Chapter 7

The Real Numbers (Reprise)

In a course on real analysis, the fundamental object of study is the set of real numbers. In Chapter 1, we introduced $\mathbb{R}$ in an intuitive and informal way. In this chapter, we show how $\mathbb{R}$ can be built using Cauchy sequences.

7.1 Cauchy Sequences in $\mathbb{Q}$

$(\mathbb{R}, |\cdot|)$ and $(\mathbb{Q}, |\cdot|)$ are both ordered fields. There is a fundamental difference between them, however: in $(\mathbb{R}, |\cdot|)$, every Cauchy sequence converges; in $(\mathbb{Q}, |\cdot|)$, some do not.

Lemma
If $(x_n) \subseteq \mathbb{Q}$ converges to $x \in \mathbb{Q}$, then (x_n^2) converges to $x^2 \in \mathbb{Q}$.

Proof: first, note that if $x \in \mathbb{Q}$, then $x^2 \in \mathbb{Q}$, since $\mathbb{Q}$ is a field. Now, let $\varepsilon > 0$. By hypothesis, $\exists N \in \mathbb{N}$ such that $n > N \implies |x_n - x| < \varepsilon$. Hence, for all $n > N$,

$$\begin{aligned}|x_n^2 - x^2| &= |x_n - x||x_n + x| < \varepsilon|x_n + x| \leq \varepsilon(|x_n| + |x|) \\ &= \varepsilon(|x_n - x + x| + |x|) \leq \varepsilon(|x_n - x| + 2|x|) < \varepsilon(\varepsilon + 2|x|).\end{aligned}$$

As ε can be made arbitrarily small, this completes the proof. ■

The following result sets the stage to show that $\mathbb{Q}$ is **incomplete** (see proof on pages 7-8).

Lemma
There is no rational number a for which $a^2 = 2$.

We build a sequence of rational numbers a_n for which $a_n^2 \to 2$:

$$a_1 = \frac{1}{1}, \quad a_2 = \frac{14}{10}, \quad a_3 = \frac{141}{100}, \quad a_4 = \frac{1414}{1000}, \quad \ldots$$

We can show by induction that

$$0 < a_1 < a_2 < \cdots < a_{n-1} < a_n < \cdots < 2 \quad \text{and} \quad 0 < a_1^2 < a_2^2 < \cdots < a_{n-1}^2 < a_n^2 < \cdots < 2.$$

For $n \in \mathbb{N}$, write $b_n = a_n + \frac{1}{10^{n-1}}$. Then $b_n^2 > 2 > a_n^2$ for all n.

Consequently, $a_n^2 \to 2$ since

$$|a_n^2 - 2| \leq |b_n^2 - a_n^2| = |b_n - a_n||b_n + a_n| \leq \frac{1}{10^{n-1}}\left(2a_n + \frac{1}{10^{n-1}}\right) \to 0.$$

It is easy to see that (a_n) is a Cauchy sequence in $\mathbb{Q}$; indeed, $|a_n - a_m| < 10^{-n}$ whenever $m \geq n$.

However, (a_n) cannot be a convergent sequence in $\mathbb{Q}$: were it to converge to a number $a \in \mathbb{Q}$, we would have $a_n^2 \to a^2 = 2 \in \mathbb{Q}$ according to the first Lemma, but $a \notin \mathbb{Q}$ according to the second Lemma.

A metric space (E, d) in which every Cauchy sequence also converges in (E, d) is termed **complete.**[1] The previous discussion shows that $(\mathbb{Q}, |\cdot|)$ is not complete.

7.2 Building $\mathbb{R}$ by Completing $\mathbb{Q}$

Is the fact that $\mathbb{Q}$ incomplete problematic? Not in the sense that arithmetic in $\mathbb{Q}$ is compromised. But it is still fairly inconvenient.

If we take a closer look at the formal definition, we notice that we can only claim a sequence to be convergent **once we know what its limit is**. But if we already know that the sequence has a limit, then it automatically converges.

