derivative of** f **at** c, denoted by $f'(c) = L$, if

$$\forall \varepsilon > 0, \exists \delta_\varepsilon > 0 \text{ s.t. } x \in I \text{ and } 0 < |x-c| < \delta_\varepsilon \implies \left| \frac{f(x)-f(c)}{x-c} - L \right| < \varepsilon.$$

In that case, we say that f is **differentiable** at c.[1] While $f'(c)$ (if it exists) is a real number, , $f' : I \to \mathbb{R}$ is a function – the **derivative function**.

Example: let $f : I \to \mathbb{R}$ be defined by $f(x) = x^3$. Set $c \in I$. Then

$$f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = \lim_{x\to c} \frac{x^3-c^3}{x-c} = \lim_{x\to c}(x^2+cx+c^2) = 3c^2.$$

The corresponding derivative function is $f' : I \to \mathbb{R}$, $f'(x) = 3x^2$. □

As we learn in calculus courses, there is a link between differentiability and continuity.

[1] This definition simply states that $f'(c)$ exists if $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists, and that, in that case,

$$f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}.$$

Theorem 39
If $f : I \to \mathbb{R}$ has a derivative at c, then f is continuous at c.

Proof: let $x, c \in I$, $x \neq c$. Then $f(x) - f(c) = \left(\frac{f(x)-f(c)}{x-c}\right)(x-c)$ and so

$$\lim_{x\to c}(f(x) - f(c)) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c} \cdot \lim_{x\to c}(x - c),$$

if all the limits exist. But $x - c \to 0$ when $x \to c$, and

$$\lim_{x\to c} \frac{f(x) - f(c)}{x - c} = f'(c)$$

by hypothesis, so

$$\lim_{x\to c}(f(x) - f(c)) = f'(c) = 0 = 0 \Longrightarrow \lim_{x\to c} f(x) = f(c),$$

which means that f is continuous at c. ■

The converse of Theorem 39 does not always hold, however. The function $|\cdot| : \mathbb{R} \to \mathbb{R}$, for instance, is continuous at $x = 0$, but it has no derivative there as $|x|/x$ has no limit when $x \to 0$. Continuity is a **necessary condition** for differentiability, but it is **not sufficient**.

Example (WEIERSTRASS' MONSTER) Weierstrass provided the first example of such a function in 1872: $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \sum_{n\in\mathbb{N}} \frac{\cos(3^n x)}{2^n}.$$

That it took so long to find an example is mostly due to the fact that the definition of a function has evolved a fair amount over the last 200 years. □

The usual rules of differentiability are easily demonstrated.

Theorem 40
Let $c \in I$, I an interval, $\alpha \in \mathbb{R}$, $f, g : I \to \mathbb{R}$ be differentiable at c, with $g(c) \neq 0$. Then

1. *αf is differentiable at c and $(\alpha f)'(c) = \alpha f'(c)$;*
2. *$f + g$ is differentiable at c and $(f + g)'(c) = f'(c) + g'(c)$;*
3. *fg is differentiable at c and $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$;*
4. *f/g is differentiable at c and $(f/g)'(c) = \dfrac{f'(c)g(c) - f(c)g'(c)}{[g(c)]^2}$.*

Proof: in all instances, we compute the limit of the differential quotient, taking into account the fact that f and g are differentiable at c.

1. If αf is differentiable at c, then

$$(\alpha f)'(c) = \lim_{x\to c} \frac{(\alpha f)(x) - (\alpha f)(c)}{x-c} = \lim_{x\to c} \frac{\alpha(f(x) - f(c))}{x-c} = \alpha \lim_{x\to c} \frac{f(x)-f(c)}{x-c}.$$

   But f is differentiable at c, so the last limit exists, validating the string of equations, and is equal to $f'(c)$, and so $(\alpha f)'(c) = \alpha f'(c)$.

2. If $f+g$ is differentiable at c, then

$$\begin{aligned}(f+g)'(c) &= \lim_{x\to c} \frac{(f+g)(x) - (f+g)(c)}{x-c} = \lim_{x\to c} \frac{f(x)+g(x)-f(c)-g(c)}{x-c} \\ &= \lim_{x\to c} \frac{f(x)-f(c)}{x-c} + \lim_{x\to c} \frac{g(x)-g(c)}{x-c}.\end{aligned}$$

   But both f,g is differentiable at c, so the sum of limits exists, validating the string of equations, and is equal to $f'(c)+g'(c)$, and so $(f+g)'(c) = f'(c)+g'(c)$.

3. If fg is differentiable at c, then

$$\begin{aligned}(fg)'(c) &= \lim_{x\to c} \frac{(fg)(x) - (fg)(c)}{x-c} = \lim_{x\to c} \frac{f(x)g(x) - f(c)g(c)}{x-c} \\ &= \lim_{x\to c} \frac{f(x)g(x) - f(c)g(x) + f(c)g(x) - f(c)g(c)}{x-c} \\ &= \lim_{x\to c} \frac{f(x)-f(c)}{x-c} g(x) + \lim_{x\to c} f(c) \frac{g(x)-g(c)}{x-c} \\ &= \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \cdot \lim_{x\to c} g(x) + f(c) \lim_{x\to c} \frac{g(x)-g(c)}{x-c}\end{aligned}$$

   But both f,g is differentiable at c, so the differential quotient limits exist, validating the string of equations. Furthermore, g is continuous at c, being differentiable at c (acccording to Theorem 39). Hence

$$(fg)'(c) = f'(c) \cdot \lim_{x\to c} g(x) + f(c)g'(c) = f'(c)g(c) + f(c)g'(c).$$

4. Set $h = f/g$; then $f(c) = g(c)h(c)$ and $f'(c) = g'(c)h(c) + g(c)h'(c)$ by the previous rule. Thus

$$(f/g)'(c) = h'(c) = \frac{f'(c) - g'(c)h(c)}{g(c)} = \frac{f'(c) - g'(c)f(c)/g(c)}{g(c)} = \frac{f'(c)g(c) - g'(c)f(c)}{[g(c)]^2},$$

   which completes the proof. ■

Be careful! Although what we wrote in the proof for the fourth rule is undeniably true, we still need to show that h is differentiable at c under the given conditions before we can use the product rule. The proof could instead look like the following (reprise).

4. Since g is continuous at c (being differentiable at c) and $g(c) \neq 0$, $\exists$ an interval $J \subseteq I$ such that $c \in J$ an $g \neq 0$ on J. If f/g is differentiable at c, then

$$\begin{aligned}(f/g)'(c) &= \lim_{x\to c} \frac{(f/g)(x) - (f/g)(c)}{x-c} = \lim_{x\to c} \frac{f(x)/g(x) - f(c)/g(c)}{x-c} \\ &= \lim_{x\to c} \frac{f(x)g(c) - f(c)g(x)}{g(x)g(c)(x-c)} = \lim_{x\to c} \frac{f(x)g(c) - f(c)g(c) + f(c)g(c) - f(c)g(c)}{g(x)g(c)(x-c)},\end{aligned}$$

so that

$$\begin{aligned}(f/g)'(c) &= \lim_{x\to c} \frac{1}{g(x)g(c)} \left[\frac{f(x)-f(c)}{x-c} g(c) - f(c)\frac{g(x)-g(c)}{x-c}\right] \\ &= \lim_{x\to c} \frac{1}{g(x)g(c)} \cdot \left[\lim_{x\to c} \frac{f(x)-f(c)}{x-c} g(c) - f(c) \lim_{x\to c} \frac{g(x)-g(c)}{x-c}\right].\end{aligned}$$

But both f, g is differentiable at c, so the differential quotient limits exist, validating the string of equations.

Furthermore, g is continuous at c, being differentiable at c (cf. Theorem 39), and $g \neq 0$ on J, so that $\frac{1}{g(x)} \to \frac{1}{g(c)}$ when $x \to c$. Thus

$$(f/g)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{[g(c)]^2}. \qquad \blacksquare$$

Using mathematical induction, we can easily show that

$$\Big[\sum_{i=1}^n f_i\Big]'(c) = \sum_{i=1}^n f_i'(c) \quad \text{and} \quad \Big[\prod_{i=1}^n f_i\Big]'(c) = \sum_{i=1}^n \Big(\prod_{j\neq i} f_j(c)\Big) f_i'(c),$$

if $f_1, \ldots, f_n$ are all differentiable at c. In particular, if $f_1 = \cdots = f_n$, then

$$(f^n)'(c) = nf^{n-1}(c) \cdot f'(c).$$

If we consider the identity function f, then for $c \in \mathbb{R}$, we have

$$f'(c) = \lim_{x\to c} \frac{x-c}{x-c} = 1 \Longrightarrow (f^n)'(x) = nf^{n-1}(x) \cdot f'(x) = nx^{n-1}$$

for all $x \in \mathbb{R}, n \in \mathbb{N}$; this can be extended to $n \in \mathbb{Z}$ using Theorem 40.4.

Theorem 41 (Carathéodory)
Let $I = [a,b]$ and $f : I \to \mathbb{R}$. Then f is differentiable at $c \in I$ if and only if $\exists \varphi_c : I \to \mathbb{R}$, continuous at c such that $f(x) - f(c) = \varphi_c(x)(x-c)$, for all $x \in I$. In that case, $\varphi_c(c) = f'(c)$.

Proof: let $c \in I$. Assume that $f'(c)$ exists. Define $\varphi_c : I \to \mathbb{R}$ by

$$\varphi_c(x) = \begin{cases} \frac{f(x)-f(c)}{x-c}, & x \neq c \\ f'(c), & x = c \end{cases}$$

Then φ_c is continuous at c since

$$\lim_{x\to c} \varphi_c(x) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = f'(c) = \varphi_c(c).$$

If $x = c$, then $f(x) = f(c)$ and

$$f(x) - f(c) = f(c) - f(c) = 0 = \varphi_c(c)(c-c) = \varphi_c(x)(x-c).$$

If $x \neq c$ and $x \in I$, then, by definition, $f(x) - f(c) = \varphi_c(x)(x-c)$. Assume now that $\exists \varphi_c : I \to \mathbb{R}$, continuous at c, and such that $f(x) - f(c) = \varphi_c(x)(x-c)$, for all $x \in I$.

If $x \neq c$, then

$$\varphi_c(x) = \frac{f(x)-f(c)}{x-c}$$

and, since φ_c is continuous at c,

$$\varphi_c(c) = \lim_{x\to c} \varphi_c(x) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$$

exists. Then $\varphi_c(c) = f'(c)$ and f is differentiable at c. ■

It is important to recognize that φ_c is not, as a function, the same as f', in general **– it is only at c that they can be guaranteed to coincide**, although in certain cases (such as when f is a linear function), $f'(x) = \varphi_c(x)$ for all c in I. Carethéodory's Theorem can be used to prove an important rule of calculus.

Theorem 42 (Chain Rule)
Let I, J be closed bounded intervals, $g : I \to \mathbb{R}$ and $f : I \to \mathbb{R}$ be functions such that $f(J) \subseteq I$, and let $c \in J$, with $d = f(c)$. If f is differentiable at c and g is differentiable at d, then the composition $g \circ f : J \to \mathbb{R}$ is differentiable at c and

$$(g \circ f)'(c) = g'(f(c))f'(c) = g'(d)f'(c)$$

.

Proof: since $f'(c)$ exists, Carathéodory's Theorem implies that $\varphi_c : J \to \mathbb{R}$ such that φ_c is continuous at $c \in J$ with

$$\varphi_c(c) = f'(c), \quad \text{and } f(x) - f(c) = \varphi_c(x)(x-c), \quad \text{for all } x \in J.$$

Since $g'(d)$ exists, $\exists \psi_d : I \to \mathbb{R}$ such that ψ_d is continuous at $d \in I$, with

$$\psi_d(d) = g'(d), \quad \text{and } g(y) - g(d) = \psi_d(y)(y-d), \quad \text{for all } y \in I.$$

Thus, if $y = f(x)$ and $d = f(c)$, we have

$$\begin{aligned}(g\circ f)(x) - (g\circ f)(c) &= g(f(x)) - g(f(c)) = \psi_d(f(x))(f(x) - f(c)) \\ &= \psi_d(f(x))\varphi_c(x)(x-c) = \big[(\psi_d \circ f)(x)\cdot\varphi_c(x)\big](x-c),\end{aligned}$$

for all $x \in J$ such that $f(x) \in I$.

However $(\psi_d \circ f)\cdot\varphi_c$ is continuous at c, being the product of two functions which are continuous at c. According to Carathéodory, $(\psi_d \circ f)(c)\cdot\varphi_c(c) = (g\circ f)'(c)$. But

$$(\psi_d \circ f)(c)\cdot\varphi_c(c) = \psi_d(f(c))\varphi_c(c) = g'(f(c))f'(c) = g'(d)f'(c),$$

which completes the proof. ■

The chain rule can be used to determine some of the other classical rules of differentiation.

Examples

- Suppose that $f : I \to \mathbb{R}$ is differentiable at c and that $f, f' \neq 0$ on I. If h is defined by $h(y) = \frac{1}{y}$, $y \neq 0$, then $h'(y) = -\frac{1}{y^2}$. Thus

$$(1/f)'(x) = (h\circ f)'(x) = h'(f(x))\cdot f'(x) = -\frac{f'(x)}{(f(x))^2}, \quad \text{for all } x \in I.$$

- Let $g = |\cdot|$. Then $g'(c) = \text{sgn}(c)$ for all $c \neq 0$. Indeed,

$$\lim_{x\to c}\frac{|x|-|c|}{x-c} = \begin{cases}\lim_{x\to c}\dfrac{x-c}{x-c}, & c>0\\ -\lim_{x\to c}\dfrac{x-c}{x-c}, & c<0\end{cases} = \begin{cases}1, & c>0\\ -1, & c<0\end{cases} = \text{sgn}(c),$$

but $g'(0)$ does not exist (even though $\text{sgn}(0) = 0$). If $f : [a,b] \to \mathbb{R}$ is differentiable, the chain Rule states that $|f|'(x) = \text{sgn}(f(x))\cdot f'(x)$. What happens if $f(c) = 0$? Is $|f|$ differentiable at c?

4.1.1 Mean Value Theorem

With basic calculus in the bag, we can now tackle some of the heavy analysis hitters. Let I be an interval; a function $f : I \to \mathbb{R}$ has a **relative maximum** at $c \in I$ if $\exists \delta > 0$ s.t.

$$f(x) \leq f(c), \quad \forall x \in V_\delta(c) = (c-\delta, c+\delta);$$

it has a **relative minimum** at $c \in I$ if $\exists \delta > 0$ such that

$$f(x) \geq f(c), \quad \forall x \in V_\delta(c) = (c-\delta, c+\delta).$$

If f has either a relative maximum or a relative minimum at c, we say that it has a **relative extremum** at c.[2]

Theorem 43
Let $f : [a,b] \to \mathbb{R}$, $c \in (a,b)$. If f has a relative extremum at c and if f is differentiable at c, then $f'(c) = 0$.

