Chapter 8

Metric Spaces and Sequences

> One of the natural ways we can extend the concepts we have discussed in the previous chapters is by moving from $\mathbb{R}$ to $\mathbb{R}^m$. Some of the notions that generalize nicely to vectors and functions on vectors include **norms** and **distances**, **sequences**, and **continuity**.

The symbol $\mathbb{K}$ is sometimes used to denote either $\mathbb{R}$ or $\mathbb{C}$; $\mathcal{C}_{\mathbb{R}}([0,1])$ represents the $\mathbb{R}-$vector space of continuous functions $[0,1] \to \mathbb{R}$, and $\mathcal{F}_{\mathbb{R}}([0,1])$ represents the $\mathbb{R}-$vector space of functions $[0,1] \to \mathbb{R}$.

8.1 Preliminaries

Most of the results of the previous chapters rely heavily on the properties of the absolute value. Its fundamental role in $\mathbb{R}$ is as a measure of the **magnitude** of a real number: $|x|$ is the distance from the real number x to the origin.

We can generalize the concept of the absolute value to higher-dimensional spaces in various ways. In this chapter, we discuss **norms** and **metrics**, and the **topologies** they induce.

8.1.1 Norms, Metrics, and Topology

Let E be a $\mathbb{K}-$vector space, such as $\mathbb{R}$, $\mathbb{C}^n$ or $\mathcal{C}_{\mathbb{R}}([0,1])$, say. A **norm** over E is a mapping $\|\cdot\| : E \to \mathbb{R}$ for which the following properties hold:

1. $\forall \mathbf{x} \in E$, $\|\mathbf{x}\| \geq 0$;
2. $\|\mathbf{x}\| = 0 \iff \mathbf{x} = \mathbf{0}$;
3. $\forall \mathbf{x} \in E, \forall \lambda \in \mathbb{K}$, $\|\lambda \mathbf{x}\| = |\lambda| \|\mathbf{x}\|$, and
4. $\forall \mathbf{x}, \mathbf{y} \in E$, $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

If the 4 properties hold, we say that $(E, \|\cdot\|)$ is a **normed space**.

Examples

1. $\mathbb{R}$ is a normed space together with the absolute value $|\cdot|$.
2. $\mathbb{C}$ is a normed space together with the modulus $|\cdot|$.
3. $\mathbb{R}^n$ is a normed space together with the **Euclidean norm**

$$\|\mathbf{x}\|_2 = \|(x_1,\ldots,x_n)\|_2 = \sqrt{x_1^2+\cdots+x_n^2}.$$

   The Euclidean norm over $\mathbb{R}^n$ will play a special role in our explorations: note that it is intimately linked to the **inner product**

$$(\cdot \mid \cdot):\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R},\quad \text{defined by}\quad (\mathbf{x}\mid\mathbf{y})=\sum x_iy_i \Longrightarrow \|\mathbf{x}\|=(\mathbf{x}\mid\mathbf{x})^{1/2}.$$

4. $E=\mathcal{C}_{\mathbb{R}}([0,1])$ together with the **sup norm** $\|f\|_\infty=\sup_{x\in[0,1]}|f(x)|$ is another important normed space.
5. For $p\geq 1$, the $p-$**norm** over $\mathbb{R}^n$ is defined as follows:

$$\|\mathbf{x}\|=\left(\sum_{i=1}^n |x_i|^p\right)^{1/p}.$$

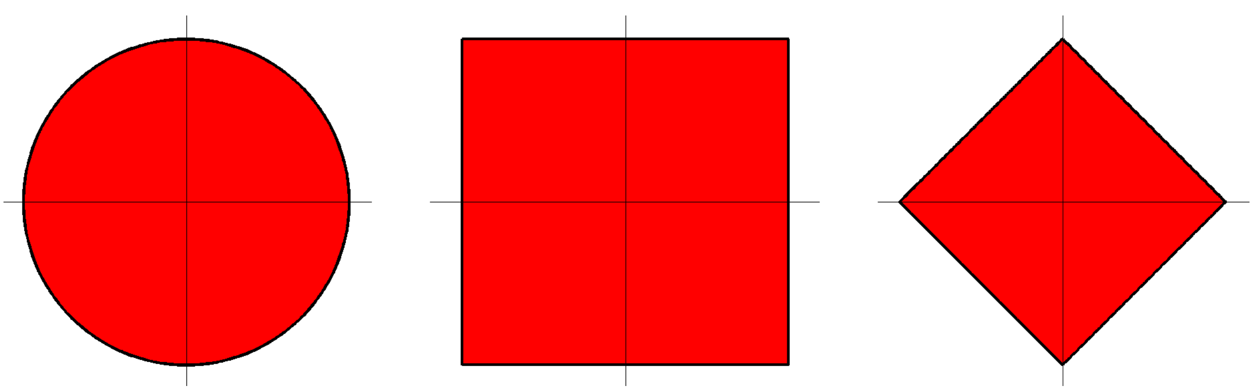
   Special cases of the $p-$norm over $\mathbb{R}^n$ include the Euclidean norm ($p=2$), the sup norm ($p=\infty$) and the $1-$norm:

$$\|\mathbf{x}\|_\infty=\max_{1\leq i\leq n}|x_i|,\qquad \|\mathbf{x}\|_\infty=\sum_{i=1}^n|x_i|. \qquad \square$$

The **open ball of radius** 1 **induced by the** $p-$**norm around the origin in** $\mathbb{R}^n$ is the set

$$B^p(\mathbf{0},1)=\{\mathbf{x}\in\mathbb{R}^n \mid \|\mathbf{x}\|_p<1\};$$

different values of p leading to different geometrical sets $B^p(\mathbf{0},1)$: $p=2,\infty,1$ (left to right).[1]



[1]We can also talk of **closed balls**, or of **general balls** of radius r centered at some point $\mathbf{a}\in\mathbb{R}^n$.

The open balls have different shapes (only the regions in red, not the boundaries), but we will see that they are all equivalent, in the sense that they all induce the same topologies.

Since there are similarities between summation and integration (the Riemann-integral of a function over an interval is, essentially, the limit of a sum), it could tempting to conclude that there are equivalent $p-$norms over $\mathcal{F}_{\mathbb{R}}([0,1])$: something along the lines of

$$\|f\|_p = \Big(\int_{[0,1]} |f|^p \, dm \Big)^{1/p} \tag{8.1}$$

where m is the Lebesgue measure (see Chapters 21 and 26), but these mappings are not in fact norms on $\mathcal{F}_{\mathbb{R}}([0,1])$.