At this stage, the main advantage a complete metric space holds over a non-complete one is simply that it allows one to talk about the convergence of a sequence without knowing a thing about its limit, save that it exists. But this does not change the fact that $\mathbb{Q}$ is not complete. What can we do about that?

The sequence (a_n) described previously does not converge in $\mathbb{Q}$, but its values get closer and closer to one of the "holes" of $\mathbb{Q}$.

If we fill up that hole (in effect starting the process of "completing" $\mathbb{Q}$), we may expect that the sequence would now converge in the bigger set. This leads to the following definition of the **real numbers** $\mathbb{R}$:

1. any Cauchy sequence in $\mathbb{Q}$ corresponds to a real number;

2. two Cauchy sequences (x_n) and (y_n) in $\mathbb{Q}$ define the same real number if $(x_n) \sim (y_n)$:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } n > N \Longrightarrow |x_n - y_n| < \varepsilon.$$

[1]We will discuss metric spaces in the coming chapters – for now, we simply think of it as a space in which we can compute the "distance" between points.

It is not too difficult to show that $\sim$ is an equivalence relation on the set of Cauchy sequences in $\mathbb{Q}$ (see exercises), and so we define $\mathbb{R}$ as the quotient set

$$\mathbb{R} = \{(x_n) \mid (x_n) \text{ is a Cauchy sequence in } \mathbb{Q}\}/\sim .$$

How does this definition of $\mathbb{R}$ compare with our usual intuition?

For starters, there should be an **addition** and a **multiplication** in $\mathbb{R}$ that behave as we expect them to (**commutative**, **associative**, **invertible, and so on**). We achieve this by endowing our definition of $\mathbb{R}$ with the following operations: if $\alpha = [(a_n)], \beta = [(b_n)] \in \mathbb{R}$, define

$$\alpha + \beta = [(a_n + b_n)] \quad \text{and} \quad \alpha\beta = [(a_n b_n)].$$

In order for this definition to make sense, we need to verify that if (a_n) and (b_n) are Cauchy sequences, then so are $(a_n + b_n)$ and $(a_n b_n)$, and that the choice or representative in the equivalence classes are irrelevant:

$$(a_n) \sim (a'_n) \text{ and } (b_n) \sim (b'_n) \Longrightarrow (a_n + b_n) \sim (a'_n + b'_n) \text{ and } (a_n b_n) \sim (a'_n b'_n).$$

The proof is left as an exercise, and relies on the following inequalities:

$$|(a_n + b_n) - (a'_n + b'_n)| \leq |a_n - a'_n| + |b_n - b'_n|$$

and

$$|a_n b_n - a'_n b'_n| \leq |a_n||b_n - b'_n| + |b'_n||a_n - a'_n|$$

and on Cauchy sequences being bounded in $\mathbb{Q}$.

Finally, in order for $\mathbb{Q}$ to be a subset of $\mathbb{R}$, we complete its definition as follows: if $\alpha \in \mathbb{R}$ is such that

$$\alpha = [(a, a, a, \ldots)], \quad a \in \mathbb{Q},$$

we identify α with $a \in \mathbb{Q}$. Consequently, if a Cauchy sequence (b_n) converges to $b \in \mathbb{Q}$, the real number $\beta = [(b_n)]$ **is** the rational number b.

7.3 An Order Relation on $\mathbb{R}$

To show that $\mathbb{R}$ is indeed complete, we next need to introduce an order on $\mathbb{R}$. If (a_n) and (b_n) are Cauchy sequences in $\mathbb{Q}$, there are three possibilities:

1. $\exists N \in \mathbb{N}$ such that $(n > N \Longrightarrow a_n \geq b_n)$;
2. $\exists N \in \mathbb{N}$ such that $(n > N \Longrightarrow a_n \leq b_n)$, or
3. (a_n) and (b_n) "overlap" infinitely often, in which case we must have $(a_n) \sim (b_n)$.

Write $\alpha = [(a_n)]$ and $\beta = [(b_n)]$. We define an **order** $<$ on $\mathbb{R}$ as follows:

1. $\alpha \geq \beta$ if cases 1 or 3 hold;

2. $\alpha \leq \beta$ if cases 2 or 3 hold.

