Chapter 5

Sequences of Functions

We now look at **sequences of functions**, which arise naturally in analysis and its applications. In particular, we discuss two **types of convergence** (**pointwise** and **uniform**and prove the **limit interchange theorems**.

5.1 Pointwise and Uniform Convergence

Let $A \subseteq \mathbb{R}$ and $(f_n)_n$ be a **sequence of functions** $f_n : A \to \mathbb{R}$. The sequence $(f_n(x))_n$ may converge for some $x \in A$ and diverge for others. Let $A_0 = \{x \in A \mid (f_n(x))_n \text{ converges}\} \subseteq A$. For each $x \in A_0$, $(f_n(x))$ converges to a unique limit

$$f(x) = \lim_{n\to\infty} f(x),$$

the **pointwise limit** of (f_n); we denote the situation by $f_n \to f$ on A_0.

Examples

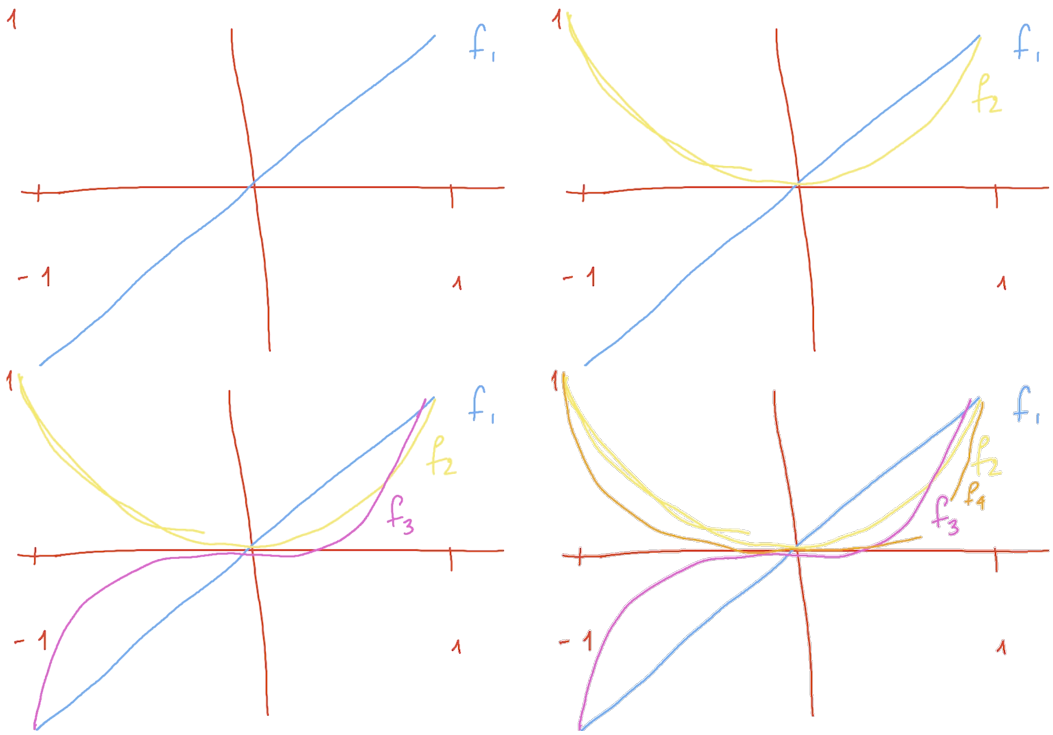
1. Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = \frac{x}{n}$ for all $n \in \mathbb{N}, x \in \mathbb{R}$, and let f be the zero function on $\mathbb{R}$. Show that $f_n \to f$ on $\mathbb{R}$.

   Proof: let $\varepsilon > 0$ and $x \in \mathbb{R}$. By the Archimedean property, $\exists N_{\varepsilon,x} > \frac{|x|}{\varepsilon}$ so that
   $$n > N_{\varepsilon,x} \implies \left|\tfrac{x}{n} - 0\right| < \tfrac{|x|}{n} < \tfrac{|x|}{N_{\varepsilon,x}} < \varepsilon,$$
   thus $f_n \to 0$ on $\mathbb{R}$. ■

2. Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = x^n$ for all $n \in \mathbb{N}, x \in \mathbb{R}$, and let f be the zero function on $\mathbb{R}$, except at $x = 1$ where $f(1) = 1$. Show that $f_n \to f$ on $(-1, 1]$.