Indeed, consider the Dirichlet function $\chi_{\mathbb{Q}} \in \mathcal{F}_{\mathbb{R}}([0,1])$, say. It can be shown that $\|f\|_1 = 0$. However, $\chi_{\mathbb{Q}} \neq 0$ which contradicts the second property of norms (in fact, $\|\cdot\|_p$ is a **seminorm** on $\mathcal{F}_{\mathbb{R}}([0,1])$).

If we instead restrict the function space to $\mathcal{C}_{\mathbb{R}}([0,1])$, $\|\cdot\|_p$ is indeed a norm for all $p \geq 1$, but unfortunately, $(\mathcal{C}_{\mathbb{R}}([0,1]), \|\cdot\|_p)$ is not complete (more on this later).

Let E be any set. A **metric** over E is a mapping $d : E \times E \to \mathbb{R}$ for which the following properties hold:

1. $\forall \mathbf{x}, \mathbf{y} \in E$, $d(\mathbf{x}, \mathbf{y}) \geq 0$;
2. $\forall \mathbf{x} \in E$, $d(\mathbf{x}, \mathbf{x}) = 0$;
3. $d(\mathbf{x}, \mathbf{y}) = 0 \Longleftrightarrow \mathbf{x} = \mathbf{y}$;
4. $\forall \mathbf{x}, \mathbf{y} \in E$, $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$, and
5. $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in E$, $d(\mathbf{x}, \mathbf{y}) \leq d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y})$.

If the 5 properties hold, we say that (E, d) is a **metric space**.

An important property of such spaces is that every normed space gives rise to a metric space.

Theorem 88
Let $(E, \|\cdot\|)$ be a normed space, and define $d : E \times E \to \mathbb{R}$ by

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|.$$

Then (E, d) is a metric space.

Proof: we show that all the metric space properties hold. Property 1, for instance, is a direct consequence of norm property 1:

$$\forall \mathbf{x}, \mathbf{y} \in E,\ d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| \geq 0.$$

Properties 2 and 3 are a direct consequence of norm property 2:

$$\forall \mathbf{x} \in E, d(\mathbf{x},\mathbf{x}) = \|\mathbf{x}-\mathbf{x}\| = \|\mathbf{0}\| = 0;$$
$$\forall \mathbf{x},\mathbf{y} \in E, d(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\| = 0 \Longleftrightarrow \mathbf{x}-\mathbf{y} = \mathbf{0} \Longleftrightarrow \mathbf{x} = \mathbf{y}.$$

Property 4 is a direct consequence of norm property 3:

$$\forall \mathbf{x},\mathbf{y} \in E,\ d(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\| = |-1| \cdot \|\mathbf{x}-\mathbf{y}\| = \|\mathbf{y}-\mathbf{x}\| = d(\mathbf{y},\mathbf{x}).$$

Property 5 is a direct consequence of norm property 5:

$$\begin{aligned}\forall \mathbf{x},\mathbf{y},\mathbf{z} \in E,\ d(\mathbf{x},\mathbf{y}) &= \|\mathbf{x}-\mathbf{y}\| = \|\mathbf{x}-\mathbf{z}+\mathbf{z}-\mathbf{y}\| \\ &\leq \|\mathbf{x}-\mathbf{z}\| + \|\mathbf{z}-\mathbf{y}\| = d(\mathbf{x},\mathbf{z}) + d(\mathbf{z},\mathbf{y}).\end{aligned}$$

Thus (E,d) is a metric space. ■

Not every metric space arises from a norm, however.

Examples

1. Let E be any set and define $d : E \times E \to \mathbb{R}$ by

$$d(\mathbf{x},\mathbf{y}) = \begin{cases} 0 & \text{if } \mathbf{x} = \mathbf{y} \\ 1 & \text{otherwise} \end{cases} \tag{8.2}$$

   Then (E,d) is a metric space in which every point is considered to be *far* from every other distinct point. We call such metric spaces **discrete**.

2. Let $E = \mathbb{R}^n$ and define $d : E \times E \to \mathbb{R}$ by $d_2(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\|_2$. Then (E,d_2) is a metric space, which we usually refer to has having the **standard topology.** □

Let (E,d) be a metric space. The **open ball centered at a** $\in E$ **with radius** $r > 0$ is the set

$$B(\mathbf{a},r) = \{\mathbf{x} \in E \mid d(\mathbf{a},\mathbf{x}) < r\};$$

the **closed ball centered at a** $\in E$ **with radius** $r > 0$ is the set

$$D(\mathbf{a},r) = D_d(\mathbf{a},r) = \{\mathbf{x} \in E \mid d(\mathbf{a},\mathbf{x}) \leq r\},$$

and the **sphere centered at a** $\in E$ **with radius** $r > 0$ is the set

$$S(\mathbf{a},r) = S_d(\mathbf{a},r) = D(\mathbf{a},r) \setminus B(\mathbf{a},r) = \{\mathbf{x} \in E \mid d(\mathbf{a},\mathbf{x}) = r\}.$$

Examples

1. Let $a \in E = \mathbb{R}$ and define $d(x,y) = |x-y|$ for all $x,y \in E$. Then, for $r > 0$, the balls reduce to intervals:

$$B(a,r) = (a-r, a+r), \quad D(a,r) = [a-r, a+r],$$

and the sphere to a discrete set $S(a,r) = \{a-r, a+r\}$.

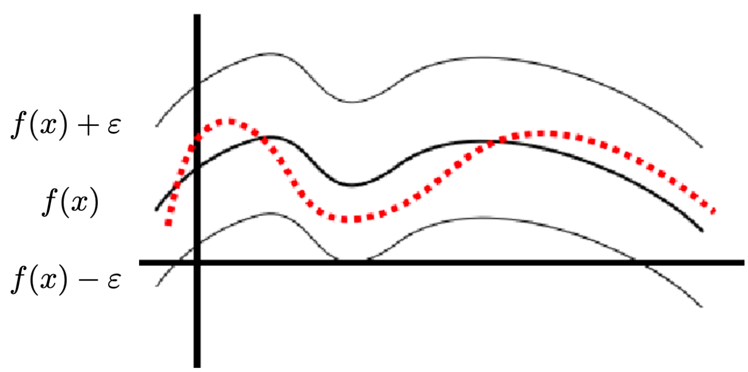
2. Let (E,d) be a discrete metric space and $\mathbf{a} \in E$. Then

$$B(\mathbf{a}, r) = \begin{cases} \{\mathbf{a}\}, & \text{if } r < 1 \\ E, & \text{otherwise} \end{cases}$$

3. Let $E = \mathcal{C}_{\mathbb{R}}([0,1])$, $d_\infty(f,g) = \|f-g\|_\infty$. Then, for $\varepsilon > 0$,

$$B(f,\varepsilon) = \{g \in E \mid \|f-g\|_\infty < \varepsilon\} = \left\{g \in E \mid \sup_{x\in[0,1]} |f(x)-g(x)| < \varepsilon\right\}$$
$$= \{g \in E \mid |f(x)-g(x)| < \varepsilon \ \forall x \in [0,1]\}$$

We see $B(f,\varepsilon)$ in the image below; f is the solid curve in the middle, the two bounding curves are ε away from f, and the red dashes show a function g in $B(f,\varepsilon)$.



