Chapter 6

Series of Functions

In the final chapter of this part, we discuss a specific type of sequence: the **series** (series of **numbers**, series of **functions**, and **power** series). Note that the latter is more naturally expressed using a complex analysis framework (see Chapter 22), but we present it here, as well as important theorems for regular series, in the real analysis framework.

6.1 Series of Numbers

Let $(x_n) \subseteq \mathbb{R}$. The **series** associated with (x_n), denoted by

$$S_{(x_n)} = \sum_{n=1}^{\infty} x_n,$$

is the sequence (s_n), where

$$s_1 = x_1, \quad s_2 = x_1 + x_2, \quad s_3 = x_1 + x_2 + x_3, \quad \ldots$$

If the sequence of partial sums s_n converges to S, we say the series $S_{(x_n)}$ **converges to the sum** S. When the context is clear, we may also write $\sum x_n (= S)$.

We start by producing a necessary condition for convergence.

Theorem 70
If $\sum_{n=1}^{\infty} x_n$ *converges, then* $x_n \to 0$.

Proof: let S be the limit of the partial sums. Then

$$\lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = S - S = 0,$$

with the second equation being guaranteed by Theorem 14 and the convergence of the series. ■

We can bypass the need to know the limit in order to prove convergence.

Theorem 71 (CAUCHY CRITERION FOR SERIES)
The series $\sum_{n=1}^{\infty} x_n$ *converges if and only if* $\forall \varepsilon > 0$, $\exists N_\varepsilon \in \mathbb{N}$ *such that*

$$m > n > N_\varepsilon \Longrightarrow |x_{n+1} + \cdots + x_m| < \varepsilon.$$

Proof: let (s_n) be the series of partial sums. If (s_n) converges, it is a Cauchy sequence, so that $\exists N_\varepsilon \in \mathbb{N}$ such that $m > n > N_\varepsilon \Longrightarrow |s_m - s_n| < \varepsilon$. But $|s_m - s_n| = |x_m + \cdots + x_{n+1}|$, so Cauchy's criterion holds.

Conversely, if Cauchy's criterion holds, the sequence of partial terms is a Cauchy sequence, and so the series converges by completeness of $\mathbb{R}$. ■

Other tests can be used to show the convergence of a series without knowing the limit.

Theorem 72 (COMPARISON TEST)
Let $\sum_{n=1}^{\infty} x_n$, $\sum_{n=1}^{\infty} y_n$ *be series whose terms are all non-negative. If* $\exists K \in \mathbb{N}$ *such that* $0 \leq x_n \leq y_n$ *when* $n > K$, *then*

1. $\sum_{n=1}^{\infty} y_n$ *converges* $\Longrightarrow \sum_{n=1}^{\infty} x_n$ *converges.*

2. $\sum_{n=1}^{\infty} x_n$ *diverges* $\Longrightarrow \sum_{n=1}^{\infty} y_n$ *diverges.*

Proof: we prove 1.; the proof for the other part is simply the contrapositive. Let $\varepsilon > 0$. As $\sum y_n$ converges, $\exists N_\varepsilon \in \mathbb{N}$ such that $0 \leq y_{n+1} + \cdots + y_m < \varepsilon$ according to Cauchy's criterion for series.

Hence, whenever $m \geq n > M_\varepsilon = \max\{N_\varepsilon, K\}$, then

$$0 \leq \sum_{i=n+1}^{m} x_i \leq \sum_{i=n+1}^{m} y_i < \varepsilon.$$

As such, $\sum x_n$ converges as it satisfies Cauchy's criterion for series. ■

Typical problems may look like the following.

Example: discuss the convergence of $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Solution: the limit of the partial sums of the first series converges to 1 as

$$\lim_{k\to\infty} s_k = \lim_{k\to\infty} \sum_{n=1}^{k} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \lim_{k\to\infty} \left(1 - \frac{1}{k+1}\right) = 1 - 0 = 1.$$

For the second series, since $n^2 \geq \frac{1}{2}(n^2+n) \geq 0$ for all $n \in \mathbb{N}$, then $\frac{2}{n(n+1)} \geq \frac{1}{n^2} \geq 0$ for all $n \in \mathbb{N}$, and

$$\infty > 2\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \geq \sum_{n=1}^{\infty} \frac{1}{n^2},$$

thus the series converges, according to the comparison theorem. □

When the sign of the underlying sequence terms alternates, convergence is particularly easy to establish.

Theorem 73 (Alternating Series Test)
Let (a_n) be a sequence of non-negative numbers such that $a_n \searrow 0$ (i.e., $a_n \to 0$ and $a_{n+1} \leq a_n$). Then $\sum_{n=0}^{\infty}(-1)^n a_n$ converges.

Proof: let (s_k) be the series of partial sums

$$s_k = \sum_{n=0}^{k}(-1)^n a_n.$$

The subsequence of even terms is $s_{2k} = s_{2k-2} - (a_{2k-1} - a_{2k})$; that of the odd terms is $s_{2k+1} = s_{2k-1} - (a_{2k} - a_{2k+1})$. Since $a_n \searrow 0$, $a_{n+1} \leq a_n$ for all n. Thus $s_{2k} \leq s_{2k-2}$ and $s_{2k+1} \geq s_{2k-1}$ for all $k \in \mathbb{N}$. But $s_{2k} \geq s_{2m+1}$ for all $k, m \in \mathbb{N}$ (left as an exercise), and so

$$a_0 = s_0 \geq s_2 \geq s_4 \geq \cdots \geq s_5 \geq s_3 \geq s_1 = a_0 - a_1.$$

Thus (s_{2k}) is a bounded decreasing sequence and (s_{2k-1}) is a bounded increasing sequence, and so $\lim_{k\to\infty} s_{2k}$ and $\lim_{k\to\infty} s_{2k-1}$ exist. According to Theorem 14, then, we have

$$\lim_{k\to\infty}(s_{2k} - s_{2k-1}) = \lim_{k\to\infty} a_{2k} = 0$$

since $a_n \searrow 0$, which implies that the alternating series converges:

$$\lim_{k\to\infty} \sum_{n=0}^{2k}(-1)^n a_n = \lim_{k\to\infty} s_{2k} = \lim_{k\to\infty} s_{2k+1} = \lim_{k\to\infty} \sum_{n=0}^{2k+1}(-1)^n a_n,$$

which completes the proof. ■

Even though it was not part of the statement, the proof of Theorem 73 allows us to conclude that the value of a convergent alternating series lies between a_{2k} and a_{2m+1} for all $k, m \in \mathbb{N}$.