   Proof: using various results seen in Chapters 2 and 3 (and in the solved problems and exercises), we know that

$$\lim_{n\to\infty} x^n = \begin{cases} 0 & x \in (-1,1) \\ 1 & x = 1 \\ \text{does not exist} & \text{otherwise} \end{cases}$$



Thus $f_n \to f$ on $(-1,1]$. Note that all f_n are continuous on $(1,1]$, but that f is not. ■

3. Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = \frac{x^2+nx}{n}$ for all $n \in \mathbb{N}, x \in \mathbb{R}$, and let f be the identity function on $\mathbb{R}$. Show that $f_n \to f$ on $\mathbb{R}$.

   Proof: as $f_n(x) = \frac{x^2}{n} + x \to f(x) = x, \forall x \in \mathbb{R}$, we have $f_n \to f$ on $\mathbb{R}$. ■

The last example show that there is something "incomplete" about pointwise convergence – why is continuity not preserved by the process? As it happens, we can define a different type of convergence which will preserve this important property.

A sequence of functions $f_n : A \to \mathbb{R}$ **converges uniformly on** $A_0 \subseteq A$ **to** $f : A_0 \to \mathbb{R}$, denoted by $f_n \rightrightarrows f$ on A_0, if the threshold $N_{\varepsilon,x} \in \mathbb{N}$ in the pointwise definition is in fact **independent** of $x \in A_0$:

$$\forall \varepsilon > 0,\ \exists N_\varepsilon \in \mathbb{N} \text{ such that } n > N_\varepsilon \text{ and } x \in A_0 \Longrightarrow |f_n(x) - f(x)| < \varepsilon.$$

The distinction between pointwise and uniform convergence is not unlike that between continuity and uniform continuity: convergence is uniform if the threshold is the same for all $x \in A_0$.

Clearly, if $f_n \rightrightarrows f$ on A_0, then $f_n \to f$ on A_0, but the converse is not necessarily true.

Examples

1. Show that the sequence $f_n : [1,2] \to \mathbb{R}$ defined by $f_n(x) = \frac{\sin x}{nx}$ for $n \in \mathbb{N}$ converges uniformly to the zero function on $[1,2]$.

   Proof: let $\varepsilon > 0$. According to the Archimedean property, $\exists N_\varepsilon > \frac{1}{\varepsilon}$ so that

   $$n > N_\varepsilon \text{ and } x \in [1,2] \Longrightarrow \left|\frac{\sin x}{nx} - 0\right| = \left|\frac{\sin x}{nx}\right| \leq \frac{1}{nx} \leq \frac{1}{n} < \frac{1}{N_\varepsilon} < \varepsilon,$$

   thus $f_n \rightrightarrows 0$ on $[1,2]$. ■

2. Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = x^n$ for all $n \in \mathbb{N}, x \in \mathbb{R}$, and let f be the zero function on $\mathbb{R}$, except at $x = 1$ where $f(1) = 1$. Show that $f_n \not\rightrightarrows f$ on $(-1,1]$.

   Proof: we use the negation of the definition. Let $\varepsilon_0 = \frac{1}{4}$, and set $x_k = \frac{1}{2^{1/k}}$ and $(n_k) = (k)$. Then

   $$|f_{n_k}(x_k) - f(x_k)| = \left|\tfrac{1}{2} - 0\right| = \frac{1}{2} \geq \varepsilon_0,$$

   which completes the proof. ■

A sequence of functions f_n does not converge uniformly to f on A_0 if

$$\exists \varepsilon_0 > 0 \text{ with } (f_{n_k}) \subseteq (f_n) \text{ and } (x_k) \subseteq A_0 \text{ s.t. } |f_{n_k}(x_k) - f(x_k)| \geq \varepsilon_0,\ \forall k \in \mathbb{N}.$$

Example: let $f_n : [0,1] \to \mathbb{R}$ be the sequence of functions defined by

$$f_n(x) = \begin{cases} nx, & x \in [0,1/n] \\ 2 - nx, & x \in [1/n, 2/n] \\ 0 & x \in [2/n, 1] \end{cases}$$

for all $n \in \mathbb{N}$. Let $f : [0,1] \to \mathbb{R}$ be the zero function on $[0,1]$. Show that $f_n \to f$ on $[0,1]$ but $f_n \not\rightrightarrows f$ on $[0,1]$.

Proof: if $x=0$, $f_n(0)=0$ for all n so $(f_n(0))$ converges to 0. If $x\in(0,1]$, $\exists N_x>2/x$ by the Archimedean property. Thus, for $n>N_x$, $f_n(x)=0$ since $x>\frac{2}{N}>\frac{2}{n}$, so $f_n(x)\to 0$ on $(0,1]$. Combining these results, $f_n\to f$ on $[0,1]$.