4. Let $A, B \neq \varnothing$ be subsets of a metric space (E,d). The **distance** between A and B is defined by

$$d(A,B) = \inf_{\mathbf{x}\in A, \mathbf{y}\in B} \{d(\mathbf{x},\mathbf{y})\}. \tag{8.3}$$

Unfortunately, d does not define a metric on $\wp(E) \setminus \varnothing$ (see exercise 10). When $A = \{\mathbf{x}\}$, we write $d(A,B) = d(\mathbf{x}, B)$. □

Lemma 89
Let (E, d) be a metric space, $\mathbf{x}, \mathbf{a} \in E$, $r > 0$ and $\mathbf{x} \notin B(\mathbf{a}, r)$. Show that $d(\mathbf{x}, B(\mathbf{a}, r)) \geq d(\mathbf{x}, \mathbf{a}) - r$.

Proof: for all $\mathbf{y} \in B(\mathbf{a}, r)$, we have $d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{a}) \geq d(\mathbf{x}, \mathbf{a})$, whence

$$d(\mathbf{x}, \mathbf{y}) \geq d(\mathbf{x}, \mathbf{a}) - d(\mathbf{y}, \mathbf{a}) \geq d(\mathbf{x}, \mathbf{a}) - r.$$

Consequently,

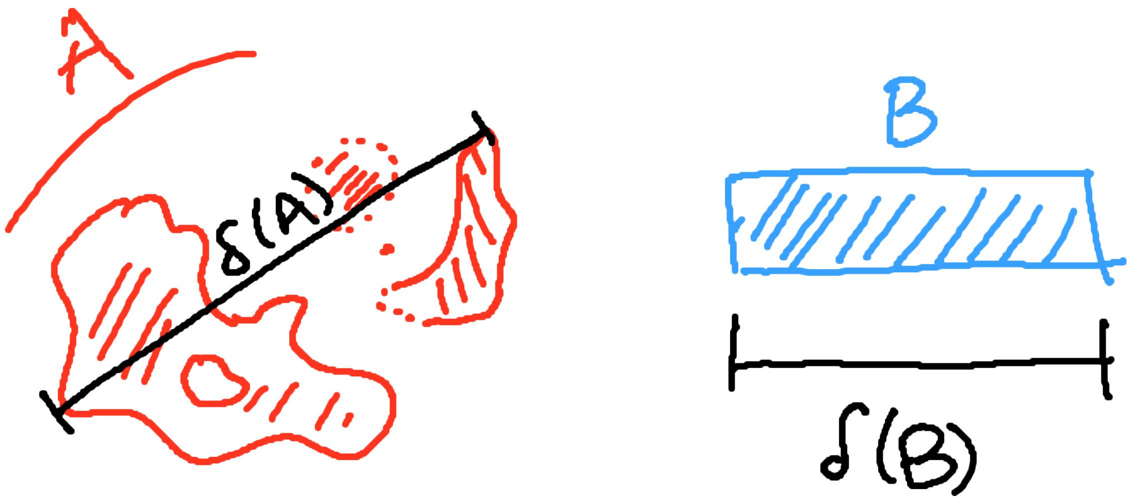
$$d(\mathbf{x}, B(\mathbf{a}, r)) = \inf_{\mathbf{y} \in B(\mathbf{a}, r)} \{d(\mathbf{x}, \mathbf{y})\} \geq d(\mathbf{x}, \mathbf{a}) - r$$

whenever $\mathbf{x} \notin B(\mathbf{a}, r)$. ■

Let (E, d) be a metric space and let $\varnothing \neq A \subseteq E$. The **diameter** of A under d is defined by

$$\delta_d(A) = \sup_{\mathbf{x}, \mathbf{y} \in A} \{d(\mathbf{x}, \mathbf{y})\}.$$

For instance, in $(\mathbb{R}^n, d_2)$, we have $\delta_{d_2}(B(\mathbf{a}, r)) = 2r$; the diameter of two subsets $A, B \subseteq \mathbb{R}^2$ is illustrated below.

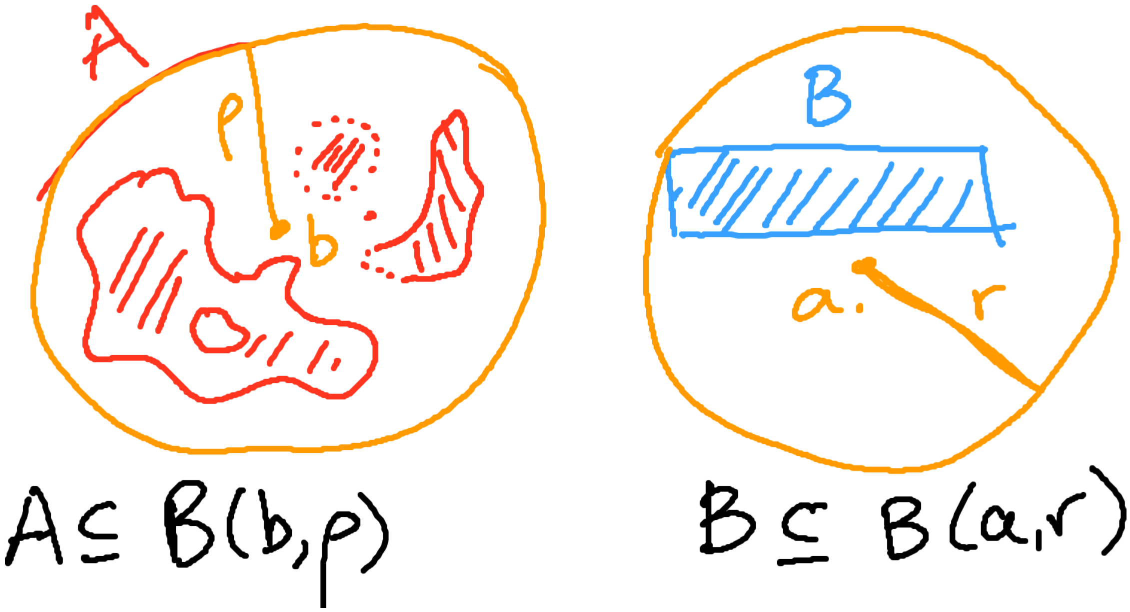


We say that A is **bounded in** (E, d) if $\delta_d(A) < \infty$.