Example: the alternating harmonic series $-1 + 1/2 - 1/3 + \cdots$ converges. Indeed, consider the sequence $(a_n) = (\frac{1}{n}) = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$. As $\frac{1}{n} \to 0$ and $\frac{1}{n+1} \leq \frac{1}{n}$ for all n, then the corresponding alternating series converges. Its value lies between $s_0 = 1$ and $s_1 = 1 - \frac{1}{2} = \frac{1}{2}$, $s_1 = \frac{1}{2}$ and $s_2 = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$, $s_2 = \frac{5}{6}$ and $s_3 = \frac{5}{6} - \frac{1}{4} = \frac{7}{12}$, etc. $\square$

Two other convergence tests are often used in practice: the ratio test and the root test.

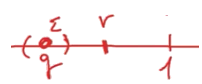
Theorem 74 (Ratio Test)
Let (a_n) be a sequence of positive real numbers.

1. *If $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} < 1$, then $\sum\limits_{n=1}^{\infty} a_n$ converges.*

2. *If $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} > 1$, then $\sum\limits_{n=1}^{\infty} a_n$ diverges.*

Proof:

1. Assume $0 \leq \frac{a_{n+1}}{a_n} \to q < 1$. Let $r = \frac{q+1}{2}$. Thus $q < r < 1$ and there are only finitely many indices n for which $\frac{a_{n+1}}{a_n} > r$. Indeed, let $\varepsilon \in (0, \frac{1-q}{2})$.



Then, $\exists N_\varepsilon \in \mathbb{N}$ such that

$$n > N_\varepsilon \Longrightarrow \frac{a_{n+1}}{a_n} - q < \varepsilon < \frac{1-q}{2} \Longrightarrow \frac{a_{n+1}}{a_n} \leq \frac{q+1}{2} = r.$$

Then

$$n > N_\varepsilon \Longrightarrow a_n = \frac{a_n}{a_{n-1}} \cdot \cdots \cdot \frac{a_{N+1}}{a_N} \cdot a_N \leq r^{n-N} a_N.$$

The tail of the original series converges, as

$$\sum_{n=N+1}^{\infty} a_n \leq \sum_{n=N+1}^{\infty} a_N r^{n-N} = \frac{a_N}{r^N} \sum_{n=N+1}^{\infty} r^n = \frac{a_N}{r^N}\left(\frac{r^{N+1}}{1-r}\right) < \infty,$$

where the last equation is left as an exercise. As $a_0 + \cdots + a_N$ is also finite, the full series converges.

2. Assume $\frac{a_{n+1}}{a_n} \to q > 1$. Using a similar argument as in part 1., we can show that $\exists r > 1$ and $N \in \mathbb{N}$ such that $\frac{a_{n+1}}{a_n} \geq r > 1$ for all $n \in \mathbb{N}$, so that $a_{n+1} > a_n$ for all $n \geq 1$.

   Thus $a_n \not\to 0$, and so $\displaystyle\sum_{n=0}^{\infty} a_n$ diverges, according to Theorem 70. ■

If $\frac{a_{n+1}}{a_n} \to 1$, then the series may converge or may diverge, depending on the specific nature of a_n. The key parts of the proof (namely, the convergence of the tail in the first case and the condition $a_n \not\to 0$ in the second) are also valid if the statement is relaxed to some extent.

Theorem 74 (Ratio Test Reprise)
Let (a_n) be a sequence of real numbers with $a_n \neq 0$ for all n.

1. *If* $\displaystyle\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, *then* $\displaystyle\sum_{n=1}^{\infty} a_n$ *converges.*
2. *If* $\displaystyle\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$, *then* $\displaystyle\sum_{n=1}^{\infty} a_n$ *diverges.*

The root test is similar (we will not prove it).

Theorem 75 (Root Test)
Let (a_n) be a sequence of positive real numbers.

1. *If* $\displaystyle\limsup_{n\to\infty} \sqrt[n]{a_n} < 1$, *then* $\displaystyle\sum_{n=1}^{\infty} a_n$ *converges.*
2. *If* $\displaystyle\liminf_{n\to\infty} \sqrt[n]{a_n} > 1$, *then* $\displaystyle\sum_{n=1}^{\infty} a_n$ *diverges.*

This general result also has a stricter version, replacing lim sup and lim inf by lim. In either version, if the limit is 1, then the series may converge or diverge, depending on the specific nature of the terms a_n.

Examples: discuss the convergence of

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n2^n}, \quad \sum_{n=1}^{\infty} \frac{3^n}{n2^n}, \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^p},\ p > 0.$$

1. The terms are all non-zero. We compute

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{(n+1)2^{n+1}}\cdot\frac{n2^n}{(-1)^n}\right| = \frac{1}{2}\lim_{n\to\infty}\left|\frac{n}{n+1}\right| = \frac{1}{2} < 1,$$

   so the series converges according to the ratio test. □

2. The terms are all positive. We compute

$$\lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\sqrt[n]{\frac{3^n}{n2^n}} = \frac{3}{2}\lim_{n\to\infty}\frac{1}{n^{1/n}} = \frac{3}{2} > 1,$$

   so the series diverges according to the root test. □

3. The terms are all positive. For all $p > 0$, we compute

$$\lim_{n\to\infty}\left|\frac{1}{(n+1)^p}\cdot\frac{n^p}{1}\right| = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^p \to 1^p = 1.$$

   Thus we cannot use the ratio test to determine if the series converges. If $p = 1$, the harmonic series is bounded below by a divergent series