Proof: without loss of generality, assume that f has a relative maximum at c; the proof for a relative minimum follows the same lines. Let $\tilde{\delta}$ be the quantity whose existence is guaranteed by the definition:

$$f(x) \leq f(c), \quad \forall x \in V_{\tilde{\delta}}.$$

If $f'(c) > 0$, then $\exists \delta > 0$ such that $\frac{f(x)-f(c)}{x-c} > 0$ whenever $0 < |x-c| < \delta$. Indeed, according to the definition of the derivative, if $\varepsilon = \frac{1}{2}f'(c) > 0$, $\exists \delta_\varepsilon > 0$ such that

$$\left| \frac{f(x)-f(c)}{x-c} - f'(c) \right| < \varepsilon = \frac{1}{2}f'(c)$$

whenever $0 < |x-c| < \delta_\varepsilon$. Set $\delta = \min\{\delta_\varepsilon, \tilde{\delta}\}$. Then

$$-\frac{1}{2}f'(c) < \frac{f(x)-f(c)}{x-c} - f'(c) < \frac{1}{2}f'(c), \quad \text{whenever } 0 < |x-c| < \delta,$$

and so

$$0 < \frac{1}{2}f'(c) < \frac{f(x)-f(c)}{x-c}, \quad \text{whenever } 0 < |x-c| < \delta.$$

But if $x \in V_\delta(c)$ with $x > c$, then

$$f(x) - f(c) = \underbrace{(x-c)}_{>0} \cdot \underbrace{\frac{f(x)-f(c)}{x-c}}_{>0} > 0,$$

and so $f(x) > f(c)$, which contradicts the fact that f has a relative maximum at c. Thus, $f'(c) \not> 0$. We can prove that $f'(c) \not< 0$ using a similar argument. As neither $f'(c) > 0$ nor $f'(c) < 0$, we must have $f'(c) = 0$. ■

[2] Note that the definition of relative extremum makes no mention of continuity or differentiability.

This result justifies the common practice of looking for relative extrema at **roots of the derivative**. Since c is not an endpoint of I, we must also include a and b in the search for extrema.[3] The next theorem has far-reaching consequences.

Theorem 44 (ROLLE)
Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = 0$ and $f(b) = 0$, $\exists c \in (a, b)$ such that $f'(c) = 0$.

Proof: if $f \equiv 0$ on $[a, b]$, then the conclusion holds for any $c \in (a, b)$. If $\exists x^*$ such that $f(x^*) \neq 0$, we may suppose, without loss of generality, that $f(x^*) > 0$. According to the max/min theorem, f reaches its maximum

$$\sup\{f(x) \mid x \in [a, b]\} > 0$$

somewhere in $[a, b]$. But since $f(a) = f(b) = 0$, the maximum must be reached in (a, b). Denote that point by c. Then $f'(c)$ exists and since f has a relative maximum at c, Theorem 43 implies that $f'(c) = 0$. ■

This subsection's main result is an easy corollary of Rolle's Theorem.

Theorem 45 (MEAN VALUE THEOREM)
Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. If f is differentiable on (a, b), $\exists c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$.

Proof: let $\varphi : [a, b] \to \mathbb{R}$ be defined by

$$\varphi(x) = f(x) - (a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Then

$$\varphi(a) = f(a) - f(a) - \frac{f(b) - f(a)}{b - a}(a - a) = 0, \quad \text{and}$$
$$\varphi(b) = f(b) - f(a) - \frac{f(b) - f(a)}{b - a}(b - a) = f(b) - f(a) - (f(b) - f(a)) = 0.$$

But φ is continuous on $[a, b]$ as f and $x \mapsto x - a$ are continuous on $[a, b]$. According to Rolle's Theorem, $\exists c \in (a, b)$ such that $\varphi'(c) = 0$. But

$$\varphi'(x) = f'(x) - \frac{f(b) - f(a)}{b - a},$$

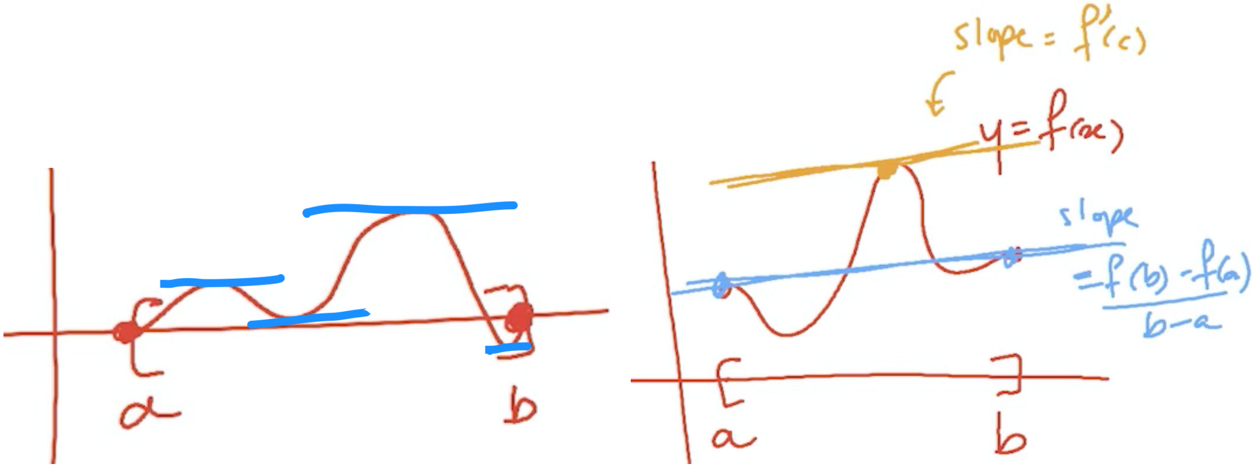
so that $f'(c) - \frac{f(b)-f(a)}{b-c} = 0$, which completes the proof. ■

[3] What happens if f is not differentiable at c in Theorem 43?

Among other things, this tells us something about functions whose derivatives is identically zero on $[a,b]$.

> **Theorem 46**
> *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If $f'\equiv 0$ on (a,b), then f is constant on $[a,b]$.*
>
> **Proof:** let $x\in(a,b]$. According to the mean value theorem, $\exists c\in(a,x)$ such that $f(x)-f(a)=f'(c)(x-a)$. But $f'(c)=0$, so that $f(x)-f(a)=0$ for all $x\in[a,b]$. ■

Illustrations of Rolle's theorem (left) and the mean value theorem (right) are shown below.



4.1.2 Taylor Theorem

This subsection's main result is used extensively in applications. It is, in a way, an extension of the mean value theorem to higher order derivatives. We can naturally obtain the **higher-order derivatives** of a function f by formally applying the differentiation rules repeatedly. Hence, $f^{(2)}=f''=(f')'$, $f^{(3)}=f'''=(f'')'=((f')')'$, etc. Suppose $f=f^{(0)}$ can be differentiated n times at $x=x_0$. The n**th Taylor polynomial of** f **at** $x=x_0$ is

$$P_n(x;f,x_0)=\sum_{i=0}^{n}\frac{f^{(i)}(x_0)}{i!}(x-x_0)^i.$$

> **Theorem 47** (Taylor)
> *Let $n\in\mathbb{N}$ and $f:[a.b]\to\mathbb{R}$ be such that f and its derivatives $f',f'',\ldots,f^{(n)}$ are continuous on $[a,b]$, and $f^{(n+1)}$ exists on (a,b). If $x_0\in[a,b]$, then for all $x\neq x_0\in[a,b]$, $\exists c$ between x and x_0 such that*
>
> $$f(x)=P_n(x;f,x_0)+\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}.$$

Proof: let $x \in [a,b]$. If $x_0 < x$, set $J = [x_0, x]$. Otherwise, set $J = [x, x_0]$. Let $F : J \to \mathbb{R}$ be defined by

$$F(t) = f(x) - P_n(t; f, x) = f(x) - f(t) - f'(t)(x-t) - \cdots - \frac{f^{(n)}(t)}{n!}(x-t)^n.$$

Note that F is continuous on J as f and its n higher-order derivatives are continuous on J, and that

$$\begin{aligned} F'(t) = -f'(t) &- \Big[f''(t)(x-t) - f'(t)\Big] - \Big[\frac{f'''(t)}{2!}(x-t)^2 - f''(t)(x-t)\Big] \\ &- \cdots - \\ &- \Big[\frac{f^{(n+1)}(t)}{n!}(x-t)^n - f^{(n)}(t)(x-t)^{n-1}\Big]. \end{aligned}$$

Thus $F'(t) = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n$. Let $G : J \to \mathbb{R}$ be defined by

$$G(t) = F(t) - \left(\frac{x-t}{x-x_0}\right)^{n+1} F(x_0).$$

Then

$$G(x_0) = F(x_0) - \left(\frac{x-x_0}{x-x_0}\right)^{n+1} F(x_0) = 0$$
$$G(x) = F(x) - \left(\frac{x-x}{x-x_0}\right)^{n+1} F(x_0) = F(x).$$

But

$$F(x) = f(x) - f(x) - f'(x)(x-x) - \cdots - \frac{f^{(n)(x)}}{n!}(x-x)^n = 0.$$

Thus $G(x) = 0$. Note that G is continuous on J. Furthermore, G is differentiable on J since

$$G'(t) = F'(t) + \frac{(n+1)}{x-x_0}\left(\frac{x-t}{x-x_0}\right)^n F(x_0) = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n + \frac{(n+1)}{x-x_0}\left(\frac{x-t}{x-x_0}\right)^n F(x_0).$$

As G satisfies the hypotheses of Rolle's theorem, $\exists c$ between x and x_0 such that $G'(c) = 0$. Thus

$$\frac{f^{(n+1)}(c)}{n!}(x-c)^n = (n+1)\frac{(x-c)^n}{(x-x_0)^{n+1}}F(x_0) \quad \Longrightarrow$$
$$F(x_0) = \frac{f^{(n+1)}(c)}{n!(n+1)}\frac{(x-c)^n}{(x-c)^n}(x-x_0)^{n+1} = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}.$$

But

$$F(x_0) = f(x) - P_n(x_0; f, x) \Longrightarrow f(x) = P_n(f; x_0) + F(x_0),$$

which completes the proof. ■

One of the obvious uses of Taylor's theorem is for approximations.

Example: use Taylor's Theorem with $n = 2$ to approximate $\sqrt[4]{1+x}$ near $x_0 = 0$ (for $x > -1$).

Solution: let $f(x) = (1+x)^{1/4}$. Then

$$f'(x) = \frac{1}{4}(1+x)^{-3/4}, \quad f''(x) = -\frac{3}{16}(1+x)^{-7/4}, \quad f'''(x) = \frac{21}{64}(1+x)^{-11/4}$$

are all continuous in closed intervals $[-a, a]$, $1 > a > 0$, so Taylor's theorem can be brought to bear on the situation. Note that $f(0) = 1$, $f'(0) = \frac{1}{4}$ and $f''(0) = -\frac{3}{16}$. According to Taylor's Theorem, for every $x \in [-a, a]$, $1 > a > 0$, $\exists c$ between x and 0 such that

$$f(x) = P_2(x; f, 0) + \frac{f'''(c)}{3!}x^3 = 1 + \frac{1}{4}x - \frac{3}{32}x^2 + \frac{7}{128(1+c)^{11/4}}x^3.$$

For instance, $\sqrt[4]{1.4}$ can be approximated by

$$f(0.4) \approx P_2(0.4) = 1 + \tfrac{1}{4}(0.4) - \tfrac{3}{32}(0.4)^2 \approx 1.085.$$

Moreover, since $c \in (0, 0.4)$,

$$\tfrac{f'''(c)}{6}(0.4)^3 = \tfrac{7}{128}(1+c)^{-11/4}(0.4)^3 \leq \tfrac{7}{128}(0.4)^3 = 0.0035,$$

so $|\sqrt[4]{1.4} - 1.085| \leq 0.0035$, which is to say that the approximation is correct to 2 decimal places. □

4.1.3 Relative Extrema

We end the section on differentiability by giving a characterization of **relative extrema** using the derivative.

A function $f : I \to \mathbb{R}$ is **increasing** (resp. **decreasing**) if

$$f(x_1) \leq f(x_2), \quad (\text{resp. } f(x_1) \geq f(x_2)) \quad \forall x_1 \leq x_2 \in I.$$

If the inequalities are strict, then the function is **strictly increasing** (resp. **strictly decreasing**). A function that is either increasing or decreasing (exclusively) is **monotone**. If the function is also differentiable, then a link exists.

Theorem 48
Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$, differentiable on (a, b). Then f is increasing on $[a, b]$ if and only if $f' \geq 0$ on (a, b).

Proof: suppose f is increasing and let $c \in (a,b)$. For all $x < c$ in (a,b), we have $f(x) \leq f(c)$; for all $x > c$ in (a,b), we have $f(x) \geq f(c)$. Thus

$$\frac{f(x)-f(c)}{x-c} \geq 0, \quad \text{for all } x \neq c \in (a,b).$$

Since f is differentiable at c, we must have

$$f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \geq 0.$$

As c is arbitrary, we have $f'(x) \geq 0$ for all $x \in (a,b)$. If, conversely, $f'(x) \geq 0$ for all $x \in (a,b)$, let $x_1 < x_2 \in [a,b]$. By the Mean Value Theorem, $\exists c \in (x_1,x_2)$ such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1).$$

Since $f'(c) \geq 0$ an $x_2 > x_1$, then

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c) \geq 0 \Longrightarrow f(x_2)-f(x_1) \geq 0 \Longrightarrow f(x_2) \geq f(x_1),$$

which is to say, f is increasing on $[a,b]$. ■

Theorem 48 holds for decreasing functions as well (after having made the obvious changes to the statement).[4]

The next theorem is a celebrated result from calculus.