Proposition 90
Let (E, d) be a metric space and let $\varnothing \neq A \subseteq E$. Then, A is bounded in (E, d) if and only if $\exists \mathbf{x} \in E$, $\exists r > 0$ such that $A \subseteq B(\mathbf{x}, r)$.

Proof: one direction is immediate: if $\exists \mathbf{x} \in E$, $\exists r > 0$ such that $A \subseteq B(\mathbf{x}, r)$, then $d(\mathbf{y}, \mathbf{z}) < r$ for all $\mathbf{y}, \mathbf{z} \in A \subseteq B(\mathbf{x}, r)$, so that $\delta_d(A) \leq r$.

Conversely, if $\delta_d(A) \leq M$, say, then $d(\mathbf{y}, \mathbf{z}) < r = M + 1$ for all $\mathbf{y}, \mathbf{z} \in A$. Pick any $\mathbf{x} \in A$. Then for any other $\mathbf{y}$ in A, $d(\mathbf{x}, \mathbf{y}) < r$, so that $\mathbf{y} \in B(\mathbf{x}, r)$. Thus $A \subseteq B(\mathbf{x}, r)$. ■

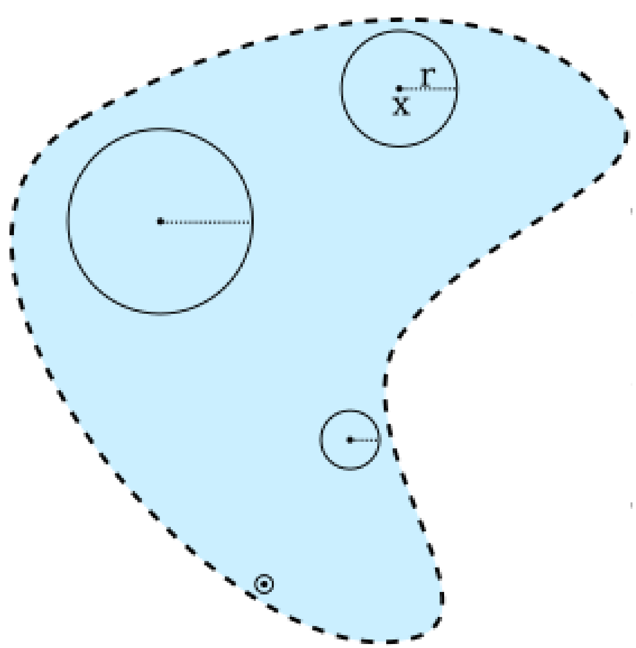


In this subsection, (E, d) is always a metric space, so we drop the d to lighten the text.

A subset $A \subseteq E$ is an **open subset of E under d** (or simply "open" if the context is clear) if either

- $A = \varnothing$, or
- $\forall \mathbf{x} \in E, \exists r > 0$ such that $B(\mathbf{x}, r) \subseteq A$.

We denote this relationship by $A \subseteq_O E$; an open subset of $\mathbb{R}^2$ in the Euclidean topology is shown below (D.J. Eck).

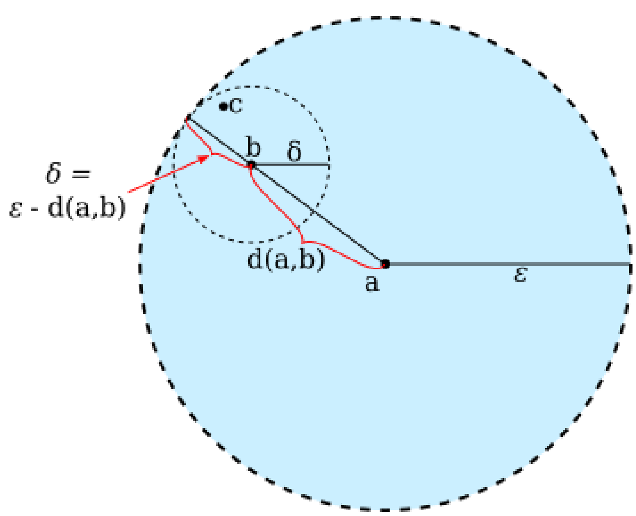


Proposition 91
Open sets in E have the following properties:

1. $E \subseteq_O E$;
2. $\forall \mathbf{a} \in E, r > 0$, *then* $B(\mathbf{a}, r) \subseteq_O E$;
3. *the union of an arbitrary family* $\{A_i\}_{i\in I}$ *of open subsets of* E *is an open subset of* E, *and*
4. *the intersection of a finite family* $\{A_i\}_{i=1}^{\ell}$ *of open subsets of* E *is an open subset of* E.

Proof:

1. Let $\mathbf{x} \in E$. Since $B(\mathbf{x}, r) \subseteq E$ for all $r > 0$, then $E \subseteq_O E$.
2. Let $B(\mathbf{a}, R)$ be an open ball in E, and let $\mathbf{x} \in B(\mathbf{a}, R)$. By definition, $d(\mathbf{a}, \mathbf{x}) < R$ implies $\exists \rho > 0$ with $\rho = \frac{R - d(\mathbf{a},\mathbf{x})}{2}$. It is not hard to show that with such a ρ, we have $B(\mathbf{x}, \rho) \subseteq B(\mathbf{a}, R)$.
3. Let $A = \bigcup A_i$. If $A = \varnothing$ then $A \subseteq_O E$. If $A \neq \varnothing$, let $\mathbf{x} \in A$. By definition, $\exists i \in I$ such that $\mathbf{x} \in A_i$. But $A_i \subseteq_O E$ and, as such, $\exists \rho > 0$ for which $B(\mathbf{x}, \rho) \subseteq A_i \subseteq \bigcup A_i = A$. Consequently, $A \subseteq_O E$.
4. It suffices to prove the result for $\ell = 2$ (why?). Let $A = A_1 \cap A_2$. If $A = \varnothing$ then $A \subseteq_O E$. If $A \neq \varnothing$, let $\mathbf{x} \in A$. Then $\mathbf{x} \in A_1$. But $A_1 \subseteq_O E$ and, as such, $\exists r_1 > 0$ for which $B(\mathbf{x}, r_1) \subseteq A_1 \subseteq A$. As well, $\mathbf{x} \in A_2$. But $A_2 \subseteq_O E$ and, as such, $\exists r_2 > 0$ for which $B(\mathbf{x}, r_2) \subseteq A_2 \subseteq A$. Set $\rho = \min\{r_1, r_2\}$. Then $B(\mathbf{x}, r) \subseteq A_1 \cap A_2$, and, consequently, $A \subseteq_O E$. ■



We have seen plenty of examples in Part I.