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n} &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\cdots\\ &\geq 1+\frac{1}{2}+\underbrace{\frac{1}{4}+\frac{1}{4}}_{=1/2}+\underbrace{\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}}_{=1/2}+\cdots = 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots=\infty,\end{aligned}$$

   and so must itself be divergent. As $\frac{1}{n^p} > \frac{1}{n}$ for all n when $p < 1$, then the series diverges for all $0 < p \leq 1$ according to the comparison theorem. If $p > 1$, the $p-$series is bounded above by a convergent series

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^p} &= 1+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}+\frac{1}{8^p}+\cdots\\ &\leq 1+\underbrace{\frac{1}{2^p}+\frac{1}{2^p}}_{\text{2 times}}+\underbrace{\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}}_{\text{4 times}}+\frac{1}{8^p}+\cdots\\ &= 1+2^1\cdot\frac{1}{(2^1)^p}+2^2\cdot\frac{1}{(2^2)^p}+\cdots=\sum_{k=0}^{\infty}2^{k(1-p)}=\sum_{k=0}^{\infty}\frac{1}{(2^{p-1})^k}.\end{aligned}$$

   But this series converges according to the root test. Indeed, all the terms are positive, and, because $p > 1$,

$$\lim_{k\to\infty}\sqrt[k]{\frac{1}{(2^{p-1})^k}} = \lim_{k\to\infty}\frac{1}{2^{p-1}} < 1.$$

   Thus the $p-$series diverges for $0 < 1 \leq p$ and converges for $p > 1$. □

The next result (provided without proof) shows that the series of the absolute values may play an important role in the convergence of the "raw" series.

Theorem 76 (Absolute Convergence)
If the series $\sum_{n=0}^{\infty} |a_n|$ *converges, so does* $\sum_{n=0}^{\infty} a_n$ *(note that this is not an "if and only if" statement).*

The final result explains when the terms of a series can be re-arranged without affecting the convergence of the original series.

Theorem 77 (Series Re-arrangement)
If the series $\sum_{n=0}^{\infty} |a_n|$ *converges, so does* $\sum_{n=0}^{\infty} a_{\varphi(n)}$*,* $\varphi : \mathbb{N} \to \mathbb{N}$ *a bijection.*

6.2 Series of Functions

Series of functions play the same role for sequences of functions that series played for sequences of numbers. Let $I \subseteq \mathbb{R}$ and $f_n : I \to \mathbb{R}$, $\forall n \in \mathbb{N}$. If the sequence of partial sums

$$s_1(x) = f_1(x), \quad s_2(x) = f_1(x) + f_2(x), \quad s_3(x) = f_1(x) + f_2(x) + f_3(x), \quad \ldots$$

converges to some function $f : I \to \mathbb{R}$ for all $x \in I$, we say that the series of functions $\sum f_n$ **converges pointwise to** f **on** I.

Example: consider the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$, with $f_n(x) = x^n$ for each $n \in \mathbb{N}$. Does the sequence of partial sums $s_k(x)$ converge to some pointwise limit over some $A \subseteq \mathbb{R}$?

Solution: formally, we have

$$(1 - x^{k+1}) = (1 - x)(1 + x + x^2 + \cdots + x^k) = (1 - x)s_k(x).$$

Thus

$$x \neq -1 \implies s_k(x) = \sum_{n=0}^{k} x^n = \frac{1 - x^{k+1}}{1 - x},$$

and so

$$\sum_{n=0}^{\infty} x^n = \lim_{k \to \infty} s_k(x) = \frac{1}{1 - x}$$

when $x \in (-1, 1)$. $\square$

If the sequence of partial sums (s_n) converges uniformly to f on I, we say that the series of functions $\sum f_n$ **converges uniformly to** f **on** I. If the convergence of the series of functions is uniform, the limit interchange theorems can be applied.

Theorem 78 (Cauchy Criterion for Series of Functions)
Let $f_n : I \to \mathbb{R}$ for all $n \in \mathbb{N}$. The series of functions with term f_n converges uniformly to some function $f : I \to \mathbb{R}$ if and only if $\forall \varepsilon > 0$, $\exists N_\varepsilon \in \mathbb{N}$ (independent of $x \in I$) such that

$$m > n > N_\varepsilon \Longrightarrow \left| \sum_{i=n+1}^{m} f_i(x) \right| < \varepsilon.$$

Proof: the proof follows directly from Theorem 66 applied to the sequence of partial sums $s_m : I \to \mathbb{R}$. ■

The next result is a powerful tool to prove uniform convergence (and as a pre-requisite to the use of the limit interchange theorems). The simplicity of its proof belies its importance.

Theorem 79 (Weierstrass $M-$Test)
Let $f_n : I \to \mathbb{R}$ and $M_n \geq 0$ for all $n \in \mathbb{N}$. Assume that $|f_n(x)| \leq M_n$ for all $x \in I$, $n \in \mathbb{N}$. Then

$$\sum_{n=1}^{\infty} M_n \text{ converges} \Longrightarrow \sum_{n=1}^{\infty} f_n \text{ converges uniformly on } I.$$

Proof: let $\varepsilon > 0$. Since $\sum M_n$ converges, its sequences of partial sums (s_k) is Cauchy and $\exists K_\varepsilon \in \mathbb{N}$ such that

$$m > n > K_\varepsilon \Longrightarrow \sum_{i=n+1}^{m} M_i < \varepsilon.$$

But

$$m > n > K_\varepsilon \Longrightarrow \left| \sum_{i=n+1}^{m} f_i(x) \right| \leq \sum_{i=n+1}^{m} |f_i(x)| \leq \sum_{i=n+1}^{m} M_i < \varepsilon;$$

since K_ε is independent of $x \in I$, $\displaystyle\sum_{n=1}^{\infty} f_n$ converges uniformly on I. ■

The following example showcases its usefulness.

Example: let $\varepsilon \in (0,1)$. Consider the sequence of functions $g_n : \mathbb{R} \to \mathbb{R}$ defined by $g_n(x) = nx^{n-1}$ for each $n \in \mathbb{N}$. Does $\sigma_k(x) \rightrightarrows \sigma(x)$ on $I_\varepsilon = (-1+\varepsilon, 1-\varepsilon)$ for some σ? If so, find σ.

Solution: consider the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) = x^n$ for each $n \in \mathbb{N}$, and the corresponding sequence of partial sums $s_k(x)$ defined by $s_k(x) = 1 + x + \cdots + x^k$.