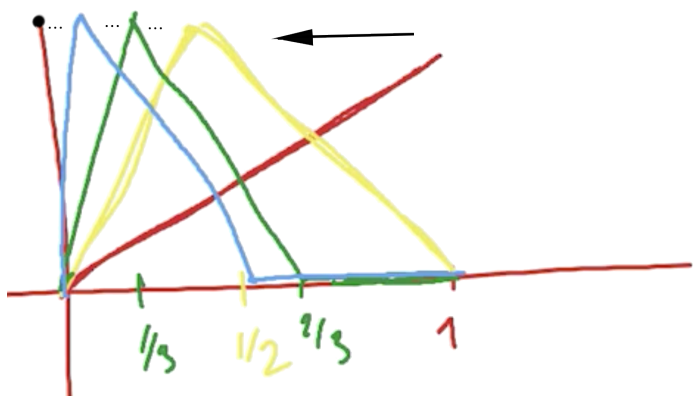
Now, let $\varepsilon_0=\frac{1}{2}$. Note that since $|f_n(\frac{1}{n})-f(\frac{1}{n})|=1$ for all $n\in\mathbb{N}$, we can never obtain

$$|f_n(x)-f(x)|<\varepsilon$$

for all $x\in[0,1]$, and so $f_n\not\rightrightarrows f$ on $[0,1]$. ■

The fact that we have to separate the proof for pointwise convergence into distinct arguments depending on the value of x is a strong indication that the convergence cannot be uniform.[1]

Intuitively, we can think of the convergence process in the last example as being a flattening process: what happens to the tents' peak as $n\to\infty$? That we have to "break" the tents to get to the pointwise limit is another indication that the convergence cannot be uniform.



The definition of uniform convergence is only ever useful if a candidate for a uniform limit is available, a situation that we have encountered before. As was the case for number sequences, the completeness of $\mathbb{R}$ comes to the rescue.

Theorem 66 (Cauchy's Criterion for Sequences of Functions)
Let $f_n:A\to\mathbb{R}$, $\forall n\in\mathbb{N}$. Then, $f_n\rightrightarrows f$ on $A_0\subseteq A$ if and only if $\forall\varepsilon>0$, $\exists N_\varepsilon\in\mathbb{N}$ (indep. of $x\in A_0$) such that $|f_m(x)-f_n(x)|<\varepsilon$ whenever $m\geq n>N_\varepsilon\in\mathbb{N}$ and $x\in A_0$.

Proof: let $\varepsilon>0$. If $f_n\rightrightarrows f$ on A_0, $\exists N_\varepsilon\in\mathbb{N}$ such that $|f_n(x)-f(x)|<\varepsilon$ when $x\in A_0$ and $n>N_\varepsilon$. Hence,

$$\begin{aligned}|f_m(x)-f_n(x)|&=|f_m(x)-f(x)+f(x)-f_n(x)|\\&\leq|f_m(x)-f(x)|+|f_n(x)-f(x)|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon\end{aligned}$$

whenever $x\in A_0$ and $m\geq n>N_\varepsilon$.

[1]Although it could be that it was possible to do a one-pass proof and that the insight escaped us.

Conversely, let $\varepsilon > 0$ and assume that $\exists N_{\varepsilon/2} \in \mathbb{N}$ (independent of $x \in A_0$) such that

$$m \geq n > N_{\varepsilon/2} \text{ and } x \in A_0 \implies -\frac{\varepsilon}{2} < f_m(x) - f_n(x) < \frac{\varepsilon}{2}.$$

Since $x \in A_0$, we know that $f_m(x) \to f$ on A_0 when $m \to \infty$. Thus,

$$m \geq n > N_{\varepsilon/2} \text{ and } x \in A_0 \implies \lim_{m\to\infty} -\frac{\varepsilon}{2} \leq \lim_{m\to\infty} (f_m(x) - f_n(x)) \leq \lim_{m\to\infty} \frac{\varepsilon}{2},$$

or

$$m \geq n > N_{\varepsilon/2} \text{ and } x \in A_0 \implies -\varepsilon < -\frac{\varepsilon}{2} \leq f(x) - f_n(x) \leq \frac{\varepsilon}{2} < \varepsilon,$$

and so $f_n \rightrightarrows f$ on A_0. ■

5.2 Limit Interchange Theorems

It is often necessary to know if the limit f of a sequence of functions (f_n) is continuous, differentiable, or Riemann-integrable. Unfortunately, we cannot guarantee that this will be the case, even when the f_n are continuous, differentiable, or Riemann-integrable, respectively.