Examples

1. Let $a \in \mathbb{R}$. Then $(-\infty, a)$ and (a, ∞) are both open in $E = \mathbb{R}$ since
$$(-\infty, a) = \bigcup_{x<a}(x, a) \quad \text{and} \quad (a, \infty) = \bigcup_{x>a}(a, x).$$

2. The intersection of an arbitrary family of open subsets of E could be open, but need not be:
$$\bigcap_{n\in\mathbb{N}}(-n, n) = (-1, 1) \subseteq_O \mathbb{R},$$
but
$$\bigcap_{n\in\mathbb{N}}(-\tfrac{1}{n}, \tfrac{1}{n}) = \{0\} \text{ is not open in } \mathbb{R};$$
we will have more to say on the topic of arbitrary intersection of open sets in Part IV and Chapter 21. □

The collection of a metric space (E, d)'s open subsets forms a **topology** τ on E:

1. $\varnothing, E \in \tau$;
2. if $U_i \in \tau$ for all $i \in I$, then $\bigcup_I U_i \in \tau$, and
3. if $U_1, U_2 \in \tau$, then $U_1 \cap U_2 \in \tau$.

Examples

1. Let (E, d) be a metric space. The collection of all open subsets of E under d forms a topology on E, the **metric space topology**.
2. Let E be any set. The collection $\tau = \{\varnothing, E\}$ forms a topology on E, the **indiscrete topology**.
3. Let E be any set. The collection $\tau = \wp(E)$ forms a topology on E, the **discrete topology**. □

A subset $A \subseteq E$ is a **closed subset of** E **under** d if $E \setminus A \subseteq_O E$. We denote this relationship by $A \subseteq_C E$.

As a consequence of the definition of closed sets in opposition to open sets, we get a whole slew of properties of closed subsets, for free, such as $\varnothing, E \subseteq_C E$. But there are more substantial ones as well.

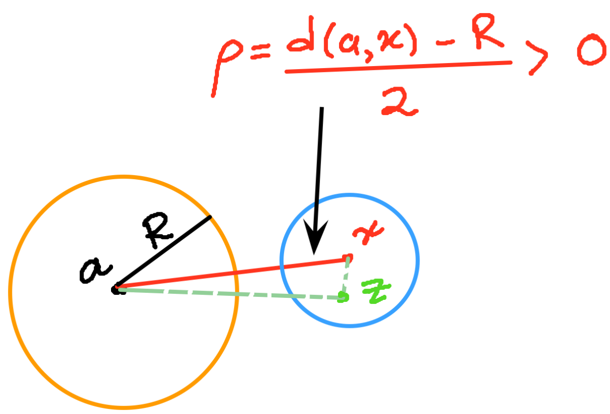
Examples

1. Every closed ball in (E, d) is closed.

   Proof: let $A = D(\mathbf{a}, R)$ be a closed ball in E and set

   $$E \setminus A = \{\mathbf{x} \in E \mid d(\mathbf{a}, \mathbf{x}) > R\}.$$

   We need to show that $E \setminus A$ is open. Let $\mathbf{x} \in E \setminus A$; by definition, $d(\mathbf{a}, \mathbf{x}) > R$ and $\rho = \frac{d(\mathbf{a},\mathbf{x})-R}{2} > 0$.



   It remains only to show that $B(\mathbf{x}, \rho) \subseteq E \setminus A$. Let $\mathbf{z} \in B(\mathbf{x}, \rho)$. Then

   $$d(\mathbf{x}, \mathbf{z}) < \rho \quad \text{and} \quad -d(\mathbf{x}, \mathbf{z}) > -\rho.$$

   Thus, according to the triangle inequality we have

   $$d(\mathbf{a}, \mathbf{z}) \geq d(\mathbf{a}, \mathbf{x}) - d(\mathbf{x}, \mathbf{z}) \geq 2\rho + R - d(\mathbf{x}, \mathbf{z}) \geq R + \rho > R;$$

   as such, $z \in E \setminus A$. This completes the proof. ■

2. Every sphere in (E, d) is closed.

   Proof: Let $S = S(\mathbf{a}, R)$. Note that

   $$E \setminus S = B(\mathbf{a}, R) \cup [E \setminus D(\mathbf{a}, R)] \subseteq_O E$$

   since it is a union of open sets. Consequently, $S \subseteq_C E$. ■

3. The intersection of an arbitrary family $\{A_i\}_{i\in I}$ of closed subsets of E is a closed subset of E. □

4. The union of a finite family $\{A_i\}_{i=1}^{\ell}$ of closed subsets of E is a closed subset of E. Note however that the union of an arbitrary family of closed subsets of E need not be closed (see exercise 18) in E. □

The **closure** of a subset $A \subseteq E$ with respect to a metric d is the smallest closed subset $\overline{A}$ of E (again, with respect to d) containing A (with possible equality).

The closure has a number of interesting properties, one of which being that $\overline{A}$ is the intersection of all closed sets containing A, and that $A \subseteq \overline{A}$ (see exercises 19 and 20).

Examples

1. In the Euclidean topology, $\overline{(0,1)} = [0,1]$.
2. In the discrete topology, $\overline{(0,1)} = (0,1)$.
3. In the Euclidean topology, $\overline{S(\mathbf{a},R)} = S(\mathbf{a},R)$. □

The closure provides us with a clear way to characterize closed subsets.

Lemma 92
Let A be a subset of E. Then $A \subseteq_C E \iff A = \overline{A}$.