We have already shown that $s_k(x) \to \frac{1}{1-x}$ pointwise on $(-1+\varepsilon, 1-\varepsilon)$. The partials sums s_k are differentiable on I_ε since

$$\sigma_k(x) = s_k'(x) = 1 + 2x + 3x^2 + \cdots + kx^{k-1}$$

are polynomials (in fact, σ_k is also continuous on I_ε). Furthermore, note that the sequence of derivatives of partial sums $\sigma_k(x)$ converge uniformly on I_ε. To show this, note that

$$|g_n(x)| = |nx^{n-1}| \leq n|1-\varepsilon|^{n-1} = M_n \quad \forall x \in I_\varepsilon,\ \forall n \in \mathbb{N}.$$

But

$$\sum_{n=0}^{\infty} M_n = \sum_{n=0}^{\infty} n(1-\varepsilon)^{n-1}.$$

Since

$$\lim_{n\to\infty} \frac{(n+1)(1-\varepsilon)^n}{n(1-\varepsilon)^{n-1}} = (1-\varepsilon) \lim_{n\to\infty} \frac{n+1}{n} = (1-\varepsilon) < 1,$$

then $\sum M_n$ converges according to the ratio test.

According to the Weierstrass $M-$test, then, $\sigma_k(x) \rightrightarrows \sigma(x)$ on I_ε for some function $\sigma : I_\varepsilon \to \mathbb{R}$. We can use the limit interchange theorem 68 to identify σ:

$$\sigma(x) = \lim_{k\to\infty} \sigma_k(x) = \lim_{k\to\infty} \frac{\mathrm{d}}{\mathrm{d}x}[s_k(x)] = \frac{\mathrm{d}}{\mathrm{d}x}\Big[\lim_{k\to\infty} s_k(x)\Big] = \frac{\mathrm{d}}{\mathrm{d}x}\Big[\frac{1}{1-x}\Big],$$

which is to say $\sigma(x) = \frac{1}{(1-x)^2}$. □

Incidentally, Theorem 68 also tells us that $s_k(x) \rightrightarrows \frac{1}{1-x}$ on I_ε, for all $0 < \varepsilon < 1$, and that for all $k \in \mathbb{N}$ and $x \in I_\varepsilon$, $\varepsilon \in (0,1)$, we have

$$\sum_{n=0}^{\infty} \frac{\mathrm{d}^k}{\mathrm{d}x^k}[x^n] = \frac{\mathrm{d}^k}{\mathrm{d}x^k} \sum_{n=0}^{\infty} x^n = \frac{\mathrm{d}^k}{\mathrm{d}x^k}\Big(\frac{1}{1-x}\Big).$$

6.3 Power Series

A **power series** around its **center** $x = x_0$ is a formal expression of the form

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n.$$

We have already seen an example of such a series, which converged uniformly on intervals containing $x_0 = 0$:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{on } I_\varepsilon = (-1+\varepsilon, 1-\varepsilon),\ \forall \varepsilon \in (0,1);$$

note, however, that the convergence is only **pointwise** on $(-1,1)$. The function $f : A \to \mathbb{R}$, $f(x) = \frac{1}{1-x}$ is defined for all $x \neq 1$, however, yet the power series $1 + x + x^2 + \cdots$ does not converge to f outside of $(-1,1)$.[1]

Examples: where do the following power series converge:

$$\sum_{n=0}^{\infty} x^n, \quad \sum_{n=1}^{\infty} (nx)^n, \quad \sum_{n=1}^{\infty} \left(\frac{x}{n}\right)^n?$$

Solution: we have seen that the first power series converges only on $(-1,1)$.

The second power series obviously converges when $x = 0$. To show that it fails to converge on $\mathbb{R} \setminus \{0\}$, note that if $|x| > 0$, then $\exists N \in \mathbb{N}$ such that $N > \frac{2}{|x|}$ by the Archimedean property. Thus,

$$n > N \Longrightarrow |(nx)^n| = n^n |x|^n > 2^n$$

and the sequence $(nx)^n$ is unbounded, which means that the terms do not go to 0, and so the series diverges.

For the third power series, let $x \in \mathbb{R}$. By the Archimedean property, $\exists N \in \mathbb{N}$ such that $N > 2|x|$. Thus,

$$n > N \Longrightarrow \left|\left(\frac{x}{n}\right)^n\right| = \frac{|x|^n}{n^n} < \frac{1}{2^n}.$$

According to the Weierstrass $M-$test and Theorem 76, the series thus converges uniformly on $\mathbb{R}$. $\square$

[1] Power series are commonly used as a formal **guessing procedure** to solve differential equations, but this is not a topic we will tackle at the moment. It is also natural to try to determine for which functions $f : A \to \mathbb{R}$ (and which A) we can find a **sequence of coefficients** (a_n) such that

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \quad \forall x \in A;$$

questions of this ilk are more naturally answered in $\mathbb{C}$; a more complete treatment would be provided in a complex analysis course.

The **radius of convergence** of a power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is

$$R = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}.$$

If the limit exists, we can replace lim sup by lim. Intuitively, for all large enough n, we have:

$$-R^{-n} \le -|a_n| \le a_n \le |a_n| \le R^{-n},$$

so that

$$-\sum_{n>N}\left(\frac{x-x_0}{R}\right)^n \le \sum_{n>N} a_n(x-x_0)^n \le \sum_{n>N}\left(\frac{x-x_0}{R}\right)^n.$$

The bounds are geometric series, and they converge when $|x-x_0| < R$. We would then expect the original power series to converge on the **interval of convergence** $|x-x_0| < R$.

Theorem 80
Let R be the radius of convergence of the power series

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n.$$

Then, if

- $R=0$, *the power series converges for $x=x_0$ and diverges for $x \neq x_0$;*
- $R=\infty$, *the power series converges absolutely on $\mathbb{R}$, and*
- $0<R<\infty$, *the power series converges absolutely on $|x-x_0|<R$, diverges on $|x-x_0|>R$; the extremities must be analyzed separately.*

Proof: follows immediately from the root test. ■

But we can provide a stronger convergence statement.