But it is not enough to write $\leq$ or $\geq$; we still need to show that the relation is indeed an **order** (this is left as an exercise).

> **Lemma**
> *Let $\varepsilon \in \mathbb{Q}$ and $N \in \mathbb{N}$. If (a_n) is a Cauchy sequence in $\mathbb{Q}$ for which $a_n \leq \varepsilon$ for all $n > N$, then $\alpha = [(a_n)] \leq \varepsilon$.*
>
> **Proof:** it suffices to identify $\varepsilon \in \mathbb{Q}$ with the equivalence class of the constant sequence
> $$[(\varepsilon, \varepsilon, \ldots)].$$
> Then the above definition of $\leq$ in $\mathbb{R}$ yields the desired conclusion. ■

We see now why we define $\mathbb{R}$ using Cauchy sequence in $\mathbb{Q}$.

> **Theorem 86**
> *Let (a_n) be a Cauchy sequence in $\mathbb{Q}$ and set $\alpha = [(a_n)] \in \mathbb{R}$. Then (a_n) converges to α in $\mathbb{R}$.*
>
> **Proof:** We want to show that given any (real) $\varepsilon > 0$, we can find an integer $N \in \mathbb{N}$ such that $|a_n - \alpha| < \varepsilon$ whenever $n > N$.
>
> For all $n \in \mathbb{N}$, the sequence $(a_n, a_n, \ldots)$ defines the real number a_n; similarly, the sequence $(a_1, a_2, \ldots)$ defines the real number α. Consequently, the sequences
> $$(a_n - a_1, a_n - a_2, \ldots, a_n - a_m, \ldots) \quad \text{and} \quad (|a_n - a_1|, |a_n - a_2|, \ldots, |a_n - a_m|, \ldots)$$
> define respectively the real numbers $a_n - \alpha$ and $|a_n - \alpha|$.
>
> Let $\varepsilon > 0$. Since (a_n) is a Cauchy sequence, there is an integer $N \in \mathbb{N}$ such that $|a_n - a_m| < \varepsilon$ (as rational numbers) for each $n, m > N$. Fix $n > N$. Then we have $|a_n - a_m| < \varepsilon$ (as rational numbers) whenever $m > N$; consequently, $|a_n - \alpha| < \varepsilon$. Since this holds whenever $n > N$, we have $a_n \to \alpha$ in $\mathbb{R}$. ■

As a corollary, every real number is the limit of a Cauchy sequence of rational numbers.

> **Theorem 87** (Completeness of $\mathbb{R}$)
> *$\mathbb{R}$ is complete.*
>
> **Proof:** let (α_n) be a Cauchy sequence in $\mathbb{R}$. We show that it converges in $\mathbb{R}$ as follows:

1. construct a sequence (a_n) in $\mathbb{Q}$ for which $|a_n - \alpha_n| < \frac{1}{10^n}$ (where a_n is viewed as the constant sequence);
2. verify that (a_n) is a Cauchy sequence in $\mathbb{Q}$ and denote the associated real number by α;
3. show that $\alpha_n \to \alpha$.

That is, once more, left as an exercise. ■

We have not put emphasis on the fact that there are multiple ways of completing sets, but the **completion** of $\mathbb{Q}$ is entirely dependent on the notion of closeness that is being used: traditionally, the metric we use is that two rational numbers are considered close to one another if their respective decimal expansions start to differ far to the **right** of the decimal point.

For instance, the distance between 23410.0001 and 23410.0008 is smaller than 10^{-3} because the decimal expansions start to differ at the 4th digit to the right of the decimal point. In base 10, if $q, r \in \mathbb{Q}$, then we can write

$$q = \sum_{i\in\mathbb{Z}} q_i 10^i, \quad r = \sum_{i\in\mathbb{Z}} r_i 10^i$$

Under the usual metric $d_{10}(q,r) = \left|\sum_{i\in\mathbb{Z}} (q_i - r_i)10^i\right|$, we have

$$\begin{aligned} d_{10}(23410.0001, 23410.0008) = \big|&\cdots + (0-0)10^n + \cdots + (0-0)10^5 \\ &+ (2-2)10^4 + (3-3)10^3 + (4-4)10^2 + (1-1)10^1 \\ &+ (0-0)10^0 + (0-0)10^{-1} + (0-0)10^{-2} \\ &+ (0-0)10^{-3} + (1-8)10^{-4} + (0-0)10^{-5} + \cdots \\ &+(0-0)10^{-n} + \cdots\big| = 7\cdot 10^{-4}. \end{aligned}$$