Examples

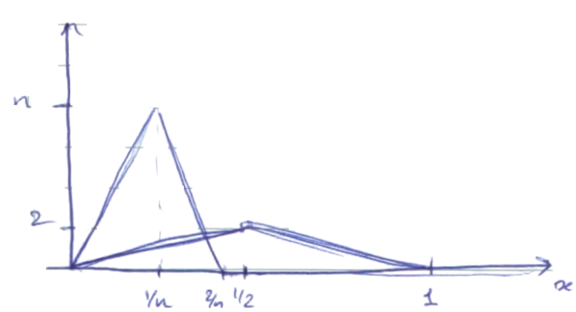
1. Consider the sequence of functions $f_n : [0,1] \to \mathbb{R}$ defined by $f_n(x) = x^n$ for $n \in \mathbb{N}$ and $f : [0,1] \to \mathbb{R}$ be the zero function except at $x = 1$ where $f(1) = 1$. Then f_n is continuous on $[0,1]$ for all $n \in \mathbb{N}$, but f is not. □

2. The same functions f_n are differentiable on $[0,1]$ for all $n \in \mathbb{N}$, but f is not (as it is not continuous at $x = 1$). □

3. Consider the functions $f_n : [0,1] \to \mathbb{R}$ defined by

$$f_n(x) = \begin{cases} n^2 x, & x \in [0, 1/n] \\ -n^2(x - 2/n), & x \in [1/n, 2/n] \\ 0 & x \in [2/n, 1] \end{cases}$$

for $n \geq 2$.



Since f_n is continuous on $[0,1]$ for all $n \geq 2$, f_n is Riemann-integrable on $[0,1]$ for all $n \geq 2$, with

$$\int_0^1 f_n = \frac{1}{2} \cdot \frac{2}{n} \cdot n = 1, \quad \text{for all } n \geq 2.$$

If $x = 0$, $f_n(0) = 0$ for all n so $(f_n(0))$ converges to 0.

If $x \in (0,1]$, $\exists N_x > 2/x$ by the Archimedean property. Thus, for $n > N_x$, $f_n(x) = 0$ since $x > \frac{2}{N} > \frac{2}{n}$, so $f_n(x) \to 0$ on $(0,1]$. So $f_n \to f$ on $[0,1]$, but

$$\int_0^1 f = 0 \neq 1 = \lim_{n\to\infty} \int_0^1 f,$$

which is to say we cannot interchange the limit and the integral here. $\square$

Note that none of the "convergences" in the previous example are uniform on $[0,1]$. When the convergence $f_n \rightrightarrows f$ on A is **uniform**, then if the f_n are

- continuous on A, so is f;
- differentiable on A, so is f, with

$$f' = \frac{\mathrm{d}}{\mathrm{d}x}\Big[\lim_{n\to\infty} f_n\Big] = \lim_{n\to\infty}\Big[\frac{\mathrm{d}}{\mathrm{d}x} f_n\Big] = \lim_{n\to\infty} f_n';$$

- Riemann-integrable on A, then so is f, with

$$\int_A f = \int_A \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_A f_n.$$

We finish this chapter by proving three **limit interchange theorems**, with applications in analysis, engineering, and mathematical physics.[2]

Theorem 67
Let $f_n : A \to \mathbb{R}$ be continuous on A for all $n \in \mathbb{N}$. If $f_n \rightrightarrows f$ on A, then f is continuous on A.

Proof: let $\varepsilon > 0$. By definition, $\exists H_{\varepsilon/3} \in \mathbb{N}$ such that

$$n > H_{\varepsilon/3} \text{ and } x \in A \Longrightarrow |f_n(x) - f(x)| < \tfrac{\varepsilon}{3}.$$

[2]Although their conclusions are often used without verifying that the convergence is indeed uniform.

Let $c \in A$. According to the triangle inequality,

$$\begin{aligned}|f(x)-f(c)| &\leq |f(x)-f_{H_{\varepsilon/3}}(x)|+|f_{H_{\varepsilon/3}}(x)-f_{H_{\varepsilon/3}}(c)|+|f_{H_{\varepsilon/3}}(c)-f(c)| \\ &< \tfrac{\varepsilon}{3}+|f_{H_{\varepsilon/3}}(x)-f_{H_{\varepsilon/3}}(c)|+\tfrac{\varepsilon}{3}\end{aligned}$$

whenever $n > H_{\varepsilon/3}$.