Theorem 81
The power series of Theorem 80 converges uniformly on any compact sub-interval

$$[a,b] \subseteq (x_0-R, x_0+R).$$

Proof: let $\ell = \max\{|a-x_0|, |b-x_0|\} < R$. For every $n \in \mathbb{N}$, set $M_n = \ell^n |a_n| \ge 0$ and $\varepsilon = \frac{1}{4}(R-\ell)$.

Since $\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n}$, $\exists N_\varepsilon \in \mathbb{N}$ such that $n > N_\varepsilon \implies |a_n| \leq (\frac{1}{R-\varepsilon})^n$. Thus, for all $n > N_\varepsilon$, we have

$$0 \leq M_n = \ell^n |a_n| = (R-4\varepsilon)^n |a_n| \leq \left(\frac{R-4\varepsilon}{R-\varepsilon}\right)^n = \Big(1 - \underbrace{\frac{3\varepsilon}{R-\varepsilon}}_{>0}\Big)^n,$$

so that

$$\begin{aligned}\sum_{n=0}^{\infty} M_n = \sum_{n=0}^{N_\varepsilon} M_n + \sum_{n>N_\varepsilon} M_n &\leq \sum_{n=0}^{N_\varepsilon} M_n + \sum_{n>N_\varepsilon}\left(1-\frac{3\varepsilon}{R-\varepsilon}\right)^n \\ &\leq \sum_{n=0}^{N_\varepsilon} M_n + \sum_{n=0}^{\infty}\left(1-\frac{3\varepsilon}{R-\varepsilon}\right)^n = \underbrace{\sum_{n=0}^{N_\varepsilon} M_n}_{\text{finite}} + \frac{R-\varepsilon}{3\varepsilon} < \infty.\end{aligned}$$

But for all $x \in [a,b]$, we have

$$|a_n(x-x_0)^n| \leq |a_n|\ell^n = M_n, \quad \text{for all } n \in \mathbb{N}.$$

According to Theorem 79, the power series converges uniformly on $[a,b]$. ■

In what follows, we let $f : (x_0 - R, x_0 + R) \to \mathbb{R}$ be the function defined by

$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n, \quad \text{and} \quad s_N(x) = \sum_{n=0}^{N} a_n (x-x_0)^n;$$

these have multiple nice properties, courtesy of the limit interchange theorems.

Theorem 82
The function f is continuous on any closed bounded interval $[a,b] \subseteq (x_0 - R, x_0 + R)$.

Proof: the functions $a_n(x-x_0)^n$ are continuous on $[a,b]$ for all n, and

$$s_N(x) = \sum_{n=0}^{N} a_n (x-x_0)^n \rightrightarrows f(x) \text{ on } [a,b] \quad \text{when } N \to \infty.$$

According to Theorem 67, f is continuous on $[a,b]$. ■

We get more than continuity, however.

Theorem 83
Let $x \in (x_0 - R, x_0 + R)$. Then f is Riemann-integrable between x_0 and x and

$$\int_{x_0}^{x} f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-x_0)^{n+1}.$$

Proof: without loss of generality, assume $x > x_0$. As in the proof of Theorem 82, $s_N(x) \rightrightarrows f(x)$ on $[x_0, x]$ when $N \to \infty$. Thus, according to Limit Interchange Theorem 69, we have

$$\begin{aligned}\int_{x_0}^{x} f(t)\,dt &= \lim_{N\to\infty}\int_{x_0}^{x} s_N(t)\,dt = \lim_{N\to\infty}\int_{x_0}^{x}\sum_{n=0}^{N} a_n(t-x_0)^n\,dt \\ &= \lim_{N\to\infty}\sum_{n=0}^{N}\int_{x_0}^{x} a_n(t-x_0)^n\,dt = \sum_{n=0}^{\infty}\frac{a_n}{n+1}(x-x_0)^{n+1},\end{aligned}$$

which completes the proof. ■

The last result shows that power series really do behave nicely on their convergence interval.

Theorem 84
The function f is differentiable on $(x_0 - R, x_0 + R)$ and

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1}.$$

Proof: as $n^{1/n} \to 1$,

$$\limsup_{n\to\infty}(n|a_n|)^{1/n} = \limsup_{n\to\infty} n^{1/n} \cdot \limsup_{n\to\infty} |a_n|^{1/n} = \frac{1}{R},$$

so the radius of convergence of both power series is identical, and so, in particular, $s_N'(x)$ converges uniformly on any closed bounded interval $[a, b] \subseteq (x_0 - R, x_0 + R)$.

Thus, according to limit interchange theorem 68, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\big[f(x)\big] &= \lim_{N\to\infty}\frac{\mathrm{d}}{\mathrm{d}x}\big[s_N(x)\big] = \lim_{N\to\infty}\frac{\mathrm{d}}{\mathrm{d}x}\sum_{n=0}^{N}\big[a_n(x-x_0)^n\big] \\ &= \lim_{N\to\infty}\sum_{n=0}^{N}\frac{\mathrm{d}}{\mathrm{d}x}\big[a_n(x-x_0)^n\big] = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1},\end{aligned}$$

which completes the proof. ■

How do we compute the power series coefficients a_n? Combining Theorems 82 and 84, we see that f is **smooth** in its interval of convergence (i.e., all of its derivatives are continuous).