Theorem 49 (First Derivative Test)
Let f be continuous on $[a,b]$ and let $c \in (a,b)$. Suppose f is differentiable on (a,c) and on (b,c), but not necessarily at c. Then

1. *if $\exists V_\delta(c) \subseteq [a,b]$ such that $f'(x) \geq 0$ for $c-\delta < x < c$ and $f'(x) \leq 0$ for $c < x < c+\delta$, then f has a relative maximum at c;*
2. *if $\exists V_\delta(c) \subseteq [a,b]$ such that $f'(x) \leq 0$ for $c-\delta < x < c$ and $f'(x) \geq 0$ for $c < x < c+\delta$, then f has a relative minimum at c.*

Proof: we only prove 1.; the proof for 2. follows the same lines. If $x \in (c-\delta, c)$, the mean value theorem states that $\exists c_x \in (x,c)$ such that

$$f(c) - f(x) = \underbrace{f'(c_x)}_{\geq 0}\underbrace{(c-x)}_{\geq 0} \geq 0,$$

so that $f(x) \leq f(c)$ for all $x \in (c-\delta, c)$.

[4] If we switch to strictly monotone functions, only one direction holds in all cases – which one?

If $x \in (c, c+\delta)$, the mean value theorem states that $\exists c_x \in (c, x)$ such that

$$f(c) - f(x) = \underbrace{f'(c_x)}_{\leq 0}\underbrace{(c-x)}_{\leq 0} \geq 0,$$

so that $f(x) \leq f(c)$ for all $x \in (c, c+\delta)$.

Combining these statements with the fact that $f(c) \leq f(c)$, we obtain $f(x) \leq f(c)$ for all $x \in V_\delta(c)$, so f has a relative maximum at c. ■

The converse of the first derivative test is not necessarily true. For instance, the function defined by

$$f(x) = \begin{cases} 2x^4 + x^4 \sin(1/x), & x \neq 0 \\ 0 & x = 0 \end{cases}$$

has an absolute minimum at $x = 0$, but it has derivatives of either sign on either side of any neighbourhood of $x = 0$.

We end this section with a rather surprising result.

Theorem 50 (DARBOUX)
Let $f : [a, b] \to \mathbb{R}$ be differentiable, continuous on $[a, b]$ and let k be strictly confined between $f'(a)$ and $f'(b)$. Then $\exists c \in (a, b)$ with $f'(c) = k$.

Proof: without loss of generality, assume $f'(a) < k < f'(b)$. Define $g : [a, b] \to \mathbb{R}$ by $g(x) = kx - f(x)$; g is then continuous and differentiable on $[a, b]$ given that both f and $x \mapsto kx$ also are.

By the max/min theorem, g reaches its maximum value at some $c \in [a, b]$. However, $g'(a) = k - f'(a) > 0$, so that $c \neq a$, and $g'(b) = k - f'(b) < 0$, so that $c \neq b$. Hence $g'(c) = 0$ for some $c \in (a, b)$, according to Theorem 43, and so $f'(c) = k$, which completes the proof. ■

Darboux's theorem states that the derivative of a continuous function, which needs not be continuous, nevertheless satisfies the intermediate value property.[5]

There are a number of other results which could be shown about differentiable functions, but they are left as exercises (see question 4).

[5]That seems like witchcraft, right? It shouldn't be possible, but the argument is sound. One of the lessons from this result is that analytical reasoning *can* be informed by intuition and geometry, but ultimately, the validity of results rests on proofs.

4.2 Riemann Integral

Calculus as a discipline only took flight after Newton announced his **theory of fluxions**. With Leibniz' independent discovery that the reversal of the process for fining tangents lead to areas under curves, integration was born. Riemann was the first to discuss integration as a process **separate** from differentiation.

We start by studying the integration of a functions $\mathbb{R} \to \mathbb{R}$. Later on, we will tackle integration of functions $\mathbb{R}^n \to \mathbb{R}$ (see Chapter 21) and of functions $\mathbb{R}^n \to \mathbb{R}^n$ (see Chapter 14).

Let $I = [a,b]$. A **partition** $P \in \mathcal{P}([a,b])$ is a subset $P = \{x_0, \dots, x_n\} \subseteq I$ such that

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

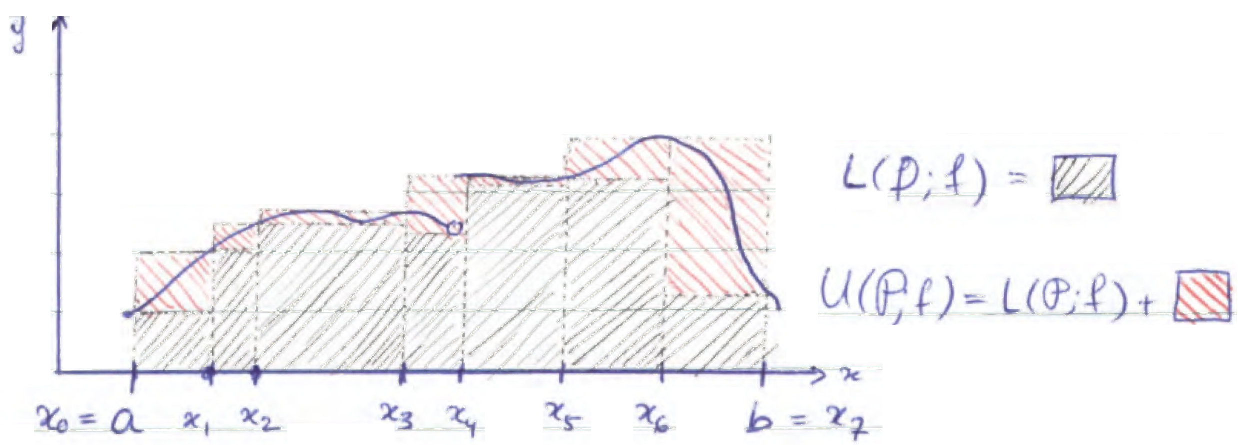
If $f : I \to \mathbb{R}$ is bounded and P is a partition of I, the sums

$$L(P;f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) < \infty, \quad U(P;f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) < \infty,$$

where

$$m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}, \quad M_i = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\}, \quad 1 \leq i \leq n$$

are the **lower** and the **upper sum of f corresponding to** P, respectively. If $f : I \to \mathbb{R}_0^+$, we can give a graphical representation of these sums; $L(P;f)$ is the area of the union of the rectangles with base $[x_{k-1}, x_k]$ and height m_k, and $U(P;f)$ is the area of the union of the rectangles with base $[x_{k-1}, x_k]$ and height M_k.



A partition Q of I is a **refinement** of a partition P of I if $P \subseteq Q$.

> **Example:** both $P = \{0,1,4,10\}$ and $Q = \{0,1,2,3,4,5,6,7,8,10\}$ are partitions of $[0,10]$; since $Q \supseteq P$, Q is a refinement of P. □

We will use the following lemma repeatedly in this section.

Lemma: let $I = [a, b]$ and $f : I \to \mathbb{R}$ be bounded. Then

1. $L(P; f) \leq U(P; f)$ for any partition P of I;
2. $L(P; f) \leq L(Q; f)$ and $U(Q; f) \leq L(Q; f)$ for any refinement $Q \supseteq P$ of I, and
3. $L(P_1; f) \leq U(P_2; f)$ for any pair of partitions P_1, P_2 of I.

Proof:

1. Let $P = \{x_0, \ldots, x_n\}$ be a partition of I. Since

$$m_i = \text{inf}\{f(x) \mid x \in [x_{i-1}, x_i]\} \leq \text{sup}\{f(x) \mid x \in [x_{i-1}, x_i]\} = M_i$$

for all $1 \leq i \leq n$, then

$$L(P; f) = \sum_{i=1}^{n} m_i(\underbrace{x_i - x_{i-1}}_{>0}) \leq \sum_{i=1}^{n} M_i(\underbrace{x_i - x_{i-1}}_{>0}) = U(P; f).$$

2. Let $Q = \{y_0, \ldots, y_m\}$ be a refinement of $P = \{x_0, \ldots, x_n\}$. Set $I_i = [x_{i-1}, x_i]$ and $\tilde{I}_j = [y_{j-1}, y_j]$, for $1 \leq i \leq n, 1 \leq j \leq m$. Write $m_i = \text{inf}\{f(x) \mid x \in I_i\}$ and $\tilde{m}_j = \text{inf}\{f(x) \mid x \in \tilde{I}_j\}$ and fix $1 \leq i \leq n$. Then $\exists j, k$ such that

$$I_i = \tilde{I}_{j+1} \cup \cdots \cup \tilde{I}_{j+k} = \bigcup_{\ell=1}^{k} \tilde{I}_{j+\ell}.$$

Then

$$\begin{aligned} m_i(x_i - x_i - 1) &= m_i(y_j + k - y_j) = m_i(y_{j+1} - y_j + \cdots + y_{j+k} - y_{j+k-1}) \\ &= m_i(y_{j+1} - y_j) + \cdots + m_i(y_{j+k} - y_{j+k-1}) \\ &= \sum_{\ell=1}^{k} m_i(y_{j+\ell} - y_{j+\ell-1}) \leq \sum_{\ell=1}^{k} \tilde{m}_{j+\ell}(y_{j+\ell} - y_{j+\ell-1}) \end{aligned}$$

since $\tilde{I}_{j+\ell} \subseteq I_i$ for all $\ell = 1, \ldots, k$. Hence

$$L(P; f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \leq \sum_{j=1}^{m} \tilde{m}_j(y_j - y_{j-1}) = L(Q; f).$$

The proof for $U(P; f) \geq U(Q; f)$ follows a similar argument.

3. Let P_1, P_2 be partitions of I. Set $Q = P_1 \cup P_2$. Then Q is a refinement of both P_1 and P_2. By the results proven in the previous parts of this lemma, we have

$$L(P_1; f) \leq L(Q; f) \leq U(Q; f) \leq U(P_2; f),$$

which completes the proof. ∎

Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be bounded. The **lower integral of f on I** is the number

$$L(f) = \sup\{L(P; f) \mid P \text{ a partition of } I\}.$$

The **upper integral of f on I** is the number

$$U(f) = \inf\{U(P; f) \mid P \text{ a partition of } I\}.$$

Since f is bounded on I, $\exists m, M$ such that $m \leq f(x) \leq M$ for all $x \in I$. Consider the **trivial partition** $P_0 = \{a, b\}$. Since any partition P of I is a refinement of P_0, we thus have

$$L(P; f) \leq U(P_0; f) \leq M(b - a) \quad \text{and} \quad U(P; f) \geq L(P_0; f) \geq m(b - a).$$

Thus $L(f), U(f)$ exist, by completeness. But we can say more.

Theorem 51
Let $f : [a, b] \to \mathbb{R}$ be bounded. Then $L(f) \leq U(f)$.

Proof: let P_1, P_2 be partitions of $[a, b]$. Then $L(P_1; f) \leq U(P_2; f)$. If we fix P_2, $U(P_2; f)$ is an upper bound for

$$A = \{L(P_1; f) \mid P_1 \text{ is a partition of } [a, b]\}.$$

Since $L(f) = \sup(A)$ and since P_2 was arbitrary, $L(f)$ is a lower bound for

$$B = \{U(P_2; f) \mid P_2 \text{ is a partition of } [a, b]\}.$$

Thus $L(f) \leq \inf(B) = U(f)$. ■

When $L(f) = U(f)$, we say that f is **Riemann-integrable** on $[a, b]$; the **integral of f on $[a, b]$** is the real number

$$L(f) = U(f) = \int_a^b f = \int_a^b f(x)\, \mathrm{d}x.$$

By convention, we define $\int_a^b f = -\int_b^a f$ when $b < a$. Note that $\int_a^a f = 0$ for all bounded functions f.

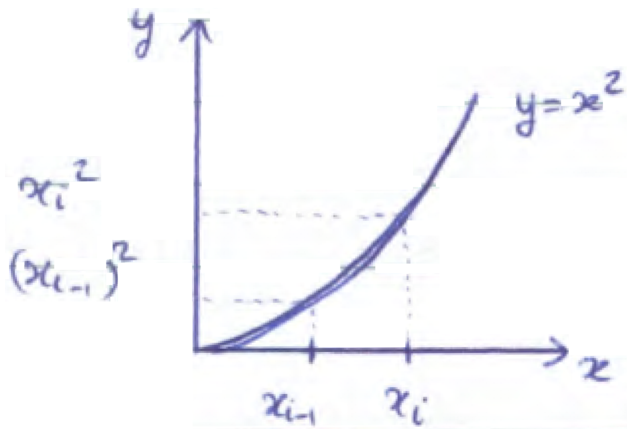
Example: show directly that the function defined by $h(x) = x^2$ is Riemann-integrable on $[a, b]$, $b > a \geq 0$. Furthermore show that $\int_a^b h = \frac{b^3 - a^3}{3}$.

Proof: let $P_n = \left\{x_i = a + \frac{b-a}{n} \cdot i \mid i = 0, \ldots, n\right\} \in \mathcal{P}([a, b])$. For $i = 1, \ldots, n$ set $m_i = \inf\{h(x) \mid x \in [x_{i-1}, x_i]\}$. With this notation, we have

$$L(P_n; h) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = \frac{b - a}{n} \sum_{i=1}^{n} m_i.$$

But $h'(x) = 2x \geq 0$ when $x \geq 0$, and so h is increasing on $[a, b]$. Consequently, for $i = 1, \ldots, n$, we have

$$m_i = x_{i-1}^2 = \left(a + \frac{b-a}{n}(i-1)\right)^2 = a^2 + 2\frac{a(b-a)}{n}(i-1) + \frac{(b-a)^2}{n^2}(i-1)^2.$$



The lower sum of h associated to P_n is thus

$$\begin{aligned} L(P_n; h) &= \frac{b-a}{n}\sum_{i=1}^{n}\left(a^2 + 2\frac{a(b-a)}{n}(i-1) + \frac{(b-a)^2}{n^2}(i-1)^2\right) \\ &= \frac{na^2(b-a)}{n} + \frac{2a(b-a)^2}{n^2}\sum_{i=1}^{n}(i-1) + \frac{(b-a)^3}{n^3}\sum_{i=1}^{n}(i-1)^2 \\ &= a^2(b-a) + \frac{2a(b-a)^2}{n^2}\cdot\frac{n(n-1)}{2} + \frac{(b-a)^3}{n^3}\cdot\frac{n(n-1)(2n-1)}{6} \\ &= a^2(b-a) + a(b-a)^2\left(1-\tfrac{1}{n}\right) + \tfrac{(b-a)^3}{6}\left(1-\tfrac{1}{n}\right)\left(2-\tfrac{1}{n}\right). \end{aligned}$$

For the lower sum of h on $[a, b]$, we have

$$\begin{aligned} L(h) &= \sup\{L(P; h) \mid P \in \mathcal{P}([a, b])\} \geq \sup_{n\in\mathbb{N}}\{L(P_n; h)\} \\ &= \sup_{n\in\mathbb{N}}\left\{a^2(b-a) + a(b-a)^2\left(1-\tfrac{1}{n}\right) + \tfrac{(b-a)^3}{6}\left(1-\tfrac{1}{n}\right)\left(2-\tfrac{1}{n}\right)\right\} \\ &= \lim_{n\to\infty}\left[a^2(b-a) + a(b-a)^2\left(1-\tfrac{1}{n}\right) + \tfrac{(b-a)^3}{6}\left(1-\tfrac{1}{n}\right)\left(2-\tfrac{1}{n}\right)\right] \\ &= a^2(b-a) + a(b-a)^2 + \tfrac{(b-a)^3}{6}\cdot 2 = \tfrac{b^3-a^3}{3}. \end{aligned}$$

Similarly, we can show that

$$U(P_n; h) = a^2(b-a) + a(b-a)^2\left(1+\tfrac{1}{n}\right) + \tfrac{(b-a)^3}{6}\left(1+\tfrac{1}{n}\right)\left(2+\tfrac{1}{n}\right).$$

For the upper sum of h on $[a,b]$, we have

$$\begin{aligned} U(h) &= \inf\{U(P;h) \mid P \in \mathcal{P}([a,b])\} \leq \inf_{n\in\mathbb{N}}\{U(P_n;h)\} \\ &= \inf_{n\in\mathbb{N}}\left\{a^2(b-a)+a(b-a)^2\left(1+\tfrac{1}{n}\right)+\tfrac{(b-a)^3}{6}\left(1+\tfrac{1}{n}\right)\left(2+\tfrac{1}{n}\right)\right\} \\ &= \lim_{n\to\infty}\left[a^2(b-a)+a(b-a)^2\left(1+\tfrac{1}{n}\right)+\tfrac{(b-a)^3}{6}\left(1+\tfrac{1}{n}\right)\left(2+\tfrac{1}{n}\right)\right] \\ &= a^2(b-a)+a(b-a)^2+\tfrac{(b-a)^3}{6}\cdot 2 = \tfrac{b^3-a^3}{3}. \end{aligned}$$

Thus $\frac{b^3-a^3}{3} \leq L(h) \leq U(h) \leq \frac{b^3-a^3}{3}$ and so $L(h) = U(h) = \int_a^b h = \frac{b^3-a^3}{3}$, which completes the proof. ■

It is clearly not the most efficient process in practice, but it works!