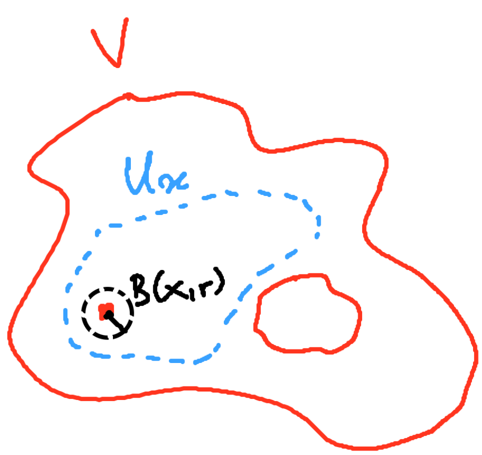
Proof: one direction is immediate. Let $A \subseteq_C E$. The smallest closed subset of E containing A is thus A itself, so $A = \overline{A}$.

Conversely, assume $A = \overline{A}$. As $\overline{A}$ is the smallest closed subset of A containing A, then $A = \overline{A}$ is closed in E. ■

A **neighbourhood** of $\mathbf{x} \in E$ is a subset $V \subseteq E$ containing an open subset $U_{\mathbf{x}} \subseteq_O E$ with $\mathbf{x} \in U_{\mathbf{x}}$. In other words, V is a neighbourhood of $\mathbf{x}$ if $\exists r > 0$ such that $B(\mathbf{x}, r) \subseteq V$ (but V is not necessarily open). The set of all neighbourhoods of $\mathbf{x}$ is denoted by

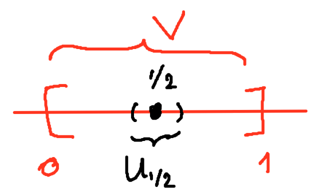
$$\mathcal{V}(\mathbf{x}) = \{V \subseteq E \mid V \text{ is a neighbourhood of } \mathbf{x}\}.$$

The image below shows a neighbourhood V of $\mathbf{x}$, with an open set $U_{\mathbf{x}}$.

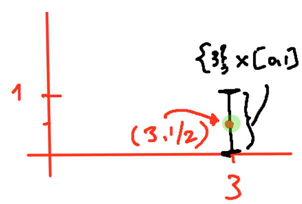


Examples

1. In $\mathbb{R}$ with the standard topology, $[0,1]$ and $(0,1]$ are neighbourhoods of $\frac{1}{2}$.



2. In $\mathbb{R}^2$ with the standard topology, $\{3\} \times [0,1]$ is not a neighbourhood of $(3, \frac{1}{2})$.



The various definitions give us an easy lemma.

Lemma 93
Let (E,d) be a metric space with $U \subseteq E$. Then U is a neighbourhood of each of its points if and only if $U \subseteq_O E$.

Proof: one direction holds as a consequence of the definition of open sets; the other as a consequence of the definition of neighbourhoods. ■

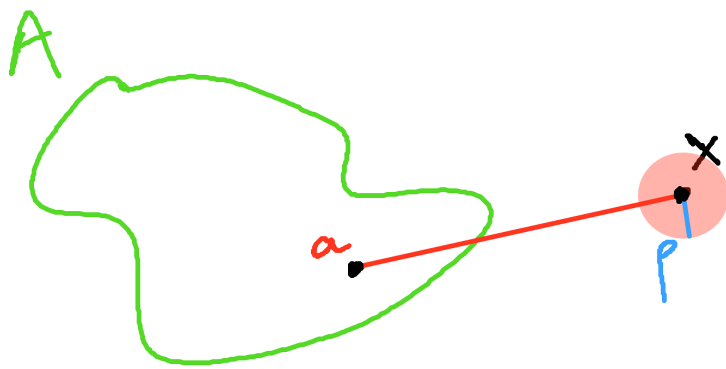
Points in $\overline{A}$ have useful (equivalent) properties.

Proposition 94
Let $A \subseteq E$. The following conditions are equivalent:

1. $\mathbf{x} \in \overline{A}$
2. $\forall \varepsilon > 0, \exists \mathbf{a} \in A$ *such that* $d(\mathbf{a}, \mathbf{x}) < \varepsilon$
3. $\forall V \in \mathcal{V}(\mathbf{x}), V \cap A \neq \varnothing$
4. $d(\{\mathbf{x}\}, A) = d(\mathbf{x}, A) = 0$

Proof: we will only prove that $1. \iff 2.$ The proof that $2. \iff 3 \iff 4.$ is left as an exercise.

Assume $\mathbf{x} \notin \overline{A}$. Then $\mathbf{x} \in E \setminus \overline{A} \subseteq_O E$. Thus $\exists \rho > 0$ such that $B(\mathbf{x}, \rho) \subseteq E \setminus \overline{A}$. Consequently, $d(\mathbf{a}, \mathbf{x}) \geq \rho, \forall \mathbf{a} \in A$.



Conversely, let $\mathbf{x} \in E$ and assume $\exists \varepsilon > 0$ such that

$$A \subseteq \underbrace{E \setminus B(\mathbf{x}, \varepsilon)}_{\text{closed}}.$$

Since $\overline{A}$ is the smallest closed set containing A, we must have

$$A \subseteq \overline{A} \subseteq E \setminus B(\mathbf{x}, \varepsilon)$$

and so $\mathbf{x} \notin \overline{A}$. ■

A subset A of E is **dense in** (E, d) if $\overline{A} = E$. A metric space (E, d) is **separable** if it has at least one dense subset.

Examples

1. $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are both dense in $\mathbb{R}$ in the usual topology. □
2. Neither of these sets are dense in $\mathbb{R}$ in the discrete topology. □
3. Every non-empty subset of E is dense in E in the indiscrete topology. □
4. **Weierstrass' Theorem:** *let P be the set of polynomial functions $[0, 1] \to \mathbb{R}$. Then P is dense in $(\mathcal{C}_\mathbb{R}([0, 1]), d_\infty)$.*

   Thus real continuous functions on $[0, 1]$ (which need not even be C^1) can be approximated as closely as desired/needed by smooth (polynomial) functions (we will discuss this further in Chapter 23).
5. $\mathbb{R}$ and $\mathbb{R}^n$ are separable in the Euclidean topology. □

A family $\mathcal{G} = \{G_\lambda\}_{\lambda\in L}$, $\varnothing \neq G_\lambda \subseteq_O E$ forms a **basis for the open subsets of** E if every non-empty open subset of E can be written as a union of members of $\mathcal{G}$.

Examples

1. $\{B(x,r) \mid x \in \mathbb{Q}, r \in \mathbb{Q}^*_+\}$ and $\{B(x,r) \mid x \in \mathbb{R}, r \in \mathbb{R}^*_+\}$ both form a basis for the open subsets of $\mathbb{R}$. □
2. $\{B(\mathbf{x},r) \mid \mathbf{x} \in \mathbb{Q}^n, r \in \mathbb{Q}^*_+\}$ forms a basis for the open subsets of $\mathbb{R}^n$. □

There is a nice way to characterize such bases.