> **Theorem 85**
> *If $R > 0$, then*
> $$a_n = \frac{f^{(n)}(x_0)}{n!}.$$
> **Proof:** if $x = x_0$, then $f(x_0) = a_0$, which corresponds to the case $n = 0$. When $n = k > 0$, then repeated application of Theorem 84 yields
> $$f^{(k)}(x) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n (x - x_0)^{n-k} \quad \text{on } (x_0 - R, x_0 - R).$$
> If we evaluate at $x = x_0$, we get $f^{(k)}(x_0) = k! a_k$, thus $a_k = \frac{f^{(k)}(x_0)}{k!}$. ■

As a corollary, if $\exists r > 0$ such that

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \text{and} \quad g(x) = \sum_{n=0}^{\infty} b_n (x - x_0)^n$$

and $f(x) = g(x)$ for all $x \in (x_0 - r, x_0 + r)$, then $a_n = b_n$ for all $n \in \mathbb{N}$.[2]

> **Example:** consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by
> $$f(x) = \begin{cases} \exp(-1/x^2), & x \neq 0 \\ 0, & x = 0 \end{cases}$$
> Show that f does not have a power series expansion.
>
> **Proof:** for all $n \in \mathbb{N}$, it can be shown that
> $$f^{(n)}(x) = \begin{cases} \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\exp(-1/x^2)\right], & x \neq 0 \\ 0, & x = 0 \end{cases}$$
> is continuous and that $f^{(n)}(0) = 0$. According to the corollary to Theorem 85, if f is equal to its power series on some interval $(-r, r)$, then all the a_n would be 0, and so $f \equiv 0$, but $f \not\equiv 0$, so f cannot be equal to its power series expansion. ■

Thus, we cannot always assume that a function is equal to its power series. There are other ways to expand a function as an infinite series, most notably *via* **Laurent Series** and **Fourier Series**. These topics are covered in courses in complex analysis and partial differential equations, respectively, although we briefly discuss the latter in Chapter 11.

[2] Attempts to strengthen this **uniqueness** result must necessarily fail.

6.4 Solved Problems

1. Answer the following questions about series.

   a) If $\sum_{k=1}^{\infty}(a_k+b_k)$ converges, what about $\sum_{k=1}^{\infty}a_k$ and $\sum_{k=1}^{\infty}b_k$?

   b) If $\sum_{k=1}^{\infty}(a_k+b_k)$ diverges, what about $\sum_{k=1}^{\infty}a_k$ and $\sum_{k=1}^{\infty}b_k$?

   c) If $\sum_{k=1}^{\infty}(a_{2k}+a_{2k-1})$ converges, what about $\sum_{k=1}^{\infty}a_k$?

   d) If $\sum_{k=1}^{\infty}a_k$ converges, what about $\sum_{k=1}^{\infty}(a_{2k}+a_{2k-1})$?

   Solution:

   a) They might both diverge. Consider $a_k=-k$ and $b_k=k$. However, if one converges, then so does the other, by the arithmetic of limits/series.

   b) At least one of them diverges because if they both converged, then the series of sums would converge as well (according to a proposition seen in the notes).

   c) Nothing. Consider $a_{2k}=k$, $a_{2k+1}=-k$, for which $\sum_{k=1}^{\infty}a_k$ diverges, but $a_{2k}=\frac{1}{k^2}$, $a_{2k+1}=0$, for which $\sum_{k=1}^{\infty}a_k$ converges.

   d) It also converges. The sequence of partial sums of the second series is
   $$(a_1+a_2, a_1+a_2+a_3+a_4, , a_1+a_2+a_3+a_4+a_5+a_6, \ldots)$$
   is a subsequence of the sequence of partial sums of the first series
   $$(a_1, a_1+a_2, a_1+a_2+a_3, a_1+a_2+a_3+a_4, \ldots).$$
   If the first series sequence of partial sums converges, so does the subsequence's series. $\square$

2. For all $r>1$, show that
   $$\frac{1}{r-1}=\frac{1}{r+1}+\frac{2}{r^2+1}+\frac{4}{r^4+1}+\frac{8}{r^8+1}+\cdots$$
   Solution: we see that
   $$\frac{1}{\ell+1}=\frac{1}{\ell-1}-\frac{2}{\ell^2-1}.$$
   Thus, for all $k\in\mathbb{N}$, if $\ell=2^k$, we have
   $$\begin{aligned}\frac{1}{r^{2^k}+1}&=\frac{1}{r^{2^k}-1}-\frac{2}{r^{2^{k+1}}-1}\\ \Longrightarrow\frac{2^k}{r^{2^k}+1}&=\frac{2^k}{r^{2^k}-1}-\frac{2^{k+1}}{r^{2^{k+1}}-1}.\end{aligned}$$

Therefore, we have a telescoping sum

$$\sum_{k=1}^{\infty} \frac{2^k}{r^{2^k}+1} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{2^k}{r^{2^k}+1} = \lim_{n\to\infty} \left(\frac{1}{r-1} - \frac{2^n}{r^{2^n}-1} \right) = \frac{1}{r-1},$$

where the last equation follows from the fact that, for $r > 1$, we have

$$\lim_{m\to\infty} \frac{m}{r^m} = 0.$$

This completes the proof. ■

3. Using Riemann integration, find the values of p for which the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.

Solution: if $p \leq 0$, then $\frac{1}{n^p} \not\to 0$ so the series diverges. In what follows, then, let $p > 0$. For $k \in \mathbb{N}$, consider the function $f_{k;p} : [1,k] \to \mathbb{R}$ defined by $f_{k;p}(x) = \frac{1}{x^p}$. Since $f'_{k;p}(x) = -\frac{p}{x^{p+1}} < 0$ for all $x \geq 1$, $f_{k;p}$ is strictly decreasing on $[1,k]$. Thus $f_{k;p}$ is Riemann-integrable on $[1,k]$. Consider the partition $P_k = \{1,2,\ldots,k,k+1\}$ of $[1,k+1]$. Since $f_{k;p}$ is Riemann-integrable,

$$L(f_{k;p};P_k) \leq \int_1^{k+1} f_{k;p} \leq U(f_{k;p};P_k).$$

As $f_{k;p}$ is decreasing on the sub-interval $[\mu,\nu]$, $f_{k;p}$ reaches its maximum at μ and its minimum at ν; Hence

$$\begin{aligned} U(f_{k;p};P_k) &= \sum_{n=1}^{k} f_{k;p}(n)(n+1-n) = \sum_{n=1}^{k} \frac{1}{n^p}, \quad \text{and} \\ L(f_{k;p};P_k) &= \sum_{n=2}^{k+1} f_{k;p}(n+1)(n+1-n) = \sum_{n=2}^{k+1} \frac{1}{n^p}. \end{aligned}$$

But

$$\sum_{n=2}^{k+1} \frac{1}{n^p} = \frac{1}{(k+1)^p} - 1 + \sum_{n=1}^{k} \frac{1}{n^p}.$$