Example: show directly that the **Dirichlet function** defined by

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

is not Riemann-integrable on $[0,1]$.

Proof: let $P = \{x_0,\ldots,x_n\} \in \mathcal{P}([0,1])$. Since both $\mathbb{Q}$ and $\mathbb{R}\setminus\mathbb{Q}$ are dense in $\mathbb{R}$, for each $1 \leq i \leq n$, $\exists q_i \in \mathbb{Q}, t_i \notin \mathbb{Q}$ such that $q_i, t_i \in [x_{i-1}, x_i]$.
But $f(q_i) = 0$ and $f(t_i) = 1$, so that $m_i = 0$, $M_i = 1$ for all $1 \leq i \leq n$. This implies that $L(P;f) = 0$ and $U(P;f) = 1$ for any partition P. Thus $L(f) = 0 \neq 1 = U(f)$, and so f is not Riemann-integrable. ■

This last example underlines some of the shortcomings of the Riemann integral – by any account the integral of Dirichlet's function should really be 0 on $[0,1]$: the set $\mathbb{R}\setminus\mathbb{Q}$ is so much larger than $\mathbb{Q}$ that whatever happens on $\mathbb{Q}$ should largely be irrelevant (see Section 1.2). There are various theories of integration – as we shall see in Chapter 21, the Lebesgue-Borel integral of f on $[0,1]$ is indeed 0.

Other issues arise with the Riemann integral, which we will discuss in the coming sections.

4.2.1 Riemann's Criterion

We focus on two **fundamental questions** associated with the Riemann integral of a function over an interval $[a,b]$: **does it exist? If so, what value does it take?**

The direct approach is cumbersome, even for the simplest of functions. The following result allows us to bypass the need to compute $L(f)$ and $U(f)$ to determine if a function is Riemann-integrable or not.

Theorem 52 (Riemann's Criterion)
Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be a bounded function. Then f is Riemann-integrable if and only if $\forall \varepsilon > 0$, $\exists P_\varepsilon$ a partition of I such that the lower sum and the upper sum of f corresponding to P_ε satisfy $U(P_\varepsilon; f) - L(P_\varepsilon; f) < \varepsilon$.

Proof: if f is Riemann-integrable, then $L(f) = U(f) = \int_a^b f$. Let $\varepsilon > 0$.

Since $\int_a^b f - \frac{\varepsilon}{2}$ is not an upper bound of $\{L(P; f) \mid P \text{ a partition of } [a, b]\}$, there exists a partition P_1 such that

$$\int_a^b f - \frac{\varepsilon}{2} < L(P_1; f) \leq \int_a^b f.$$

Using a similar argument, there exists a partition P_2 such that

$$\int_a^b f + \frac{\varepsilon}{2} \geq U(P_2; f) > \int_a^b f.$$

Set $P_\varepsilon = P_1 \cup P_2$. Then P_ε is a refinement of P_1 and P_2. Consequently,

$$\int_a^b f - \frac{\varepsilon}{2} < L(P_1; f) \leq L(P_\varepsilon; f) \leq U(P_\varepsilon; f) \leq U(P_2; f) < \int_a^b f + \frac{\varepsilon}{2}$$

which implies that

$$U(P_\varepsilon; f) - L(P_\varepsilon; f) < \varepsilon.$$

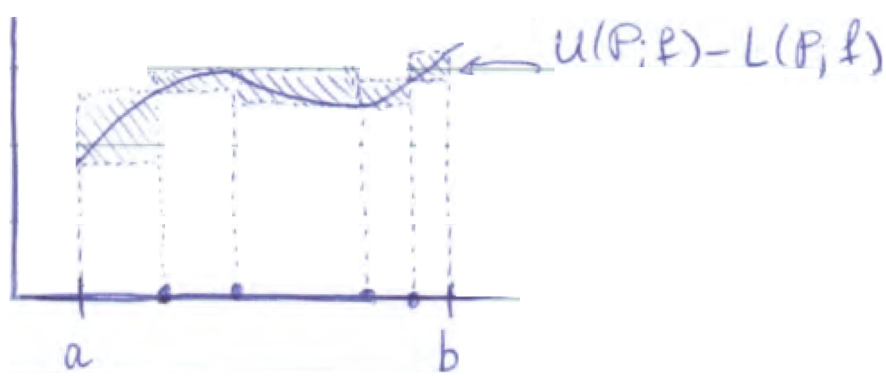
Conversely, let $\varepsilon > 0$ and P_ε be such that $U(P_\varepsilon; f) - L(P_\varepsilon; f) < \varepsilon$. Since

$$U(f) \leq U(P_\varepsilon; f) \quad \text{and} \quad L(f) \geq L(P_\varepsilon; f),$$

then

$$0 \leq U(f) - L(f) \leq U(P_\varepsilon; f) - L(P_\varepsilon; f) < \varepsilon.$$

But $\varepsilon > 0$ was arbitrary, so $U(f) - L(f) = 0$, which implies that $U(f) = L(f)$ and that f is Riemann-integrable on $[a, b]$. ■



In the illustration on the previous page (for a continuous function), the smaller the shaded area is, the closer $U(P;f)$ and $L(P;f)$ are to $\int_a^b f$.

There are 2 instances where the Riemann-integrability of a function f on $[a,b]$ is guaranteed: when f is **monotone**, and when it is **continuous**.

Theorem 53
Let $I=[a,b]$ and $f:I\to\mathbb{R}$ be a monotone function on I. Then f is Riemann-integrable on I.

Proof: we show that the result holds for increasing functions. The proof for decreasing functions is similar. Let

$$P_n=\{x_i=a+i\left(\tfrac{b-a}{n}\right)\mid i=0,\ldots,n\}$$

be the partition of I into n equal sub-intervals. Since f is increasing on I, we have, for $1\leq i\leq n$,

$$\begin{aligned} m_i &= \inf\{f(x)\mid x\in[x_{i-1},x_i]\}=f(x_{i-1}),\\ M_i &= \sup\{f(x)\mid x\in[x_{i-1},x_i]\}=f(x_i). \end{aligned}$$

Hence,

$$\begin{aligned} U(P_n;f)-L(P_n;f) &= \sum_{i=1}^n M_i(x_i-x_{i-1})-\sum_{i=1}^n m_i(x_i-x_{i-1})\\ &= \sum_{i=1}^n (M_i-m_i)(x_i-x_{i-1})\\ &= \frac{b-a}{n}\sum_{i=1}^n (f(x_i)-f(x_{i-1}))\\ &= \frac{b-a}{n}\Big[f(x_1)-f(x_0)+\cdots+f(x_n)-f(x_{n-1})\Big]\\ &= \frac{b-a}{n}(f(b)-f(a))\geq 0. \end{aligned}$$

Let $\varepsilon>0$. By the Archimedean property, $\exists N_\varepsilon\in\mathbb{N}$ such that

$$\frac{(b-a)(f(b)-f(a))}{\varepsilon}<n.$$

Set $P_\varepsilon=P_n$. Then

$$U(P_\varepsilon;f)-L(P_\varepsilon;f)<\frac{b-a}{N_\varepsilon}(f(b)-f(a))<\varepsilon,$$

and f is Riemann-integrable on $[a,b]$, according to Riemann's criterion. ■

Theorem 54 Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be continuous, with $a < b$. Then f is Riemann-integrable on I.

Proof: let $\varepsilon > 0$. According to Theorem 38, f is uniformly continuous on I. Hence $\exists \delta_\varepsilon > 0$ s.t. $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$ whenever $|x - y| < \delta_\varepsilon$ and $x, y \in [a, b]$.

Pick $n \in \mathbb{N}$ such that $\frac{b-a}{n} < \delta_\varepsilon$ and let

$$P_\varepsilon = \{x_i = a + i\left(\tfrac{b-a}{n}\right) \mid i = 0, \ldots, n\}$$

be the partition of $[a, b]$ into n equal sub-intervals.

As f is continuous on $[x_{i-1}, x_i]$, $\exists u_i, v_i \in [x_{i-1}, x_i]$ such that

$$m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\} = f(u_i), \quad M_i = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\} = f(v_i),$$

for all $1 \leq i \leq n$, according to the max/min Theorem. Since $|u_i - v_i| \leq \frac{b-a}{n} < \delta_\varepsilon$ for all i, we have:

$$\begin{aligned} U(P_\varepsilon; f) - L(P_\varepsilon; f) &= \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) = \frac{b-a}{n} \sum_{i=1}^{n} (f(v_i) - f(u_i)) \\ &< \frac{b-a}{n} \sum_{i=1}^{n} \frac{\varepsilon}{b-a} = \varepsilon, \end{aligned}$$

by uniform continuity of f. According to Theorem 52, f is Riemann-integrable. ■

4.2.2 Properties of the Riemann Integral

The Riemann integral has a whole slew of interesting properties.

Theorem 55 (Properties of the Riemann Integral)
Let $I = [a, b]$ and $f, g : I \to \mathbb{R}$ be Riemann-integrable on I. Then

1. *$f + g$ is Riemann-integrable on I, with $\int_a^b (f + g) = \int_a^b f + \int_a^b g$;*
2. *if $k \in \mathbb{R}$, $k \cdot f$ is Riemann-integrable on I, with $\int_a^b k \cdot f = k \int_a^b f$;*
3. *if $f(x) \leq g(x)\ \forall x \in I$, then $\int_a^b f \leq \int_a^b g$, and*
4. *if $|f(x)| \leq K\ \forall x \in I$, then $\left|\int_a^b f\right| \leq K(b - a)$.*

Proof: we use a variety of pre-existing results.

1. Let $\varepsilon > 0$. Since f, g are Riemann-integrable, $\exists P_1, P_2$ partitions of I such that $U(P_1; f) - L(P_1; f) < \frac{\varepsilon}{2}$ and $U(P_2; g) - L(P_2; g) < \frac{\varepsilon}{2}$.

   Set $P = P_1 \cup P_2$. Then P is a refinement of P_1 and P_2, and

   $$\begin{aligned} U(P; f+g) &\leq U(P; f) + U(P; g) \\ &< L(P; f) + L(P; g) + \varepsilon \leq L(P; f+g) + \varepsilon, \end{aligned}$$

   since, over non-empty subsets of I, we have

   $$\begin{aligned} \inf\{f(x) + g(x)\} &\geq \inf\{f(x)\} + \inf\{g(x)\} \\ \sup\{f(x) + g(x)\} &\leq \sup\{f(x)\} + \sup\{g(x)\}. \end{aligned}$$

   Hence $f + g$ is Riemann-integrable according to Riemann's criterion. Furthermore, we see from above that

   $$\int_a^b (f+g) \leq U(P; f+g) < L(P; f) + L(P; g) + \varepsilon \leq \int_a^b f + \int_a^b g + \varepsilon$$

   and

   $$\int_a^b f + \int_a^b g \leq U(P; f) + U(P; g) < L(P; f+g) + \varepsilon \leq \int_a^b (f+g) + \varepsilon.$$

   Since $\varepsilon > 0$ is arbitrary, $\int_a^b f + \int_a^b g \leq \int_a^b (f+g) \leq \int_a^b f + \int_a^b g$, from which we conclude that $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

2. The proof for $k = 0$ is trivial. We show that the result holds for $k < 0$ (the proof for $k > 0$ is similar). Let $P = \{x_0, \ldots, x_n\}$ be a partition of I.

   Since $k < 0$, we have $\inf\{kf(x)\} = k\sup\{f(x)\}$ over non-empty subsets of I, and so we have $L(P; kf) = kU(P; f)$. In particular,

   $$\begin{aligned} L(kf) &= \sup\{L(P; kf) \mid P \text{ a partition of } I\} \\ &= \sup\{kU(P; f) \mid P \text{ a partition of } I\} \\ &= k\inf\{U(P; f) \mid P \text{ a partition of } I\} = kU(f) \end{aligned}$$

   Similarly, $U(P; kf) = kL(P; f)$ and $U(fk) = kL(f)$, so

   $$L(fk) = \underbrace{kU(f) = kL(f)}_{\text{since } f \text{ is R-int.}} = U(kf).$$

   Thus kf is Riemann-integrable on I and $\int_a^b kf = L(k) = kU(f) = \int_a^b f$.