But $f_{H_{\varepsilon/3}}$ is continuous at c, so $\exists \delta_{\varepsilon/3} > 0$ such that $|f_{H_{\varepsilon/3}}(x) - f_{H_{\varepsilon/3}}(c)| < \frac{\varepsilon}{3}$ when $x \in A$ and $|x-c| < \delta_{\varepsilon/3}$. Thus $|f(x) - f(c)| < \varepsilon$ whenever $x \in A$ and $|x - c| < \delta_{\varepsilon/3}$, so f is continuous at c. As $c \in A$ is arbitrary, f is continuous on A. ■

The next two results are slightly more complicated to prove.

Theorem 68
Let $f_n : [a, b] \to \mathbb{R}$ be a sequence of differentiable functions on $[a, b]$ such that $\exists x_0 \in [a, b]$ with $f_n(x_0) \to z_0$, and $f_n'' \rightrightarrows g$ on $[a, b]$. Then $f_n \rightrightarrows f$ on $[a, b]$ for some function $f : [a, b] \to \mathbb{R}$ such that $f' = g$.

Proof: let $\varepsilon > 0$ and $x \in [a, b]$. Since $f_n' \rightrightarrows g$ on $[a, b]$, the sequence f_n' satisfies Cauchy's criterion, and so $\exists N_1 \in \mathbb{N}$ such that

$$m \geq n > N_1 \text{ and } y \in [a,b] \Longrightarrow |f_m'(y) - f_n'(y)| < \frac{\varepsilon}{2(b-a)}.$$

As $(f_n(x_0))$ converges it is also a Cauchy sequence, so $\exists N_2 \in \mathbb{N}$ such that

$$m \geq n > N_2 \Longrightarrow |f_m(x_0) - f_n(x_0)| < \frac{\varepsilon}{2}.$$

According to the mean value theorem, $\exists y$ between x and x_0 such that

$$(f_m(x) - f_n(x)) - (f_m(x_0) - f_n(x_0)) = (f_m'(y) - f_n'(y))(x - x_0).$$

Hence,

$$\begin{aligned}|f_m(x) - f_n(x)| &\leq |f_m(x_0) - f_n(x_0)| + |f_m'(y) - f_n'(y)| \cdot |x - x_0| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2(b-a)}(b-a) = \varepsilon\end{aligned}$$

for all $m \geq n > \max\{N_1, N_2\}$.

Both N_1 and N_2 are independent of x, so $N_\varepsilon = \max\{N_1, N_2\}$ also is, and thus $(f_n)_n$ satisfies Cauchy's criterion, which yields $f_n \rightrightarrows f$ on $[a, b]$.

It remains only to show that $f' = g$ on $[a,b]$. Let $\varepsilon > 0$ and $c \in [a,b]$. Since (f_n') satisfies Cauchy's criterion (as $f_n' \rightrightarrows g$), $\exists K_1 \in \mathbb{N}$ (independent of x) such that

$$m \geq n > K_1 \text{ and } y \in [a,b] \Longrightarrow |f_m'(y) - f_n'(y)| < \frac{\varepsilon}{3}.$$

But $f' \rightrightarrows g'$, so $\exists K_2 \in \mathbb{N}$ (independent of c) such that

$$n \geq K_2 \text{ and } c \in [a,b] \Longrightarrow |f_n'(c) - g(c)| < \frac{\varepsilon}{3}.$$

Set $K_\varepsilon > \max\{K_1, K_2\}$.

As $f_{K_\varepsilon}'(c)$ exists, $\exists \delta_\varepsilon > 0$ such that

$$0 < |x-c| < \delta_\varepsilon \text{ and } x \in [a,b] \Longrightarrow \left| \frac{f_{K_\varepsilon}(x) - f_{K_\varepsilon}(c)}{x-c} - f_{K_\varepsilon}'(c) \right| < \frac{\varepsilon}{3}.$$

According to the mean value theorem, $\exists y$ between x and c such that

$$(f_m(x) - f_n(x)) - (f_m(c) - f_n(c)) = (f_m'(y) - f_n'(y))(x-c).$$

If $x \neq c$, then $m \geq n > K_\varepsilon$ and $x \in [a,b] \Longrightarrow$

$$\left| \frac{f_m(x) - f_m(c)}{x-c} - \frac{f_n(x) - f_n(c)}{x-c} \right| = |f_m'(y) - f_n'(y)| < \frac{\varepsilon}{3}.$$

Letting $m \to \infty$ (i.e. $f_m \to f$ on A), we get

$$n > K_\varepsilon \text{ and } x \in [a,b] \Longrightarrow \left| \frac{f(x) - f(c)}{x-c} - \frac{f_m(c) - f_n(c)}{x-c} \right| \leq \frac{\varepsilon}{3}.$$