Thus

$$\frac{1}{(k+1)^p} - 1 + \sum_{n=1}^{k} \frac{1}{n^p} \leq \int_1^{k+1} f_{k;p} \leq \sum_{n=1}^{k} \frac{1}{n^p}.$$

Write $s_{k;p}$ for the partial sum and note that

$$\int_1^{k+1} f_{k;p} = \int_1^{k+1} \frac{dx}{x^p} = \begin{cases} \ln(k+1), & \text{when } p = 1 \\ \frac{1}{1-p}(k^{1-p}-1), & \text{when } p \neq 1 \end{cases}$$

If $p = 1$, then $\ln(k+1) \leq s_{k;1}$ for all k. Since the sequence $\{\ln(k+1)\}_k$ is unbounded, so must $\{s_{k;1}\}_k$ be unbounded, which means that the corresponding series cannot converge. If $p > 1$, then

$$\lim_{k\to\infty} \left(\frac{1}{1-p}(k^{1-p}-1) + 1 - \frac{1}{(k+1)^p} \right) = \frac{p}{p-1}.$$

Since $s_{k;p}$ is monotone (as every additional $\frac{1}{n^p}$ added to the partial sum is positive) and since $s_{k;p}$ is bounded above by the convergent sequence

$$\left\{\frac{1}{1-p}(k^{1-p}-1)+1-\frac{1}{(k+1)^p}\right\}_k,$$

$s_{k;p}$ is a convergent sequence. If $p<1$, then

$$\left\{\frac{1}{1-p}(k^{1-p}-1)\right\}_k$$

is unbounded. As $s_{k;p}\geq \frac{1}{1-p}(k^{1-p}-1)$ for all k, $\{s_{k;p}\}$ is also unbounded, which means that the corresponding series cannot converge. Thus, the series converges if and only if $p>1$. $\square$

4. Which of the following series converge?

a) $\displaystyle\sum_{n=1}^{\infty}\frac{n(n+1)}{(n+2)^2}$

b) $\displaystyle\sum_{n=1}^{\infty}\frac{2+\sin^3(n+1)}{2^n+n^2}$

c) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2^n-1+\cos^2 n^3}$

d) $\displaystyle\sum_{n=1}^{\infty}\frac{n+1}{n^2+1}$

e) $\displaystyle\sum_{n=1}^{\infty}\frac{n+1}{n^3+1}$

f) $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{n^n}$

g) $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{5^n}$

h) $\displaystyle\sum_{n=1}^{\infty}\frac{n^n}{3^{1+2n}}$

i) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{5n+3n^3}{7n^3+2}\right)^n$

Solution: we use the various tests at our disposal.

a) Since

$$\lim_{n\to\infty}\frac{n(n+1)}{(n+2)^2}=1\neq 0,$$

the series diverges .

b) Since $-1 \leq \sin^3(n+1) \leq 1$, we have

$$0 \leq \frac{2+\sin^3(n+1)}{2^n+n^2} \leq \frac{1}{2^n+n^2} \leq \frac{1}{2^n}.$$

Thus the given series converges by comparison with the geometric series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$.

c) If a_n denotes the n-th term of the series, we have

$$\frac{a_{n+1}}{a_n} = \frac{2^n - 1 + \cos^2 n^3}{2^{n+1} - 1 + \cos^2(n+1)^3} \to \frac{1}{2} < 1.$$

Thus the series converges by the ratio test.

d) We have

$$\frac{n+1}{n^2+1} \geq \frac{n}{2n^2} = \frac{1}{2n}.$$

Thus the series diverges by comparison with the harmonic series.

e) We have

$$0 \leq \frac{n+1}{n^3+1} \leq \frac{2n}{n^3} = \frac{2}{n^2}.$$

Thus the series converges by comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2}$.

f) For $n \geq 2$, we have

$$0 \leq \frac{n!}{n^n} = \frac{1}{n} \cdot \frac{2}{n} \cdot \frac{3 \cdot 4 \cdots n}{n^{n-2}} \leq \frac{2}{n^2}.$$

Thus the series converges by comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2}$.

g) If a_n denotes the n-th term in the series, we have

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{5^{n+1}} \frac{5^n}{n!} = \frac{n+1}{5} \to \infty.$$

Thus the series diverges by the ratio test.

h) We have

$$\left(\frac{n^n}{3^{1+2n}}\right)^{1/n} = \frac{n}{3^{2+1/n}} \to \infty.$$

Thus the series diverges by the root test.

i) We have

$$\left(\left(\frac{5n+3n^3}{7n^3+2}\right)^n\right)^{1/n} = \frac{5n+3n^3}{7n^3+2} \to \frac{3}{7} < 1.$$

Thus the series converges by the root test. □

5. Give an example of a power series $\sum_{k=0}^{\infty} a_k x^k$ with interval of convergence $[-\sqrt{2}, \sqrt{2})$.

Proof: consider the series

$$\sum_{k=1}^{\infty} \frac{x^k}{k}.$$

We have

$$\limsup_{k\to\infty} \sqrt[k]{\frac{|x|^k}{k}} = \limsup_{k\to\infty} \frac{|x|}{\sqrt[k]{k}} = |x|.$$

Therefore, by the root test, the series converges when $|x| < 1$ and diverges for $|x| > 1$. For $x = 1$, the series is the harmonic series, which diverges. For $x = -1$, it is the alternating harmonic series, which converges. Thus, the series converges precisely on the interval $[-1, 1)$.

Now, replace x by $x/\sqrt{2}$. The corresponding power series is thus

$$\sum_{k=0}^{\infty} \frac{1}{\sqrt{2}^k k} x^k.$$

We have

$$\limsup_{k\to\infty} \sqrt[k]{\frac{|x|^k}{\sqrt{2}^k k}} = \limsup_{k\to\infty} \frac{|x|}{\sqrt{2}\sqrt[k]{k}} = \frac{|x|}{\sqrt{2}}.$$

The series converges on $\frac{|x|}{\sqrt{2}} < 1$ and diverges on $\frac{|x|}{\sqrt{2}} > 1$. For $x = \sqrt{2}$, the series is the harmonic series, which diverges. For $x = -\sqrt{2}$, it is the alternating harmonic series, which converges.