Proposition 95
A family $\mathcal{G} = \{G_\lambda\}_{\lambda\in L}$ is a basis for the open subsets of E if and only if $\forall \mathbf{x} \in E$, $\forall V \in \mathcal{V}(\mathbf{x})$, $\exists \lambda \in L$ such that $\mathbf{x} \in G_\lambda \subseteq V$.

Proof: the direction $\Longrightarrow$ holds as a consequence of the definition of neighbourhood and of a base.

Conversely, let $\varnothing \neq U \subseteq_O E$. Note that, being open, U is a neighbourhood of all its points. Then, by hypothesis, $\forall \mathbf{x} \in U$ $\exists \lambda(\mathbf{x}) \in L$ such that $\mathbf{x} \in G_{\lambda(\mathbf{x})} \subseteq U$. However,

$$U = \bigcup_{\mathbf{x}\in U} \{\mathbf{x}\} \subseteq \bigcup_{\mathbf{x}\in U} G_{\lambda(\mathbf{x})} \subseteq U,$$

so that U is the union of elements of $\mathcal{G}$. ■

By analogy with the closure, the **interior** of a subset $A \subseteq E$ is the largest open subset of E contained in A; we denote that subset by $\text{int}(A)$ (or sometimes A°). It is not difficult to show that $\text{int}(A)$ is the union of all the open subsets of E contained in A, and that $A \subseteq_O E$ if and only if $\text{int}(A) = A$ (see exercises).

Examples

1. In the discrete topology, $\text{int}([0,1]) = [0,1]$; while in the Euclidean topology, $\text{int}([0,1]) = (0,1)$. □
2. In the Euclidean topology, $\text{int}(S(\mathbf{a},R)) = \varnothing$ and $\text{int}(D(\mathbf{a},R)) = B(\mathbf{a},R)$. □
3. While $\text{int}(\overline{(a,b)}) = (a,b)$ and $\overline{\text{int}([a,b])} = [a,b]$ in $(\mathbb{R},d_2)$, $\text{int}(\overline{W}) \neq W$, in general, as we can see with $W = (0,\frac{1}{2}) \cup (\frac{1}{2},1) \subseteq (\mathbb{R},d_2)$. □

The next concepts are not crucial to our study, but still nice to have: $U \subseteq E$ is a **regular open subset of** E if $\text{int}(\overline{U}) = U$; $B \subseteq E$ is a **regular closed subset of** E if $\overline{\text{int}(B)} = B$.

Not all metrics are derived from a norm (the discrete metric fails in that regard, for instance), but **normed vector spaces** have a very nice property when it comes to closure and balls.

Lemma 96
If (E, d) is a normed vector space, then $D(\mathbf{0}, 1) = \overline{B(\mathbf{0}, 1)}$.

Proof: since $B(\mathbf{0}, 1) \subseteq D(\mathbf{0}, 1) \subseteq_C E$, we have $\overline{B(\mathbf{0}, 1)} \subseteq D(\mathbf{0}, 1)$ as $\overline{B(\mathbf{0}, 1)}$ since the **smallest** closed subset of E containing $B(\mathbf{0}, 1)$.

As $D(\mathbf{0}, 1) = B(\mathbf{0}, 1) \cup S(\mathbf{0}, 1)$, we only need to show that $S(\mathbf{0}, 1) \subseteq \overline{B(\mathbf{0}, 1)}$ as $B(\mathbf{0}, 1) \subseteq \overline{B(\mathbf{0}, 1)}$. Let $\mathbf{x} \in S(\mathbf{0}, 1)$; then $\|x\| = 1$. Let $1 > \varepsilon > 0$ and set $\mathbf{z} = (1 - \frac{\varepsilon}{2})\mathbf{x}$.

Then $\mathbf{z} \in B(\mathbf{0}, 1)$, since $\|\mathbf{z}\| = |1 - \frac{\varepsilon}{2}| \cdot \|\mathbf{x}\| < 1$; we note further that $d(\mathbf{z}, \mathbf{x}) = \|\mathbf{z} - \mathbf{x}\| = \frac{\varepsilon}{2}\|\mathbf{x}\| = \frac{\varepsilon}{2} < \varepsilon$ and so, according to Proposition 94 with $\mathbf{a} = \mathbf{z}$ and $A = B(\mathbf{0}, 1)$, we indeed have $\mathbf{x} \in \overline{B(\mathbf{0}, 1)}$. ■

We can use this lemma to show that the discrete metric is not derived from a norm: were it so, we would have $D(\mathbf{0}, 1) = \overline{B(\mathbf{0}, 1)}$. However, in $\mathbb{R}^n$ we have

$$B(\mathbf{0}, 1) = \{\mathbf{0}\} \subseteq_C \mathbb{R} \text{ and } D(\mathbf{0}, 1) = \mathbb{R} \Longrightarrow \overline{B(\mathbf{0}, 1)} = \{\mathbf{0}\} \neq \mathbb{R} = D(\mathbf{0}, 1).$$

Proposition 97
Let $A \subseteq E$. The following conditions are equivalent:

1. $\mathbf{x} \in \text{int}(A)$
2. $A \in \mathcal{V}(\mathbf{x})$
3. $\exists \varepsilon > 0$ *such that* $B(\mathbf{x}, \varepsilon) \subseteq A$.

Proof: by definition, we have $2. \iff 3.$ It remains only to show that $1. \iff 3.$

$3. \Longrightarrow 1.$: Let $\varepsilon > 0$ and $B(\mathbf{x}, \varepsilon) \subseteq A$. Since $\text{int}(A)$ is the largest open subset of E contained in A and since $B(\mathbf{x}, \varepsilon)$ is an open subset of E contained in A, we must have $B(\mathbf{x}, \varepsilon) \subseteq \text{int}(A)$, whence $\mathbf{x} \in \text{int}(A)$.

$1. \Longrightarrow 3.$: Let $\mathbf{x} \in \text{int}(A) \subseteq_O E$. By definition, there must exist some $\varepsilon > 0$ such that $B(\mathbf{x}, \varepsilon) \subseteq \text{int}(A) \subseteq A$. ■

As an example of the usefulness of this result, note that by the density of $\mathbb{Q}$ and its complement $\mathbb{R} \setminus \mathbb{Q}$ in $\mathbb{R}$, we automatically get $\text{int}(\mathbb{Q}) = \text{int}(\mathbb{R} \setminus \mathbb{Q}) = \varnothing$ with the usual topology on $\mathbb{R}$.