Combining all of these inequalities, for $0 < |x-c| < \delta_\varepsilon$, $x \in [a,b]$, and $k > K_\varepsilon$, we have

$$\begin{aligned} \left| \frac{f(x) - f(c)}{x-c} - g(c) \right| &= \left| \frac{f(x) - f(c)}{x-c} - \frac{f_k(x) - f_k(c)}{x-c} \right. \\ &\qquad \left. + \frac{f_k(x) - f_k(c)}{x-c} - f_k'(c) + f_k'(c) - g(c) \right| \\ &\leq \left| \frac{f(x) - f(c)}{x-c} - \frac{f_k(x) - f_k(c)}{x-c} \right| + \left| \frac{f_k(x) - f_k(c)}{x-c} - f_k'(c) \right| \\ &\qquad + |f_k'(c) - g(c)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

which is to say that $f'(c) = g(c)$. ■

Theorem 69
Let $f_n : [a,b] \to \mathbb{R}$ be Riemann-integrable on $[a,b]$ for all $n \in \mathbb{N}$. If $f_n \rightrightarrows f$ on $[a,b]$, then f is Riemann-integrable on $[a,b]$ and

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n.$$

Proof: let $\varepsilon > 0$. Since $f_n \rightrightarrows f$ on $[a,b]$, $\exists K_\varepsilon \in \mathbb{N}$ (independent of x) such that

$$n \geq K_\varepsilon \Longrightarrow |f_n(x) - f(x)| < \frac{\varepsilon}{4(b-a)}.$$

Since f_{K_ε} is Riemann-integrable, $\exists P_\varepsilon = \{x_0, \ldots, x_n\}$ a partition of $[a,b]$ such that

$$U(P_\varepsilon; f_{K_\varepsilon}) - L(P_\varepsilon; f_{K_\varepsilon}) < \frac{\varepsilon}{2},$$

according to the Riemann criterion.

For all $1 \leq i \leq n$, set

$$\begin{aligned} m_i(f) &= \inf\{f(x) \mid x \in [x_{i-1}, x_i]\},\ m_i(f_{K_\varepsilon}) = \inf\{f_{K_\varepsilon}(x) \mid x \in [x_{i-1}, x_i]\}, \\ M_i(f) &= \sup\{f(x) \mid x \in [x_{i-1}, x_i]\},\ M_i(f_{K_\varepsilon}) = \sup\{f_{K_\varepsilon}(x) \mid x \in [x_{i-1}, x_i]\}. \end{aligned}$$

Then according to the reverse triangle inequality, we have

$$\begin{aligned} |f(x)| < |f_{K_\varepsilon}(x)| + \frac{\varepsilon}{4(b-a)} &\Longrightarrow |f(x)| < M_i(f_{K_\varepsilon}) + \frac{\varepsilon}{4(b-a)} \text{ on } [x_{i-1}, x_i] \\ &\Longrightarrow M_i(f) < M_i(f_{K_\varepsilon}) + \frac{\varepsilon}{4(b-a)} \text{ on } [x_{i-1}, x_i]. \end{aligned}$$

Similarly, $m_i(f) \geq m_i(f_{K_\varepsilon}) - \frac{\varepsilon}{4(b-a)}$ on $[x_{i-1}, x_i]$. Thus,

$$\begin{aligned} U(P_\varepsilon; f) &= \sum_{i=1}^n M_i(f)(x_i - x_{i-1}) \\ &\leq \sum_{i=1}^n M_i(f_{K_\varepsilon})(x_i - x_{i-1}) + \frac{\varepsilon}{4(b-a)} \sum_{i=1}^n (x_i - x_{i-1}) = U(P_\varepsilon; f_{K_\varepsilon}) + \frac{\varepsilon}{4}. \end{aligned}$$

Similarly, $L(P_\varepsilon; f) \geq L(P_\varepsilon; f_{K_\varepsilon}) - \frac{\varepsilon}{4}$. Hence

$$U(P_\varepsilon; f) - L(P_\varepsilon; f) \leq U(P_\varepsilon; f_{K_\varepsilon}) - L(P_\varepsilon; f_{K_\varepsilon}) + \frac{\varepsilon}{2} < \varepsilon.$$

Thus, according to the Riemann criterion, f is Riemann-integrable.