Thus, the series converges precisely on the interval $[-\sqrt{2}, \sqrt{2})$. ■

6. Find the values of x for which the following series converge:

a) $\sum_{n=1}^{\infty} (nx)^n$;

b) $\sum_{n=1}^{\infty} x^n$;

c) $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$;

d) $\sum_{n=1}^{\infty} \frac{x^n}{n!}$.

Solution:

a) The series diverges whenever $x \neq 0$ since the terms $(nx)^n$ do not tend to zero when $n \to \infty$. (For large enough n, we have $n|x| \geq 1$.) Thus, this power series converges *only* at its centre.

b) The geometric series converges precisely on the interval $(-1, 1)$, and the series takes on the value $\frac{1}{1-x}$ there.

c) For $|x| \leq 1$, we have

$$\left|\frac{x^n}{n^2}\right| \leq \frac{1}{n^2},$$

and thus the series converges for these values of x. If $|x| > 1$, the terms $|x^n/n^2| \to \infty$, and so the series diverges. Hence the series converges precisely on the interval $[-1, 1]$.

d) Let $x \in \mathbb{R}$. Using the ratio test we have

$$\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} = \frac{x}{n+1} \to 0.$$

Thus the series converges for all $x \in \mathbb{R}$ (and takes on the value e^x). ■

7. If the power series $\sum a_k x^k$ has radius of convergence R, what is the radius of convergence of the series $\sum a_k x^{2k}$?

Solution: the new series can be written as $\sum_{k=0}^{\infty} b_k x^k$, where $b_k = a_{k/2}$ if k is even and $b_k = 0$ if k is odd. Thus

$$\begin{aligned}\limsup_{k\to\infty} \sqrt[k]{|b_k|} &= \lim_{k\to\infty} \sqrt[k]{|a_{k/2}|} = \lim_{k\to\infty} \sqrt[2k]{|a_k|} = \lim_{k\to\infty} \left(\sqrt[k]{|a_k|}\right)^{1/2} \\ &= \left(\lim_{k\to\infty} \sqrt[k]{|a_k|}\right)^{1/2} = R^{1/2}.\end{aligned}$$

Therefore, the radius of convergence of the new series is $\sqrt{R}$. □

8. Obtain power series expansions for the following functions.

a) $\dfrac{x}{1+x^2}$;

b) $\dfrac{x}{(1+x^2)^2}$;

c) $\dfrac{x}{1+x^3}$;

d) $\dfrac{x^2}{1+x^3}$;

e) $f(x) = \displaystyle\int_0^1 \frac{1-e^{-sx}}{s}\, ds$, about $x = 0$.

Solution:

a) Since

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k,$$

we have

$$\frac{x}{1+x^2} = x\sum_{k=0}^{\infty}(-x^2)^k = \sum_{k=0}^{\infty}(-1)^k x^{2k+1}.$$

b) We know that, for $x \in (-1,1)$, $\dfrac{1}{1-x} = \displaystyle\sum_{k=1}^{\infty} x^k$. For any $-1 < a < b < 1$, the series $\displaystyle\sum_{k=1}^{\infty} kx^{k-1}$ converges uniformly on $[a,b]$. Indeed, let $c = \max\{|a|,|b|\} < 1$. Then, for all $x \in [a,b]$, we have

$$|kx^{k-1}| \leq kc^{k-1}.$$

Now,

$$\frac{(k+1)c^k}{kc^{k-1}} = \frac{k+1}{k}c \to c \quad \text{as } k \to \infty.$$

Since $c < 1$, the ratio test tells us that $\displaystyle\sum_{k=1}^{\infty} kc^{k-1}$ converges. Thus, $\displaystyle\sum_{k=1}^{\infty} kx^{k-1}$ converges uniformly by the Weierstrass M-test. Consequently, we have

$$\sum_{k=1}^{\infty} kx^{k-1} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2},$$

and so for any $x \in [a,b] \subseteq (-1,1)$:

$$\frac{x}{(1+x^2)^2} = x\sum_{k=1}^{\infty} k(-x^2)^{k-1} = \sum_{k=1}^{\infty}(-1)^{k-1}kx^{2k-1}.$$

c) Using the geometric series, we have

$$\frac{x}{1+x^3} = x\sum_{k=0}^{\infty}(-x^3)^k = \sum_{k=0}^{\infty}(-1)^k x^{3k+1}.$$

d) Using the geometric series, we have

$$\frac{x^2}{1+x^3} = x^2\sum_{k=0}^{\infty}(-x^3)^k = \sum_{k=0}^{\infty}(-1)^k x^{3k+2}.$$

e) Since

$$e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!},$$

we have

$$\frac{1-e^{-sx}}{s} = -\frac{1}{s}\sum_{k=1}^{\infty}\frac{(-sx)^k}{k!} = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{s^{k-1}x^k}{k!}.$$

This series converges absolutely for all s and all x (use the ratio test or compare it to the series for e^x). Therefore, viewing it as a power series in s (for some fixed x), its interval of convergence is ∞, and its centre is 0. Thus the series can be integrated term by term:

$$\begin{aligned}\int_0^1 \frac{1-e^{-sx}}{s}\,ds &= \int_0^1 \sum_{k=1}^{\infty}(-1)^{k+1}\frac{s^{k-1}x^k}{k!}\,ds \\ &= \sum_{k=1}^{\infty}(-1)^{k+1}\left(\int_0^1 s^{k-1}\,ds\right)\frac{x^k}{k!} \\ &= \sum_{k=1}^{\infty}(-1)^{k+1}\left[\frac{s^k}{k}\right]_{s=0}^{s=1}\frac{x^k}{(k!)} = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k(k!)}.\end{aligned}$$

□

6.5 Exercises

1. Prepare a 2-page summary of this chapter, with important definitions and results.
2. Prove the relaxed version of Theorem 74.
3. Prove Theorem 75, as well as its relaxed version.
4. Prove Theorem 76.
5. Prove Theorem 77.
6. Explain the infinite sums paradoxes of Chapter 2 in light of Theorems 76 and 77.