But that is an artificial convention. What would happen if we defined a metric the other way? Two rational numbers would be considered close to one another if their respective decimal expansions start to differ far to the **left** of the decimal point, say.

Under this new metric $\tilde{d}_{10}(q,r) = \left|\sum_{i\in\mathbb{Z}} (q_i - r_i)10^{-i}\right|$, we have

$$\begin{aligned} \tilde{d}_{10}(23410.0001, 23410.0008) = \big|&\cdots + (0-0)10^{-n} + \cdots + (0-0)10^{-5} \\ &+ (2-2)10^{-4} + (3-3)10^{-3} + (4-4)10^{-2} + (1-1)10^{-1} \\ &+ (0-0)10^0 + (0-0)10^1 + (0-0)10^2 + (0-0)10^3 \\ &+(1-8)10^4 + (0-0)10^5 + \cdots + (0-0)10^n + \cdots\big| = 7\cdot 10^4, \end{aligned}$$

so that 23410.0001 and 23410.0008 are actually far apart, whereas 20000000012 and 12 are very close to one another since $\tilde{d}_{10}(20000000012, 12) = 2 \cdot 10^{-10}$.

If $\tilde{d}_{10}$ is indeed a metric on $\mathbb{Q}$ (see exercise 10), then Cauchy sequences in $(\mathbb{Q}, d)$ will not have a lot in common with Cauchy sequences in $(\mathbb{Q}, \tilde{d})$. There is no reason to expect that the completion of $\mathbb{Q}$ will be the same in both instances, and in fact, it is not.

When we complete $\mathbb{Q}$ using the metric $\tilde{d}_p$, where p is a prime integer, the resulting set we obtain is called the **field of $p-$adic numbers**, and it is distinct from $\mathbb{R}$. Just about everything we will do in these course notes could also apply to these new sets.

The moral of the story is that different metrics lead to different completions of $\mathbb{Q}$, and that neither of those is intrinsically superior to the others.

7.4 Exercises

1. Prepare a 2-page summary of this chapter, with important definitions and results.
2. Show that the relation $(x_n) \sim (y_n)$ is an equivalence relation on the space of Cauchy sequences in $\mathbb{Q}$ (i.e., show that it is reflexive, symmetric, and transitive).
3. If (a_n) and (b_n) are Cauchy sequences in $\mathbb{Q}$, show that so are $(a_n + b_n)$ and $(a_n b_n)$.
4. If (a_n), (b_n), (a_n') and (b_n') are Cauchy sequences in $\mathbb{Q}$ such that $(a_n) \sim (a_n')$ and $(b_n) \sim (b_n')$, show that $(a_n + b_n) \sim (a_n' + b_n')$ and $(a_n b_n) \sim (a_n' b_n')$.
5. Show that $\mathbb{R}$ is a field.
6. If (a_n) and (b_n) are Cauchy sequences which "overlap" infinitely often, show that $(a_n) \sim (b_n)$.
7. Let $\alpha, \beta, \gamma \in \mathbb{R}$. If $\alpha \leq \beta$ and $\beta \leq \gamma$, show that $\alpha \leq \gamma$.
8. Let $\alpha, \beta \in \mathbb{R}$. If $\alpha \leq \beta$ and $\beta \leq \alpha$, show that $\alpha = \beta$.
9. Fill the details in the proof of Theorem 7.3.
10. Show that $\tilde{d}_{10}$ is a metric on $\mathbb{Q}$ (use the definition in Section 8.1.1).
11. Let p be a prime integer. What can you say about the field of $p-$adic numbers?