3. We start by showing that if $h : I \to \mathbb{R}$ is integrable on I and $h(x) \geq 0$ for all $x \in I$, then $\int_a^b h(x) \geq 0$. Let $P_0 = \{a, b\} = \{x_0, x_1\}$ and $m_1 = \inf\{h(x) \mid x \in [a, b]\} \geq 0$. Then,

$$0 \leq m_1(b-a) = L(P_0; h) \leq L(P; h)$$

for any partition P of I, as $P \supseteq P_0$. But h is Riemann-integrable by assumption, thus

$$\int_a^b h = \sup\{L(P; h) \mid P \text{ a partition of } I\} \geq L(P_0; h) \geq 0.$$

Then, set $h = g - f$. By hypothesis, $h(x) = g(x) - f(x) \geq 0$. Then

$$\int_a^b h = \int_a^b (g - f) = \int_a^b g - \int_a^b f \geq 0,$$

which implies that $\int_a^b g \geq \int_a^b f$.

4. Let $P_0 = \{a, b\} = \{x_0, x_1\}$. As always, set $m_1 = \inf\{f(x) \mid x \in [a, b]\}$, and $M_1 = \sup\{f(x) \mid x \in [a, b]\}$. Then for any partition P of I, we have

$$\begin{aligned} m_1(b-a) &= L(P_0; f) \leq L(P; f) \leq L(f) = \int_a^b f \\ &= U(f) \leq U(P; f) \leq U(P_0; f) = M_1(b-a). \end{aligned}$$

In particular,

$$m_1(b-a) \leq \int_a^b f \leq M_1(b-a).$$

Now, if $|f(x)| \leq K$ for all $x \in I$, then $-K \leq m_1$ and $M_1 \leq K$ so that

$$-K(b-a) \leq m_1(b-a) \leq \int_a^b f \leq M_1(b-a) \leq K(b-a),$$

so that $|\int_a^b f| \leq K(b-a)$. ■

When all the functions involved are non-negative, these results and the next one are compatible with the calculus interpretation of the Riemann integral as the **area under the curve**.

Theorem 56 (Additivity of the Riemann Integral)
Let $I = [a, b]$, $c \in (a, b)$, and $f : I \to \mathbb{R}$ be bounded on I. Then f is Riemann-integrable on I if and only if it is Riemann-integrable on $I_1 = [a, c]$ and on $I_2 = [c, b]$. When that is the case, $\int_a^b f = \int_a^c f + \int_c^b f$.

Proof: we start by assuming that f is Riemann-integrable on I. Let $\varepsilon > 0$. According to the Riemann criterion, $\exists P_\varepsilon$ a partition of I such that $U(P_\varepsilon; f) - L(P_\varepsilon; f) < \varepsilon$. Now, set $P = P_\varepsilon \cup \{c\}$. Then P is a refinement of P_ε so that

$$U(P; f) - L(P; f) \le U(P_\varepsilon; f) - L(P_\varepsilon; f) < \varepsilon.$$

Set $P_1 = P \cap I_1$ and $P_2 = P \cap I_2$. Then P_i is a partition of I_i, and

$$\begin{aligned}\varepsilon > U(P; f) - L(P; f) &\ge U(P_1; f) + U(P_2; f) - L(P_1; f) - L(P_2; f)\\ &= \big[U(P_1; f) - L(P_1; f)\big] + \big[U(P_2; f) - L(P_2; f)\big]\end{aligned}$$

Consequently, $U(P_i; f) - L(P_i; f) < \varepsilon$ for $i = 1, 2$ and f is Riemann-integrable on I_1 and I_2, according to the Riemann criterion.

Now assume that f is Riemann-integrable on I_1 and I_2. Let $\varepsilon > 0$. According to the Riemann criterion, for $i = 1, 2$, $\exists P_i$ a partition of I_i such that

$$U(P_i; f) + L(P_i; f) < \frac{\varepsilon}{2}.$$

Set $P = P_1 \cup P_2$. Then P is a partition of I. Furthermore,

$$\begin{aligned}U(P; f) - L(P; f) &= U(P_1; f) + U(P_2; f) - L(P_1; f) - L(P_2; f)\\ &= U(P_1; f) - L(P_1; f) + U(P_2; f) - L(P_2; f) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\end{aligned}$$

thus f is Riemann-integrable on I according the Riemann criterion.

Finally, let's assume that f is Riemann-integrable on I (and so on I_1, I_2), or vice-versa. Let P_1, P_2 be partitions of I_1, I_2, respectively, such that

$$U(P_i; f) - L(P_i; f) < \frac{\varepsilon}{2}, \quad i = 1, 2.$$

Set $P = P_1 \cup P_2$. Then P is a partition of I and

$$\begin{aligned}\int_a^b f &\le U(P; f) = U(P_1; f) + U(P_2; f)\\ &< L(P_1; f) + L(P_2; f) + \varepsilon = \int_a^c f + \int_c^b f + \varepsilon.\end{aligned}$$

Similarly,

$$\int_a^b f \ge L(P_1; f) + L(P_2; f) > U(P_1; f) + U(P_2; f) - \varepsilon \ge \int_a^c f + \int_c^b f - \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, $\int_a^b f = \int_a^c f + \int_c^b f$. ■

The next theorem is the crowning achievement of what has come before, combining results from previous chapters and sections. Its proof constitutes the first "real" example of what we might as well refer to as **analytical reasoning**.

Theorem 57 (Composition Theorem for Integrals)
Let $I = [a,b]$ and $J = [\alpha,\beta]$, $f : I \to \mathbb{R}$ Riemann-integrable on I, $\varphi : J \to \mathbb{R}$ continuous on J and $f(I) \subseteq J$. Then $\varphi \circ f : I \to \mathbb{R}$ is Riemann-integrable on I.

Proof: let $\varepsilon > 0$, $K = \sup\{|\varphi(x)| \mid x \in J\}$ (wich is guaranteed to exist according to the max/min theorem) and $\varepsilon' = \frac{\varepsilon}{b-a+2K}$.

Since φ is uniformly continuous on J (being continuous on a closed, bounded interval), $\exists \delta_\varepsilon > 0$ s.t.

$$|x-y| < \delta_\varepsilon,\ x, y, \in J \Longrightarrow |\varphi(x) - \varphi(y)| < \varepsilon'.$$

Without loss of generality, pick $\delta_\varepsilon < \varepsilon'$.

Since f is Riemann-integrable on I, there is a partition $P = \{x_0, \ldots, x_n\}$ of $I = [a,b]$ such that

$$U(P;f) - L(P;f) < \delta_\varepsilon^2$$

(according to Riemann's criterion).

We show that $U(P;\varphi \circ f) - L(P;\varphi \circ f) < \varepsilon$, and so that $\varphi \circ f$ is Riemann-integrable according to Riemann's criterion. On $[x_{i-1}, x_i]$ for $i = 1, \ldots, n$, set

$$m_i = \inf\{f(x)\},\ M_i = \sup\{f(x)\},\ \tilde{m}_i = \inf\{\varphi(f(x))\},\ \tilde{M}_i = \sup\{\varphi(f(x))\}.$$

With those, set $A = \{i \mid M_i - m_i < \delta_\varepsilon\}$, $B = \{i \mid M_i - m_i \geq \delta_\varepsilon\}$.

- If $i \in A$, then

$$x, y \in [x_{i-1}, x_i] \Longrightarrow |f(x) - f(y)| \leq M_i - m_i < \delta_\varepsilon,$$

  so $|\varphi(f(x)) - \varphi(f(y)| < \varepsilon'\ \forall x, y \in [x_{i-1}, x_i]$. In particular, $\tilde{M}_i - \tilde{m}_i \leq \varepsilon'$.

- If $i \in B$, then

$$x, y \in [x_{i-1}, x_i] \Longrightarrow |\varphi(f(x)) - \varphi(f(y))| \leq |\varphi(f(x))| + |\varphi(f(y))| \leq 2K.$$

  In particular, $\tilde{M}_i - \tilde{m}_i \leq 2K$, since $-K \leq \tilde{m}_i \leq \varphi(z) \leq \tilde{M}_i \leq K$ for all $z \in [x_{i-1}, x_i]$.

Then

$$\begin{aligned}U(P;\varphi\circ f)-L(P;\varphi\circ f) &= \sum_{i=1}^{n}(\tilde{M}_i-\tilde{m}_i)(x_i-x_{i-1})\\ &= \sum_{i\in A}(\tilde{M}_i-\tilde{m}_i)(x_i-x_{i-1})+\sum_{i\in B}(\tilde{M}_i-\tilde{m}_i)(x_i-x_{i-1})\\ &\leq \varepsilon'\sum_{i\in A}(x_i-x_{i-1})+2K\sum_{i\in B}(x_i-x_{i-1})\\ &\leq \varepsilon'(b-a)+2K\sum_{i\in B}\frac{(M_i-m_i)}{\delta_\varepsilon}(x_i-x_{i-1})\\ &= \varepsilon'(b-a)+\frac{2K}{\delta_\varepsilon}\sum_{i=1}^{n}(M_i-m_i)(x_i-x_{i-1}).\end{aligned}$$

By earlier work in the proof, we have

$$\sum_{i=1}^{n}(M_i-m_i)(x_i-x_{i-1})\leq U(P;f)-L(P;f)<\delta_\varepsilon^2,$$

so that

$$\begin{aligned}U(P;\varphi\circ f)-L(P;\varphi\circ f) &< \varepsilon'(b-a)+\frac{2K}{\delta_\varepsilon}\cdot\delta_\varepsilon^2\\ &= \varepsilon'(b-a)+2K\delta_\varepsilon<\varepsilon'(b-a)+2K\varepsilon'\\ &= \varepsilon'(b-a+2K)=\varepsilon,\end{aligned}$$

which completes the proof. ■

The proof of the composition theorem requires the intervals I and J to be closed, as the following example shows.

Example: let $f,\varphi:(0,1)\to\mathbb{R}$ be defined by $f(x)=x$ and $\varphi(x)=\frac{1}{x}$. Then f is Riemann-integrable on $(0,1)$, φ is continuous on $(0,1)$, but $\varphi\circ f:(0,1)\to\mathbb{R}$, $(\varphi\circ f)(x)=1/x$, is not Riemann-integrable on $(0,1)$. □

Note, however, that there are examples of functions defined on open intervals for which the conclusion of the composition theorem still holds.

Example: let $f,\varphi:(0,1)\to\mathbb{R}$ be defined by $f(x)=x$ and $\varphi(x)=x$. Then f is Riemann-integrable on $(0,1)$, φ is continuous on $(0,1)$, and $\varphi\circ f:(0,1)\to\mathbb{R}$, $(\varphi\circ f)(x)=x$, is Riemann-integrable on $(0,1)$. □

Theorem 57 is rather technical, but it can be used to show a variety of results.

Theorem 58
Let $I = [a,b]$ and $f, g : I \to \mathbb{R}$ be Riemann-integrable on I. Then fg and $|f|$ are Riemann-integrable on I, and $\left|\int_a^b f\right| \leq \int_a^b |f|$.

Proof: the function defined by $\varphi(t) = t^2$ is continuous. by the Composition theorem, $\varphi \circ (f+g) = (f+g)^2$ and $\varphi \circ (f-g) = (f-g)^2$ are both Riemann-integrable on I. But the product fg can be re-written as

$$fg = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right].$$

According to Theorem 55, fg is Riemann-integrable on I.

Now, consider the function defined by $\varphi(t) = |t|$. It is continuous, so $\varphi \circ f = |f|$ is Riemann-integrable on I according to the composition theorem.

Pick $c \in \{\pm 1\}$ such that $c\int_a^b f \geq 0$. Hence

$$\left|\int_a^b f\right| = c\int_a^b f = \int_a^b cf \leq \int_a^b |f|,$$

since $cf(x) \leq |f(x)|$ for all $x \in I$. ■

Note that even if the product of Riemann-integrable functions is itself Riemann-integrable there is no simple way to express $\int_a^b fg$ in terms of $\int_a^b f$ and $\int_a^b g$.

Given all that has come so far, we might suspect that the composition of Riemann-integrable functions is also Riemann-integrable. The following counter-example shows that this **need not be the case**.

Example: let $I = [0,1]$ and let $f : I \to \mathbb{R}$ be **Thomae's function**:

$$f(x) = \begin{cases} 1, & x = 0 \\ 1/n, & x = m/n \in \mathbb{Q},\ \text{gcd}(m,n) = 1 \\ 0, & x \notin \mathbb{Q} \end{cases}$$

It can be shown that f is Riemann-integrable on $[0,1]$ and that $\int_0^1 f = 0$. Consider the function $g : [0,1] \to \mathbb{R}$ defined by $g(x) \equiv 1$ on $(0,1]$ and $g(0) = 0$. Then g is Riemann-integrable on $[0,1]$, with $\int_0^1 g = 1$, but $g \circ f : [0,1] \to \mathbb{R}$ is the **Dirichlet function**, and is therefore not Riemann-integrable on $[0,1]$. □

4.2.3 Fundamental Theorem of Calculus

With Descartes' creation of analytical geometry, it became possible to find the **tangents** to curves that are **algebraically** described.[6] Fermat then showed the connection between that problem and the problem of finding the **maximum/minimum** of a (continuous) function. In the 1680s, Newton and Leibniz eventually discovered that computing the **area underneath a curve** is exactly the opposite of finding the tangent. **Calculus** provided a general framework to solve problems that had hitherto been very difficult to solve.[7] In this section, we study the connection between these concepts.

Theorem 59 (FUNDAMENTAL THEOREM OF CALCULUS, 1ST VERSION)
Let $I = [a,b]$, $f : I \to \mathbb{R}$ be Riemann-integrable on I, and $F : I \to \mathbb{R}$ be such that F is continuous on I and differentiable on (a,b). If $F'(x) = f(x)$ for all $x \in (a,b)$, then $\int_a^b f = F(b) - F(a)$.

Proof: let $\varepsilon > 0$. Since f is Riemann-integrable on I, $\exists P_\varepsilon \in \mathcal{P}(I)$ such that

$$U(P_\varepsilon; f) - L(P_\varepsilon; f) < \varepsilon.$$

Applying the mean value theorem to F on $[x_{i-1}, x_i]$ for each $1 \leq i \leq n$, we conclude that $\exists t_i \in (x_{i-1}, x_i)$ such that

$$\frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}} = F'(t_i) = f(t_i), \quad 1 \leq i \leq n.$$

Let $\tilde{m}_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}$, $\tilde{M}_i = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\}$ for $1 \leq i \leq n$. Then

$$L(P_\varepsilon; f) = \sum_{i=1}^n \tilde{m}_i (x_i - x_{i-1}) \leq \sum_{i=1}^n f(t_i)(x_i - x_{i-1}) = \sum_{i=1}^n (F(x_i) - F(x_{i-1})) = F(b) - F(a),$$

and, similarly, $U(P_\varepsilon; f) \geq F(b) - F(a)$. Then $L(P_\varepsilon; f) \leq F(b) - F(a) \leq U(P_\varepsilon; f)$ for all $\varepsilon > 0$. Since we have

$$L(P_\varepsilon; f) \leq \int_a^b f \leq U(P_\varepsilon; f)$$

and $U(P_\varepsilon; f) - L(P_\varepsilon; f) < \varepsilon$, for all $\varepsilon > 0$, we must also have

$$\left| \int_a^b f - (F(b) - F(a)) \right| < \varepsilon, \quad \text{for all } \varepsilon > 0,$$

so that $\int_a^b f = F(b) - F(a)$. ■

[6]That is, curves who can be expressed in $\mathbb{R}^2$ as $f(x,y) = 0$ for algebraic functions f.
[7]And even then, only in specific circumstances.