Finally, let $\varepsilon > 0$. As $f_n \rightrightarrows f$ on $[a,b]$, $\exists \hat{K}_\varepsilon$ (independent of x) such that

$$n > \hat{K}_\varepsilon \text{ and } x \in [a,b] \implies |f_n(x) - f(x)| < \frac{\varepsilon}{2(b-a)}.$$

Consequently, $\int_a^b f_n \to \int_a^b f$, since $n > \hat{K}_\varepsilon \implies$

$$\left| \int_a^b f_n - \int_a^b f \right| = \left| \int_a^b (f_n - f) \right| \leq \int_a^b |f_n - f| \leq \int_a^b \frac{\varepsilon}{2(b-a)} = \frac{\varepsilon}{2} < \varepsilon,$$

which completes the proof. ■

5.3 Solved Problems

1. Show that $\lim_{n\to\infty} \dfrac{nx}{1+n^2x^2} = 0$ for all $x \in \mathbb{R}$.

   Proof: if $x = 0$, then $\frac{nx}{1+n^2x^2} = 0 \to 0$. If $x \neq 0$, let $\varepsilon > 0$. By the Archimedean property, $\exists N_\varepsilon > \frac{1}{\varepsilon|x|}$ (depending on x) s.t.

   $$\left| \frac{nx}{1+n^2x^2} - 0 \right| = \frac{n|x|}{1+n^2x^2} < \frac{n|x|}{n^2x^2} = \frac{1}{n|x|} < \frac{1}{N_\varepsilon|x|} < \varepsilon$$

   whenever $n > N_\varepsilon$, i.e. $\frac{nx}{1+n^2x^2} \to 0$ on $\mathbb{R}$. ■

2. Show that if $f_n(x) = x + \frac{1}{n}$ and $f(x) = x$ for all $x \in \mathbb{R}$, $n \in \mathbb{N}$, then $f_n \rightrightarrows f$ on $\mathbb{R}$ but $f_n^2 \not\rightrightarrows g$ on $\mathbb{R}$ for any function g.

   Proof: let $\varepsilon > 0$. By the Archimedean property, $\exists N_\varepsilon > \frac{1}{\varepsilon}$ (independent of x) s.t.

   $$|f_n(x) - f(x)| = \left| x + \frac{1}{n} - x \right| = \frac{1}{n} < \frac{1}{N_\varepsilon} < \varepsilon$$

   whenever $n > N_\varepsilon$, i.e. $f_n \rightrightarrows 0$ on $\mathbb{R}$.

   Now, $(f_n(x))^2 = x^2 + \frac{2x}{n} + \frac{1}{n^2} \to x^2$ for all $x \in \mathbb{R}$. Hence, $f_n^2 \to g$ on $\mathbb{R}$, where $g(x) = x^2$. If f_n^2 converges uniformly to any function, it will have to do so to g. But let $\varepsilon_0 = 2$ and $x_n = n$. Then

   $$\left| (f_n(x_n))^2 - g(x_n) \right| = \left| \frac{2x_n}{n} + \frac{1}{n^2} \right| = 2 + \frac{1}{n^2} \geq 2 = \varepsilon_0$$

   for all $n \in \mathbb{N}$ (this is the negation of the definition of uniform convergence). Hence f_n^2 does not converge uniformly on $\mathbb{R}$. ■

3. Let $f_n(x) = \frac{1}{(1+x)^n}$ for $x \in [0,1]$. Denote by f the pointwise limit of f_n on $[0,1]$. Does $f_n \rightrightarrows f$ on $[0,1]$?

Proof: first note that $1 \leq 1+x$ on $[0,1]$. In particular, $\frac{1}{1+x} \leq 1$ on $[0,1]$. If $x \in (0,1]$, then $\frac{1}{(1+x)^n} \to 0$, according to one of the chapter's examples.

If $x = 0$,

$$\frac{1}{(1+x)^n} = \frac{1}{1^n} = 1 \to 1;$$

i.e. $f_n \to f$ on $[0,1]$, where

$$f(x) = \begin{cases} 0, & x \in (0,1] \\ 1, & x = 0 \end{cases}.$$

However, $f_n \not\rightrightarrows f$ by theorem 67, since f_n is continuous on $[0,1]$ for all $n \in \mathbb{N}$, but f is not. ■

4. Let (f_n) be the sequence of functions defined by $f_n(x) = \frac{x^n}{n}$, for $x \in [0,1]$ and $n \in \mathbb{N}$. Show that (f_n) converges uniformly to a differentiable function $f : [0,1] \to \mathbb{R}$, and that the sequence (f_n') converges pointwise to a function $g : [0,1] \to \mathbb{R}$, but that $g(1) \neq f'(1)$.