This classical calculus result is quite useful in applications,[8] as is its cousin.

Theorem 60 (Fundamental Theorem of Calculus, 2nd version)
Let $I = [a, b]$, $f : I \to \mathbb{R}$ be Riemann-integrable on I. Define a function $F : I \to \mathbb{R}$ by $F(x) = \int_a^x f$. Then F is continuous on I. Furthermore, if f is continuous at $c \in (a, b)$, then F is differentiable at c and $F'(c) = f(c)$.

Proof: since f is Riemann-integrable on I, then f is bounded on I. Let $K > 0$ be such that $|f(x)| < K$ for all $x \in I$. Let $x \in I$ and $\varepsilon > 0$. Set $\delta_\varepsilon = \frac{\varepsilon}{K}$. Then whenever $|x - y| < \delta_\varepsilon = \frac{\varepsilon}{K}$ and $y \in I$, we have

$$|F(y) - F(x)| = \left|\int_a^y f - \int_a^x f\right| = \left|\int_x^y f\right| \leq K|x - y| < \varepsilon.$$

Then F is uniformly continuous on I, and so is continuous on I. Now assume that f is continuous at c and let $\varepsilon > 0$. Then $\exists \delta_\varepsilon > 0$ such that $|f(x) - f(c)| < \varepsilon$ whenever $|x - c| < \delta_\varepsilon$ and $x \in I$.

Thus, if $0 \leq |h| = |x - c| < \delta_\varepsilon$ and $x \in I$, we have

$$\begin{aligned}
\left|\frac{F(c+h) - F(c)}{h} - f(c)\right| &= \left|\frac{1}{h}\int_a^{c+h} f - \frac{1}{h}\int_a^c f - f(c)\right| \\
&= \left|\frac{1}{h}\int_c^{c+h} f - \frac{1}{h}\int_c^{c+h} f(c)\right| = \left|\frac{1}{h}\int_c^{c+h}(f - f(c))\right| \\
&\leq \frac{1}{|h|}\left|\int_c^{c+h}(f - f(c))\right| < \frac{1}{|h|}\cdot\varepsilon\left|\int_c^{c+h} 1\right| = \frac{1}{|h|}\cdot\varepsilon|h| = \varepsilon,
\end{aligned}$$

which is to say, $F'(c) = f(c)$. ■

The first version of the fundamental theorem of calculus provides a justification of the **method used to evaluate definite integrals** in calculus; the second version, which allows the upper bound of the Riemann integral to vary, provides a basis for finding antiderivatives.

Let $I = [a, b]$ an $f : I \to \mathbb{R}$. An **antiderivative** of f on I is a differentiable function $F : I \to \mathbb{R}$ such that $F'(x) = f(x)$ for all $x \in I$. If f is Riemann-integrable on I, the function $F : I \to \mathbb{R}$ defined by $F(x) = \int_a^x f$ for $x \in I$ is the **indefinite integral of f on I.** If f is Riemann-integrable on I and if F is an antiderivative of f on I, then

$$\int_a^b f = F(b) - F(a).$$

However, Riemann-integrable functions on I may not have antiderivatives on I (such as the signum and Thomae's functions), and functions with antiderivatives may not be Riemann-integrable on I (such as the reciprocal of the square root function on $[0, 1]$).

[8] We will see in Chapter 14 that the Theorem 59 (1st version) is a special case of a more general result.

If f is Riemann-integrable on I, then $F(x) = \int_a^x f$ exists. Moreover, if f is continuous on I, than F is an antiderivative of f on I, since $F'(x) = f(x)$ for all $x \in I$. Continuous functions thus **always have antiderivatives**.[9]

But if f is not continuous on I, the indefinite integral F may not be an antiderivative of f on I – it may fail to be differentiable at certain points of I, or F' may exists but be different from f at various points of I.

4.2.4 Evaluation of Integrals

We complete this chapter by presenting some common methods used to evaluate integrals, and the proof for two of them.

Theorem 61 (Integration by Parts)
Let $f, g : [a,b] \to \mathbb{R}$ both be Riemann-integrable on $[a,b]$, with antiderivatives $F, G : [a,b] \to \mathbb{R}$, respectively. Then

$$\int_a^b Fg = F(b)G(b) - F(a)G(a) - \int_a^b fG.$$

Proof: Let $H : [a,b] \to \mathbb{R}$ be defined by $H = FG$. As F and G are both differentiable, so is H: $H' = F'G + FG' = fG + Fg$.

Then $\int_a^b H' = H(b) - H(a)$, so

$$\int_a^b (fG + Fg) = F(b)G(b) - F(a)G(a) \Longrightarrow \int_a^b Fg = H(b) - H(a) - \int_a^b fG.$$

This completes the proof. ■

Theorem 62 (First Substitution Theorem)
Let $J = [\alpha, \beta]$, and $\varphi \to \mathbb{R}$ be a function with a continuous derivative on J. If $f : I \to \mathbb{R}$ is continuous on $I = [a,b] \supseteq \varphi(J)$, then

$$\int_\alpha^\beta (f \circ \varphi)\varphi' = \int_{\varphi(\alpha)}^{\varphi(\beta)} f.$$

Proof: Since f is continuous on I, it is Riemann-integrable on I and so we can define a function $F : I \to \mathbb{R}$ through

$$F(x) = \int_{\varphi(\alpha)}^x f, \quad x \in I.$$

By construction F is continuous and differentiable on I. Furthermore, $F' = f$ on I, according to the second version of the fundamental theorem of calculus.

[9] Even if they can't be expressed using elementary functions.

Define $H : J \to \mathbb{R}$ by $H = F \circ \varphi$. Then H is differentiable on I, being the composition of two differentiable functions on I, and $H' = (F' \circ \varphi)\varphi' = (f' \circ \varphi)\varphi'$ is Riemann-integrable since $\varphi, f \circ \varphi$ are Riemann-integrable (being continuous) on I, according to Theorem 58. The first version of the Fundamental Theorem of Calculus then yields

$$\int_\alpha^\beta (f \circ \varphi)\varphi' = \int_\alpha^\beta H' = H(\beta) - H(\alpha) = F(\varphi(\beta)) - F(\varphi(\alpha)) = \int_{\varphi(\alpha)}^{\varphi(\beta)} f,$$

which completes the proof. ■

The proofs of the last three theorems are left as an exercise.

Theorem 63 (Second Substitution Theorem)
Let $J = [\alpha, \beta]$, and $\varphi \to \mathbb{R}$ be a function with a continuous derivative on J and such that $\varphi' \neq 0$ on J. Let $I = [a, b] \supseteq \varphi(J)$, and $\psi : I \to \mathbb{R}$ be the inverse of φ (which exists as φ is montoone). If $f : I \to \mathbb{R}$ is continuous on I, then

$$\int_\alpha^\beta f \circ \varphi = \int_{\varphi(\alpha)}^{\varphi(\beta)} f\psi'.$$

Theorem 64 (Mean Value Theorem for Integrals)
Let $I = [a, b]$, $f : I \to \mathbb{R}$ be continuous on I, and $p : I \to \mathbb{R}$ be Riemann-integrable on I, with $p \geq 0$ on I. Then $\exists c \in (a, b)$ such that

$$\int_a^b fp = f(c) \int_a^b p.$$

Theorem 65 (Squeeze Theorem for Integrals)
Let $I = [a, b]$ and $f \leq g \leq h : I \to \mathbb{R}$ be bounded on I. If f, h are Riemann-integrable on I with $\int_a^b f = \int_a^b h$, then g is Reimann-integrable on I and $\int_a^b g = \int_a^b f = \int_a^b h$.

4.3 Solved Problems

1. Use the definition to find the derivative of the function defined by $g(x) = \frac{1}{x}$, $x \in \mathbb{R}$, $x \neq 0$.

   Solution: from calculus, we "know" that $g'(x) = -\frac{1}{x^2}$. Let $c \in \mathbb{R}$ s.t. $c \neq 0$. Set $a_c = \frac{c}{2}$ and $b_c = \frac{3c}{2}$. Clearly, if $c > 0$, $0 < a_c < c < b_c$, whereas $b_c < c < a_c < 0$ if $c < 0$. In both cases, $\frac{1}{|x|} \leq \frac{1}{|a_c|}$ whenever x lies between a_c and b_c. We restrict g on the interval between a_c and b_c (denote this interval by A).

   Let $\varepsilon > 0$ and set $\delta_\varepsilon = |a_c|c^2\varepsilon$. Then whenever $0 < |x - c| < \delta_\varepsilon$ and $x \in A$, we have

   $$\left|\frac{\frac{1}{x} - \frac{1}{c}}{x - c} + \frac{1}{c^2}\right| = \left|\frac{c - x}{xc(x - c)} + \frac{1}{c^2}\right| = \left|\frac{1}{c^2} - \frac{1}{xc}\right| = \frac{|x - c|}{|x|c^2} \leq \frac{|x - c|}{|a_c|c^2} < \frac{\delta_\varepsilon}{|a_c|c^2} = \varepsilon,$$

   which validates our calculus guess. □

2. Prove that the derivative of an even differentiable function is odd, and vice-versa.

   Proof: if f is even, then $f(x) = f(-x)$ for all $x \in \mathbb{R}$. Let $g(x) = f(-x)$. Then g is differentiable by the chain rule and $f(x) = g(x)$ for all $x \in \mathbb{R}$. Furthermore,

   $$f'(x) = g'(x) = (f(-x))' = f'(-x) \cdot -1,$$

   that is, $-f'(-x) = f'(-x)$, or f' is odd. The other statement is proved similarly. ■

3. Let $a > b > 0$ and $n \in \mathbb{N}$ with $n \geq 2$. Show that $a^{1/n} - b^{1/n} < (a - b)^{1/n}$.

   Proof: consider the continuous function $f : [1, \infty) \to \mathbb{R}$ defined by $f(x) = x^{1/n} - (x - 1)^{1/n}$, whose derivative is

   $$f'(x) = \frac{1}{n}\left(x^{\frac{1-n}{n}} - (x - 1)^{\frac{1-n}{n}}\right).$$

   Now,

   $$0 \leq x - 1 < x, \quad \forall x \geq 1 \Longrightarrow 0 \leq (x - 1)^n < x^n, \quad \forall x \geq 1, n \geq 2$$
   $$\therefore 0 \leq (x - 1)^{\frac{n}{n-1}} < x^{\frac{n}{n-1}}, \quad \forall x \geq 1, n \geq 2$$

   and so

   $$\frac{1}{x^{\frac{n}{n-1}}} < \frac{1}{(x - 1)^{\frac{n}{n-1}}},$$

   or $x^{\frac{1-n}{n}} < (x - 1)^{\frac{1-n}{n}}$ for all $x \geq 1, n \geq 2$.

   Hence $f'(x) < 0$ for all $x \in [1, \infty)$, that is f is strictly decreasing over $[1, \infty)$. But $f(\frac{a}{b}) < f(1)$, as $\frac{a}{b} > 1$. But

   $$f\left(\frac{a}{b}\right) = \left(\frac{a}{b}\right)^{\frac{1}{n}} - \left(\frac{a}{b} - 1\right)^{\frac{1}{n}} = \frac{1}{b^{\frac{1}{n}}}\left(a^{\frac{1}{n}} - (a - b)^{\frac{1}{n}}\right)$$

and $f(1) = 1$, so

$$\frac{1}{b^{\frac{1}{n}}}\left(a^{\frac{1}{n}} - (a-b)^{\frac{1}{n}}\right) < 1,$$

that is $a^{\frac{1}{n}} - (a-b)^{\frac{1}{n}} < b^{\frac{1}{n}}$, which completes the proof. ■

4. Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Show that if $\lim_{x\to a} f'(x) = A$, then $f'(a)$ exists and equals A.

   Proof: let $x \in (a,b)$. By the Mean Value Theorem, $\exists c_x \in (a,x)$ s.t.

   $$\frac{f(x)-f(a)}{x-a} = f'(c_x).$$

   When $x \to a$, $c_x \to a$ (indeed, let $\varepsilon > 0$ and set $\delta_\varepsilon = \varepsilon$; then $|c_x - a| < |x-a| < \delta_\varepsilon = \varepsilon$ whenever $0 < |x-a| < \delta_\varepsilon$). Then

   $$\lim_{x\to a} f'(c_x) = \lim_{c_x \to a} f'(c_x) = A$$

   by hypothesis. Hence $\lim_{x\to a} f'(x)$ exists and so

   $$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} f'(x) = A$$

   exists. ■

5. If $x > 0$, show $1 + \frac{1}{2}x - \frac{1}{8}x^2 \leq \sqrt{1+x} \leq 1 + \frac{1}{2}x$.

   Proof: let $x_0 = 0$ and $f(x) = \sqrt{1+x}$. According to Taylor's theorem, since f is C^3 when $x > 0$, $f(x) = P_1(x) + R_1(x)$ and $f(x) = P_2(x) + R_2(x)$, where

   $$\begin{aligned}
   P_1(x) &= f(x_0) + f'(x_0)(x-x_0) = \sqrt{1+0} + \frac{1}{2\sqrt{1+0}}x = 1 + \frac{1}{2}x \\
   P_2(x) &= P_1(x) + \frac{f''(x_0)}{2}(x-x_0)^2 = 1 + \frac{1}{2}x - \frac{1}{8\sqrt[3]{1+0}}x^2 = 1 + \frac{1}{2}x - \frac{1}{8}x^2 \\
   R_1(x) &= \frac{f''(c_1)}{2}(x-x_0)^2 = -\frac{1}{8\sqrt[3]{1+c_1}}x^2, \quad \text{for some } c_1 \in [0,x] \\
   R_2(x) &= \frac{f'''(c_2)}{6}(x-x_0)^3 = \frac{3}{48\sqrt[5]{1+c_2}}x^3, \quad \text{for some } c_2 \in [0,x].
   \end{aligned}$$

   When $x > 0$, $R_1(x) \leq 0$ and $R_2(x) \geq 0$, so $P_2(x) \leq f(x) \leq P_1(x)$. ■

6. Let $a \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be defined by

   $$f(x) = \begin{cases} x^2 & \text{if } x \geq 0, \\ ax & \text{if } x < 0. \end{cases}$$

   For which values of a is f differentiable at $x = 0$? For which values of a is f continuous at $x = 0$?

Solution: We have

$$f'_+(0) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{x^2}{x} = \lim_{x\to 0^+} x = 0$$

and

$$f'_-(0) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{ax}{x} = \lim_{x\to 0^+} a = a.$$

Thus, f is differentiable at $x = 0$ if and only if $a = 0$.

Since both x^2 and ax are continuous functions, we have

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 = 0 = f(0) = 0 = \lim_{x\to 0^-} ax = \lim_{x\to 0^-} f(x)$$

and the the function f is continuous at $x = 0$ for all values of a. □

7. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Show that f is Lipschitz if and only if f' is bounded on (a, b).

Proof: Suppose that f satisfies the Lipschitz condition on $[a, b]$ with constant M. Then, for all $x_0 \in (a, b)$, we have

$$\left|\frac{f(x)-f(x_0)}{x-x_0}\right| \le M \qquad \forall\, x \in (a,b) \setminus \{x_0\}.$$

Thus

$$|f'(x_0)| = \left|\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}\right| = \lim_{x\to x_0} \left|\frac{f(x)-f(x_0)}{x-x_0}\right| \le M,$$

where we used the fact that the absolute value function is continuous to pull the limit out of the absolute value. So the derivative of f is bounded on (a, b).

Now assume that $|f'(x)| \le M$ for all $x \in (a, b)$. Let $x, y \in [a, b]$, $x < y$. Applying the Mean Value Theorem to f on the interval $[x, y]$ yields the existence of $c \in (x, y)$ such that

$$\frac{f(y)-f(x)}{y-x} = f'(c).$$

Thus

$$\left|\frac{f(x)-f(y)}{x-y}\right| \le M \Longrightarrow |f(x)-f(y)| \le M|x-y|.$$

This completes the proof. ■

8. Prove that $\int_0^1 g = \frac{1}{2}$ if

$$g(x) = \begin{cases} 1 & x \in (\frac{1}{2}, 1] \\ 0 & x \in [0, \frac{1}{2}] \end{cases}.$$

Is that still true if $g(\frac{1}{2}) = 7$ instead?

Proof: let $\varepsilon > 0$ and define the partition $P_\varepsilon = \{0, \frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon, 1\}$. Since g is bounded on $[0,1]$, $L(g) \leq U(g)$ exist and

$$L(g) \geq L(P_\varepsilon; g) = \frac{1}{2} - \varepsilon \quad \text{and} \quad U(g) \leq U(P_\varepsilon; g) = \frac{1}{2} + \varepsilon.$$

Hence

$$\frac{1}{2} - \varepsilon \leq L(g) \leq U(g) \leq \frac{1}{2} + \varepsilon, \quad \text{for all } \varepsilon > 0.$$

Since $\varepsilon > 0$ is arbitrary, then $\frac{1}{2} \leq L(g) \leq U(g) \leq \frac{1}{2}$; by definition, g is Riemann-integrable on $[0,1]$ and $L(g) = U(g) = \int_0^1 g = \frac{1}{2}$.

If instead $g(1/2) = 7$, the exact same work as above yields

$$\frac{1}{2} - \varepsilon \leq L(g) \leq U(g) \leq \frac{1}{2} + 13\varepsilon, \quad \text{for all } \varepsilon > 0.$$

Since $\varepsilon > 0$ is arbitrary, then $\frac{1}{2} \leq L(g) \leq U(g) \leq \frac{1}{2}$; by definition, g is also Riemann-integrable on $[0,1]$ and $L(g) = U(g) = \int_a^b f = \frac{1}{2}$. ■

9. Let $f : [a,b] \to \mathbb{R}$ be bounded and such that $f(x) \geq 0, \forall x \in [a,b]$. Show $L(f) \geq 0$.

Proof: as f is bounded on $[a,b]$, $L(f)$ exists and the set

$$\{f(x) \mid x \in [a,b]\} \neq \varnothing$$

is bounded below. By completeness of $\mathbb{R}$, $m_1 = \inf\{f(x) \mid x \in [a,b]\}$ exists. Furthermore, $m_1 \geq 0$ since $f(x) \geq 0$ for all $x \in [a,b]$.

Let $P = \{x_0, x_1\} = \{a, b\}$ be the trivial partition of $[a,b]$. Then

$$L(f) \geq L(P; f) = m_1(b-a) \geq 0,$$

which completes the proof. ■

10. Let $f : [a,b] \to \mathbb{R}$ be increasing on $[a,b]$. If P_n partitions $[a,b]$ into n equal parts, show that

$$0 \leq U(P_n; f) - \int_a^b f \leq \frac{f(b) - f(a)}{n}(b-a).$$

Proof: as f is increasing, it is monotone and thus Riemann-integrable by Theorem 53. Then $L(f) = U(f) = \int_a^b f$. Let

$$P_n = \{x_i = a + i\tfrac{b-a}{n} \mid i = 0, \ldots, n\}$$

be the partition of $[a,b]$ into n equal sub-intervals. By definition, $L(P_n; f) \leq \int_a^b f$ and $U(P_n; f) \geq \int_a^b f$. Then

$$U(P_n; f) - L(P_n; f) \geq U(P_n; f) - \int_a^b f \geq \int_a^b f - \int_a^b f = 0.$$

In particular, $U(P_n; f) - \int_a^b f \geq 0$. As f is increasing on $[a,b]$,

$$M_i = \sup_{[x_{i-1},x_i]} \{f(x)\} = f(x_i), \quad m_i = \inf_{[x_{i-1},x_i]} \{f(x)\} = f(x_{i-1}), \quad \text{and}$$

$$\begin{aligned} U(P_n; f) - L(P_n; f) &= \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) \\ &= \frac{b-a}{n} \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) = \frac{b-a}{n}(f(b) - f(a)). \end{aligned}$$

Since $L(P_n; f) \leq \int_a^b f$, then

$$\frac{b-a}{n}(f(b) - f(a)) = U(P_n; f) - L(P_n; f) \geq U(P_n; f) - \int_a^b f \geq 0,$$

which completes the proof. ■

11. Let $f : [a,b] \to \mathbb{R}$ be an integrable function and let $\varepsilon > 0$. If P_ε is the partition whose existence is asserted by the Riemann Criterion, show that $U(P; f) - L(P; f) < \varepsilon$ for all refinement P of P_ε.

    Proof: let P be a refinement of P_ε. Then $U(P_{\varepsilon;f}) \geq U(P; f)$ and $L(P_\varepsilon; f) \leq L(P; f)$, and so
    $$U(P_\varepsilon; f) \geq U(P; f) \geq L(P; f) \geq L(P_\varepsilon; f).$$
    By the Riemann Criterion, $U(P_\varepsilon; f) < \varepsilon + L(P_\varepsilon; f)$. Then
    $$\varepsilon + L(P; f) \geq \varepsilon + L(P_\varepsilon; f) > U(P_\varepsilon; f) \geq U(P; f),$$
    i.e. $\varepsilon + L(P; f) > U(P; f)$, which completes the proof. ■

12. Let $a > 0$ and $J = [-a, a]$. Let $f : J \to \mathbb{R}$ be bounded and let $\mathcal{P}^*$ be the set of symmetric partitions of J that contain 0. Show $L(f) = \sup\{L(P; f) \mid P \in \mathcal{P}^*\}$.

    Proof: let $\alpha = \sup\{L(P; f) \mid P \in \mathcal{P}^*\}$. By definition,
    $$\alpha \leq L(f) = \sup\{L(P; f) \mid P \text{ is a partition of } [-a, a]\}.$$
    Let $\varepsilon > 0$ and $P_\varepsilon = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[-a, a]$ such that
    $$L(f) - \varepsilon < L(P_\varepsilon; f) \leq L(f).$$
    Such a partition exists as $L(f) - \varepsilon$ is not the supremum of the aforementioned set.

    Consider the set $\{0, \pm x_0, \ldots, \pm x_n\}$. Eliminate all the repetitions from this set and re-order its elements. Denote the new set by Q_ε.

    Then Q_ε is a refinement of P_ε and $Q_\varepsilon \in \mathcal{P}^*$; so $\alpha \geq L(Q_\varepsilon; f)$, and
    $$L(f) - \varepsilon < L(P_\varepsilon; f) \leq L(Q_\varepsilon; f) \leq \alpha \leq L(f),$$
    as $\varepsilon > 0$ is arbitrary, $L(f) = \alpha$. ■

13. Let $a > 0$ and $J = [-a, a]$. Let f be integrable on J. If f is even (i.e. $f(-x) = f(x)$ for all x), show that
$$\int_{-a}^{a} f = 2\int_{0}^{a} f.$$
If f is odd (i.e. $f(-x) = -f(x)$ for all x), show that $\int_{-a}^{a} f = 0$.

Proof: as f is integrable over $[-a, a]$, Theorem 56 implies that f is integrable over $[0, a]$. If f is even, let $P \in \mathcal{P}^*$. There is a partition $\tilde{P}$ of $[0, a]$ s.t. $L(P; f) = 2L(\tilde{P}; f)$ and vice-versa. Indeed, let

$$P = \{x_{-n}, \ldots, x_{-1}, x_0, x_1, \ldots, x_n\},$$

where $x_0 = 0$ and $x_{-i} = -x_i$ for all $i = 1, \ldots, n$. Then $P \in \mathcal{P}^*$.

Let $m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}$, for $i = -n-1, \ldots, 0, \ldots, n$. Since f is even, $m_i = m_{-i+1}$ for $i = -n-1, \ldots, 0, \ldots, n$. Then

$$L(P; f) = \sum_{i=-n-1}^{0} m_i(x_i - x_{i-1}) + \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = 2\sum_{i=1}^{n} m_i(x_i - x_{i-1}) = L(\tilde{P}; f),$$

where $\tilde{P}$ is a partition of $[0, a]$.

This, combined with the previous solved problem, yields

$$\begin{aligned}\int_{-a}^{a} f &= \sup\{L(P; f) \mid P \in \mathcal{P}^*\} = \sup\{2L(\tilde{P}; f) \mid \tilde{P} \text{ is a partition of } [0, a]\} \\ &= 2\sup\{L(\tilde{P}; f) \mid \tilde{P} \text{ is a partition of } [0, a]\} = 2\int_{0}^{a} f.\end{aligned}$$

If f is odd, consider the function $h : \mathbb{R} \to \mathbb{R}$ given by

$$h(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}.$$

The product fh is an even function, so

$$2\int_{0}^{a} f = 2\int_{0}^{a} hf = \int_{-a}^{a} hf = \int_{-a}^{0} hf + \int_{0}^{a} hf = \int_{-a}^{0} -f + \int_{0}^{a} f,$$

and so $\int_0^a f = \int_{-a}^0 -f = -\int_{-a}^0 f$. Then

$$\int_{-a}^{a} f = \int_{-a}^{0} f + \int_{0}^{a} f = -\int_{0}^{a} f + \int_{0}^{a} f = 0,$$

which completes the proof. ■

14. Give an example of a function $f : [0, 1] \to \mathbb{R}$ that is not integrable on $[0, 1]$, but such that $|f|$ is integrable on $[0, 1]$.

Solution: here is one example – $f:[0,1]\to\mathbb{R}$, defined by $f(x)=-1$ if $x\notin\mathbb{Q}$ and $f(x)=1$ if $x\in\mathbb{Q}$. The proof that f is not Riemann-integrable on $[0,1]$ is similar to the proof that the Dirichlet function is not Rimeann-integrable on $[0,1]$. □

15. Let $f:[a,b]\to\mathbb{R}$ be integrable on $[a,b]$. Show $|f|$ is integrable on $[a,b]$ directly.

Proof: let $\varepsilon>0$. By the Riemann criterion, there exists a partition $P_\varepsilon=\{x_0,\ldots,x_n\}$ of $[a,b]$ such that $U(P_\varepsilon;f)-L(P_\varepsilon;f)<\varepsilon$.

For all $i=1,\ldots,n$, let

$$M_i=\sup\{f(x)\mid x\in[x_{i-1},x_i]\}\quad\text{and}\quad m_i=\inf\{f(x)\mid x\in[x_{i-1},x_i]\}.$$

For all $i=1,\ldots,n$, we then have $|f(x)-f(y)|\leq M_i-m_i$ on $[x_{i-1},x_i]$. As

$$||f(x)|-|f(y)||\leq|f(x)-f(y)|\leq M_i-m_i\quad\text{for all } x,y\in[x_{i-1},x_i],$$

we have $\tilde{M}_i-\tilde{m}_i\leq M_i-m_i$, where

$$\tilde{M}_i=\sup\{|f(x)|\mid x\in[x_{i-1},x_i]\}\quad\text{and}\quad\tilde{m}_i=\inf\{|f(x)|\mid x\in[x_{i-1},x_i]\}$$

for all $i=1,\ldots,n$. Then

$$\begin{aligned}U(P_\varepsilon;|f|)-L(P_\varepsilon;|f|)&=\sum_{i=1}^n\left(\tilde{M}_i-\tilde{m}_i\right)(x_i=x_{i-1})\\&\leq\sum_{i=1}^n(M_i-m_i)(x_i=x_{i-1})=U(P_\varepsilon;|f|)-L(P_\varepsilon;|f|)<\varepsilon.\end{aligned}$$

According to the Riemann criterion, $|f|$ is thus integrable on $[a,b]$. ■

16. If f is integrable on $[a,b]$ and $0\leq m\leq f(x)\leq M$ for all $x\in[a,b]$, show that

$$m\leq\left[\frac{1}{b-a}\int_a^b f^2\right]^{1/2}\leq M.$$

Proof: by hypothesis, $m^2\leq f^2(x)\leq M^2$ for all $x\in[a,b]$. As f is integrable on $[a,b]$, so is f^2, by Theorem 58.