   Proof: the sequence $f_n(x) = \frac{x^n}{n} \to f(x) \equiv 0$ on $[0,1]$. Indeed, let $\varepsilon > 0$. By the Archimedean property, $\exists N_\varepsilon > \frac{1}{\varepsilon}$ s.t.

   $$\left| \frac{x^n}{n} - 0 \right| \leq \frac{|x|^n}{n} \leq \frac{1}{n} < \frac{1}{N_\varepsilon} < \varepsilon$$

   whenever $n > N_\varepsilon$. Note that f is differentiable and $f'(x) = 0$ for all $x \in [0,1]$. Furthermore, $f_n'(x) = \frac{nx^{n-1}}{n} = x^{n-1} \to g(x)$ on $[0,1]$, where

   $$g(x) = \begin{cases} 0, & x \in [0,1) \\ 1, & x = 1 \end{cases},$$

   by one of the examples I did in class. Then $g(1) = 1 \neq 0 = f'(1)$. ■

5. Show that $\lim\limits_{n\to\infty} \int_1^2 e^{-nx^2}\,\mathrm{d}x = 0$.

   Proof: as $\left(e^{-nx^2}\right)' = -2nxe^{-nx^2} < 0$ on $[1,2]$ for all $n \in \mathbb{N}$, e^{-nx^2} is decreasing on $[1,2]$ for all n, that is

   $$e^{-nx^2} \leq e^{-n(1)^2} = e^{-n} \quad \text{for all } n \in \mathbb{N}.$$

   Now,

   $$f_n(x) = e^{-nx^2} \rightrightarrows f(x) \equiv 0 \quad \text{on } [1,2].$$

   Indeed, let $\varepsilon > 0$. By the Archimedean property, $\exists N_\varepsilon > \ln \frac{1}{\varepsilon}$ (independent of x) s.t.

   $$\left| e^{-nx^2} - 0 \right| = e^{-nx^2} < e^{-Nx^2} \leq e^{-N} < \varepsilon$$

   whenever $n > N_\varepsilon$. Then (and only because of this uniform convergence),

   $$\lim_{n\to\infty} \int_1^2 e^{-nx^2}\,\mathrm{d}x = \int_1^2 \lim_{n\to\infty} e^{-nx^2}\,\mathrm{d}x = \int_1^2 0\,\mathrm{d}x = 0,$$

   by the limit interchange theorem for integrals. ■

6. Show that $\lim_{n\to\infty} \int_{\pi/2}^{\pi} \frac{\sin(nx)}{nx}\,\mathrm{d}x = 0$.

   Proof: for $n \in \mathbb{N}$, define $f_n : [\pi/2, \pi] \to \mathbb{R}$ by

$$f_n(x) = \frac{\sin(nx)}{nx}.$$

   Then each f_n is continuous. For all $n \in \mathbb{N}$, we have

$$\sup_{x\in[\pi/2,\pi]} \left\{ \left| \frac{\sin(nx)}{nx} \right| \right\} \leq \frac{2}{n\pi}.$$

   Since $2/n\pi \to 0$ as $n \to \infty$, we have $f_n \rightrightarrows 0$ (why?). Then the limit interchange theorem for integrals applies, and we have

$$\lim_{n\to\infty} \int_{\pi/2}^{\pi} \frac{\sin(nx)}{nx}\,\mathrm{d}x = \int_{\pi/2}^{\pi} 0\,\mathrm{d}x = 0.$$

   This completes the proof. ∎

7. Show that if $f_n \rightrightarrows f$ on $[a, b]$, and each f_n is continuous, then the sequence of functions $(F_n)_n$ defined by

$$F_n(x) = \int_a^x f_n(t)\,dt$$

   also converges uniformly on $[a, b]$.

   Proof: define $F(x) = \int_a^x f(t)\,dt$. Let $\varepsilon > 0$. Since $f_n \rightrightarrows f$, $\exists N \in \mathbb{N}$ such that, for all $n \geq N$, we have

$$|f_n(x) - f(x)| < \frac{\varepsilon}{b-a} \quad \forall\, x \in [a, b].$$

   Then, for all $n \geq N$ and $x \in [a, b]$, we have

$$\begin{aligned} |F_n(x) - F(x)| &= \left| \int_a^x f_n(t)\,dt - \int_a^x f(t)\,dt \right| \leq \int_a^x |f_n(t) - f(t)|\,dt \\ &\leq (x-a) \cdot \frac{\varepsilon}{b-a} \leq \varepsilon. \end{aligned}$$

   Thus $F_n \rightrightarrows F$ on $[a, b]$. ∎

5.4 Exercises

1. Prepare a 2-page summary of this chapter, with important definitions and results.

2. Are all the hypotheses of Theorem 68 necessary?