Then

$$\int_a^b m^2\leq\int_a^b f^2\leq\int_a^b M^2$$

by the squeeze theorem for integrals and so

$$m^2(b-a)\leq\int_a^b f^2\leq M^2(b-a).$$

We obtain the result by re-arranging the terms and extracting square roots. ■

17. If f is continuous on $[a,b]$ and $f(x) \geq 0$ for all $x \in [a,b]$, show there exists $c \in [a,b]$ such that
$$f(c) = \left[\frac{1}{b-a}\int_a^b f^2\right]^{1/2}.$$

Proof: by the max/min theorem, $\exists x_0, x_1 \in [a,b]$ such that
$$m = \inf_{[a,b]}\{f(x)\} = f(x_0),\ M = \sup_{[a,b]}\{f(x)\} = f(x_1).$$
By the preceding solved problem, we then have
$$f(x_0) \leq \left[\frac{1}{b-a}\int_a^b f^2\right]^{1/2} \leq f(x_1).$$
As f is continuous on $[x_0, x_1]$ (or $[x_1, x_0]$), the intermediate value theorem states $\exists c \in [a,b]$ such that
$$f(c) = \left[\frac{1}{b-a}\int_a^b f^2\right]^{1/2},$$
which completes the proof. ■

18. If f is continuous on $[a,b]$ and $f(x) > 0$ for all $x \in [a,b]$, show that $\frac{1}{f}$ is integrable on $[a,b]$.

Proof: since f is continuous on $[a,b]$ it is integrable on $[a,b]$; by Theorem 36, since f is continuous and $[a,b]$ is a closed bounded interval, then $f([a,b]) = [m,M]$ is also closed bounded interval. Furthermore, $0 < m \leq M$ since $f(x) > 0$ for all $x \in [a,b]$.

Let $\varphi : [m,M] \to \mathbb{R}$ be defined by $\varphi(t) = \frac{1}{t}$. Then φ is continuous and bounded on $[m,M]$ and so $\varphi \circ f : [a,b] \to \mathbb{R}$, defined by $\varphi(f(x)) = \frac{1}{f(x)}$ is integrable on $[a,b]$, by Theorem 57. ■

19. Let f be continuous on $[a,b]$. Define $H : [a,b] \to \mathbb{R}$ by
$$H(x) = \int_x^b f \quad \text{for all } x \in [a,b].$$
Find $H'(x)$ for all $x \in [a,b]$.

Proof: define $F(x) = \int_a^x f$. Since f is continuous, F is differentiable and the fundamental theorem of calculus (2nd version) yields $F'(x) = f(x)$ for all $x \in [a,b]$. Then, by the additivity theorem, we have:
$$F(x) + H(x) = \int_a^x f + \int_x^b f = \int_a^b f.$$
In particular,
$$H(x) = \int_a^b f - F(x).$$
As F is differentiable, $\int_a^b f - F(x)$ is also differentiable; so is H since $H'(x) = 0 - F'(x) = -f(x)$. ■

20. Suppose $f : [0, \infty) \to \mathbb{R}$ is continuous and $f(x) \neq 0$ for all $x > 0$. If

$$(f(x))^2 = 2\int_0^x f \quad \text{for all } x > 0,$$

show that $f(x) = x$ for all $x \geq 0$.

Proof: as f is continuous, $F(x) = \int_0^x f$ is continuous; the fundamental theorem of calculus (2nd version) then yields $F'(x) = f(x)$ for all $x \in [0, \infty)$.

Now, either $f(x) > 0$ for all $x > 0$ or $f(x) < 0$ for all $x > 0$ – otherwise f admits a root $c > 0$ by the intermediate value theorem, which would contradict $f(x) \neq 0$ $\forall x > 0$.

But

$$F(x) = \int_0^x f = \frac{(f(x))^2}{2} > 0 \quad \text{for all } x > 0,$$

so $\int_0^x f > 0$ for all $x > 0$, which is to say that $f > 0$ for all $x > 0$ – otherwise, $\int_0^x f \leq \int_0^x 0 = 0$, which contradicts one of the above inequalities.

By construction,

$$\frac{(f(0))^2}{2} = F(0) = \int_0^0 f = 0,$$

that is, $f(0) = 0$. Now, let $c > 0$. By hypothesis, $F'(c) = f(c) > 0$. Furthermore, $F(c) = \frac{(f(c))^2}{2}$. As f is continuous at c,

$$\lim_{x \to c} \frac{1}{2}(f(x) + f(c)) = f(c).$$

Thus we have:

$$1 = \frac{F'(c)}{f(c)} = \frac{\lim_{x \to c} \frac{F(x) - F(c)}{x - c}}{\lim_{x \to c} \frac{1}{2}(f(x) + f(c))} = \lim_{x \to c} \frac{(f(x))^2 - (f(c))^2}{(x - c)(f(x) + f(c))}$$
$$= \lim_{x \to c} \frac{(f(x) - f(c))(f(x) + f(c))}{(x - c)(f(x) + f(c))} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c).$$

Then, the function f is differentiable and $f'(c) = 1$ for all $c > 0$. By the fundamental theorem of calculus (1st version),

$$\int_0^x f' = f(x) - f(0) = f(x) - 0 = f(x)$$

for all $x \geq 0$. As $\int_0^x f' = \int_0^x 1 = x - 0 = x$, this completes the proof (which, incidentally, is one of my favourite analysis proofs). ■

21. Let $f, g : [a, b] \to \mathbb{R}$ be continuous and such that

$$\int_a^b f = \int_a^b g.$$

Show that there exists $c \in [a, b]$ such that $f(c) = g(c)$.

Proof: as f and g are continuous, the functions

$$F(x) = \int_a^x f \quad \text{and} \quad G(x) = \int_a^x g$$

are continuous and differentiable on $[a, b]$, with $F'(x) = f(x)$ and $G'(x) = g(x)$, according to the fundamental theorem of calculus (2nd version). Then $H(x) = F(x) - G(x)$ is continuous.

But by hypothesis, we have

$$H(a) = F(a) - G(a) = \int_a^a f - \int_a^a g = 0 - 0 = 0$$
$$H(b) = F(b) - G(b) = \int_a^b f - \int_a^b g = 0.$$

Since H is also differentiable, $\exists c \in (a, b)$ such that $H'(c) = 0$, by Rolle's theorem. As

$$H'(c) = F'(c) - G'(c) = f(c) - g(c) = 0,$$

this completes the proof. ■

22. Let $f : [0, 3] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x & x \in [0, 1) \\ 1 & x \in [1, 2) \\ x & x \in [2, 3] \end{cases}.$$

Find $F : [0, 3] \to \mathbb{R}$, where
$$F(x) = \int_0^x f.$$

Where is F differentiable? What is F' there?

Solution: the function f is increasing on $[0, 3]$ so it is Riemann-integrable there. The function F is given by
$$F(x) = \begin{cases} \frac{x^2}{2}, & x \in [0, 1) \\ x - \frac{1}{2}, & x \in [1, 2) \\ \frac{x^2-1}{2}, & x \in [2, 3] \end{cases}$$

By the fundamental theorem of calculus, F is differentiable wherever f is continuous, that is, on $[0, 2) \cup (2, 3]$, and $F' = f$ there. □

23. If $f : [0, 1] \to \mathbb{R}$ is continuous and $\int_0^x f = \int_x^1 f$ for all $x \in [0, 1]$, show that $f \equiv 0$.

Proof: as f is continuous, then $F(x) = \int_0^x f$ is continuous and differentiable on $[0, 1]$, with $F'(x) = f(x)$, by the fundamental theorem of calculus. By the additivity theorem,
$$\int_0^x f + \int_x^1 f = \int_0^1 f.$$

But $\int_0^x f = \int_x^1 f$ so $2\int_0^x f = \int_0^1 f$. In particular,

$$F(x) = \frac{1}{2}\int_0^1 f = \text{constant}.$$

Then $f(x) = F'(x) = 0$ for all $x \in [0,1]$. ■

24. Let $f : [a,b] \to \mathbb{R}$ be continuous, $f \geq 0$ on $[a,b]$, and $\int_a^b f = 0$. Show that $f \equiv 0$ on $[a,b]$.

Proof: We show the contrapositive. Suppose that there exists $z \in [a,b]$ such that $f(z) > 0$. Since f is continuous, we may assume $z \in (a,b)$, as if $f(z) = 0$ for all $z \in (a,b)$, then $f(a) = f(b) = 0$.

Then, taking $\varepsilon = f(z)/2$ in the definition of continuity, there exists a $\delta > 0$ such that

$$|x - z| < \delta \implies |f(x) - f(z)| < f(z)/2 \implies f(x) > f(z)/2.$$

Reducing δ if necessary, we may assume $\delta \leq \min\{z-a, b-a\}$. Therefore,

$$[z - \delta/2, z + \delta/2] \subseteq (z - \delta, z + \delta) \subseteq [a,b].$$

Thus

$$\int_a^b f = \int_a^{z-\delta/2} f + \int_{z-\delta/2}^{z+\delta/2} f + \int_{z+\delta/2}^b f \geq 0 + \delta f(z)/2 + 0 > 0.$$

This completes the proof. ■

25. Let $f : [a,b] \to \mathbb{R}$ be continuous and let $\int_a^b f = 0$. Show $\exists c \in [a,b]$ such that $f(c) = 0$.

Proof: we show the contrapositive. Suppose $f(c) \neq 0$ for all $c \in [a,b]$. Then, by the intermediate value theorem, either $f(x) > 0$ for all $x \in [a,b]$ or $f(x) < 0$ for all $x \in [a,b]$.

If $f(x) > 0$ for all $x \in [a,b]$, then $\int_a^b f > 0$ by the preceding solved problem. Similarly, if $f(x) < 0$ for all $x \in [a,b]$, then $\int_a^b (-f) > 0$, which implies that $-\int_a^b f > 0$. In both cases, $\int_a^b f \neq 0$. ■

26. Compute $\dfrac{\mathrm{d}}{\mathrm{d}x}\displaystyle\int_{-x}^x e^{t^2}\,dt$.

Solution: according to the additivity property of the Riemann integral and the fundamental theorem of calculus, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\int_{-x}^x e^{t^2}\,dt &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{-x}^0 e^{t^2}\,dt + \int_0^x e^{t^2}\,dt\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(-\int_0^{-x} e^{t^2}\,dt + \int_0^x e^{t^2}\,dt\right)\\ &= -\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{-x} e^{t^2}\,dt + \frac{\mathrm{d}}{\mathrm{d}x}\int_0^x e^{t^2}\,dt\\ &= -e^{x^2}\cdot(-1) + e^{x^2} = 2e^{x^2},\end{aligned}$$

where we used the chain rule in the second-to-last equation. □

27. Let $f : [a,b] \to \mathbb{R}$ be Riemann-integrable on $[a+\delta, b]$ and unbounded in the interval $(a, a+\delta)$ for every $0 < \delta < b-a$. Define

$$\int_a^b f = \lim_{\delta \to 0^+} \int_{a+\delta}^b f,$$

where $\delta \to 0^+$ means that $\delta \to 0$ and $\delta > 0$. A similar construction allows us to define

$$\int_a^b g = \lim_{\delta \to 0^+} \int_a^{b-\delta} g.$$

Such integrals are said to be **improper**; when the limits exist, they are further said to be **convergent**. How can the expression

$$\int_0^1 \frac{1}{\sqrt{|x|}}\,\mathrm{d}x$$

be interpreted as an improper integral? Is it convergent? If so, what is its value?

Solution: by definition,

$$\int_0^1 \frac{1}{\sqrt{|x|}}\,\mathrm{d}x = \lim_{a \to 0^+} \int_a^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x = \lim_{a \to 0^+} \left(2\sqrt{1} - 2\sqrt{a}\right) = 2.$$

Thus the improper integral converges to 2. □

28. Let $G : \mathbb{R} \to \mathbb{R}$ be defined according to

$$G(x) = \begin{cases} x^2 \sin\left(\frac{\pi}{x^2}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Show that G is the antiderivative of some function $g : [0,1] \to \mathbb{R}$, but that g is not Riemann-integrable on $[0,1]$.

Proof: the derivative of G is

$$G'(x) = g(x) = \begin{cases} 2x \sin\left(\frac{\pi}{x^2}\right) - \frac{2\pi}{x}\cos\left(\frac{\pi}{x^2}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

But g is not bounded on $[0,1]$, so it cannot be Riemann-integrable on $[0,1]$. ■

29. Let $f : \mathbb{R} \to \mathbb{R}$ be Thomae's function. Show that the indefinite integral of f on $[1,2]$ is not an antiderivative of f on $[1,2]$.

Proof: for any $x \in \mathbb{Q} \cap [1,2]$, the indefinite integral F is such that $F'(x) \neq f(x)$; F cannot then be an antiderivative of f on $[1,2]$.[10] ■

30. Without evaluating the integrals, show that $\displaystyle\int_1^4 e^{-8t}\,dt = \frac{1}{8}\int_4^8 te^{-t^2/2}\,dt.$

Proof: we can use the 2nd substitution theorem with $f(x) = e^{-x^2/2}$, $\varphi(t) = 4\sqrt{t}$, $\psi(t) = \frac{t^2}{16}$, $J = [1,4]$. ■

[10] Of course, this will only make sense if you've managed to find F...

4.4 Exercises

1. Prepare a 2-page summary of this chapter, with important definitions and results.

2. With the assumptions of Theorem 48, show that f is decreasing on $[a, b]$ if and only if $f' \leq 0$ on (a, b).

3. Prove part 2. of the first derivative test.

4. Let $f : (a, b) \to \mathbb{R}$ be differentiable on (a, b), with $f'(x) \neq 0$. Prove the following statements.

   a) f is monotone on (a, b) and $f((a, b))$ is an open interval (α, β);

   b) f has an **inverse** $f^{-1} : (\alpha, \beta) \to \mathbb{R}$ such that

   $$f^{-1}(f(x)) = x, \quad f(f^{-1}(y)) = y, \quad \forall x \in (a, b), y \in (\alpha, \beta),$$

   c) f^{-1} is differentiable on (α, β), with

   $$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}, \quad \forall y \in (\alpha, \beta).$$

5. Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be bounded. Then $U(Q; f) \leq L(Q; f)$ for any refinement $Q \supseteq P$ of I.

6. Prove that $f \equiv 1$ is Riemann-integrable on $[0, 1]$.

7. Show that Theorem 53 holds for decreasing functions.

8. Show that Thomae's function f is Riemann-integrable over $[0, 1]$ and that $\int_0^1 f = 0$.

9. Show that the signum function and Thomae's function do not have antiderivatives on any closed, bounded interval $I \subseteq \mathbb{R}$.

10. Show that the reciprocal of the square root function has an anti-derivative on $[0, 1]$, but that it is not Riemann-integrable on $[0, 1]$.

11. Find a function $f : [a, b] \to \mathbb{R}$ such that the indefinite integral $F : [a, b] \to \mathbb{R}$ defined by $F(x) = \int_a^x f$ is not an antiderivative of f.

12. Prove Theorems 63, 64, and 65.

13. For which values of s does the integral $\int_0^1 x^s \, \mathrm{d}x$ converge?

14. Show that the indefinite integral of sgn is not an antiderivative of sgn on $[-1, 1]$.