We end this section with a few other topological concepts:

- the **boundary** of a subset $A \subseteq E$ is simply defined by $\partial A = \overline{A} \setminus \text{int}(A)$ and the **exterior** of A is given by $\text{int}(E \setminus A)$;[2]
- we say that $\mathbf{x} \in E$ is a **cluster point** of A if

$$\forall \varepsilon > 0,\ \exists \mathbf{y}_\varepsilon \in B(\mathbf{x}, \varepsilon) \cap A \text{ such that } \mathbf{y}_\varepsilon \neq \mathbf{x};$$

- we also say that $\mathbf{x} \in E$ is an **isolated point** of A if $\exists \varepsilon > 0$ for which $B(\mathbf{x}, \varepsilon) \cap A = \{\mathbf{x}\}$.

Examples Let $A = \{\frac{1}{n} : n \geq 1\}$.

1. 0 is a cluster point of A since $B(0, \varepsilon) \cap A$ contains all $\frac{1}{n}$, where $n > \frac{1}{\varepsilon}$.
2. For all $n \geq 1$, $\frac{1}{n}$ is an isolated point of A, as $B(\frac{1}{n}, \frac{1}{2n(n+1)}) \cap A = \{\frac{1}{n}\}$.

There is a link between cluster points of a set and its closure.

Lemma 98
If $\mathbf{x}$ is a cluster point of A, then $\mathbf{x} \in \overline{A}$ and every neighbourhood of $\mathbf{x}$ contains an infinite set of points in A.

Proof: that $\mathbf{x} \in \overline{A}$ is a direct consequence of Propostion 94. The rest of the proof can be done by showing that if a neighbourhood of $\mathbf{x}$ exists which contain only a finite number of points of A, then $\mathbf{x}$ cannot be a cluster point of A. ■

Finally, if (E, d) is a metric space and $F \subseteq E$, then (F, d) is also a metric space, called a **metric subspace** of E. The topology on F is completely determined by the topology on E.

Proposition 99
Let (E, d) be a metric space and $F \subseteq E$. Then

$$B \subseteq_O F \iff \exists A \subseteq_O E \text{ such that } B = A \cap F$$

and

$$B \subseteq_C F \iff \exists A \subseteq_C E \text{ such that } B = A \cap F.$$

Proof: left as an exercise. ■

[2] In a nutshell, the exterior is the largest open subset of E which excludes A in its entirety.

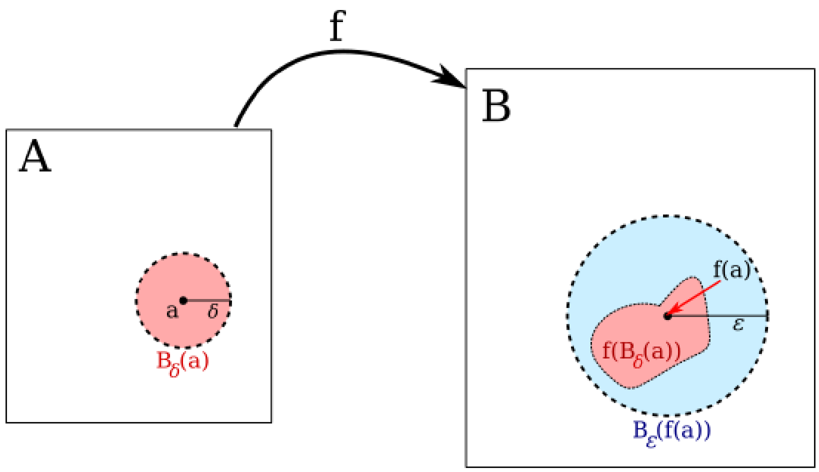
8.1.2 Continuity

The concept of **continuity** is fundamental in all aspects of analysis. Let $(A, d_A), (B, d_B)$ be metric spaces. Since we view $d_A(\mathbf{a}, \mathbf{x})$ and $d_B(f(\mathbf{a}), f(\mathbf{x}))$ as generalizations of $|a-x|$ and $|f(a)-f(x)|$, respectively, and we say that a map $f : A \to B$ is **continuous at a** $\in A$ if

$$\forall \varepsilon > 0,\ \exists \delta > 0, (\mathbf{x} \in A \text{ and } d_A(\mathbf{a}, \mathbf{x}) < \delta) \implies d_B(f(\mathbf{a}), f(\mathbf{x})) < \varepsilon;$$

or, equivalently, if for any open $\varepsilon-$ball W centered at $f(\mathbf{a})$, there is an open $\delta-$ball V centered at **a** such that $f(V) \subseteq W$; or yet again equivalently, if for any neighbourhood $W \subseteq_O B$ of $f(\mathbf{a})$, there is a neighbourhood $V \subseteq_O A$ of **a** such that $f(V) \subseteq W$.[3]

The continuity of $f : (\mathbb{R}^2, d_2) \to (\mathbb{R}^2, d_2)$ at $\mathbf{a} \in \mathbb{R}^2$ is illustrated below (D.J. Eck).



We further say that the map f is **continuous on** A if it is continuous at each $\mathbf{a} \in A$.

Proposition 100
Let $(E, d), (\tilde{E}, \tilde{d})$ be metric spaces, and let $f : E \to \tilde{E}$. The following conditions are equivalent:

1. *f is continuous on E;*
2. *for any $W \subseteq_O \tilde{E}$, $f^{-1}(W) = \{\mathbf{x} \in E | f(\mathbf{x}) \in W\} \subseteq_O E$, and*
3. *for any $Y \subseteq_C \tilde{E}$, $f^{-1}(Y) \subseteq_C E$.*

Proof: that $2. \iff 3.$ follows directly from the fact that

$$f^{-1}(\tilde{E} \setminus Y) = E \setminus f^{-1}(Y).$$

[3]That these definitions are equivalent is left as an exercise.

1. $\implies$ 2.: Let $W \subseteq_O \tilde{E}$ and $\mathbf{x} \in f^{-1}(W)$. Since W is open in $\tilde{E}$, $\exists \varepsilon > 0$ such that $B(f(\mathbf{x}), \varepsilon) \subseteq W$. By continuity, $\exists \delta > 0$ such that $f(B(\mathbf{x}, \delta)) \subseteq B(f(\mathbf{x}), \varepsilon) \subseteq W$. But this means that

$$B(\mathbf{x}, \delta) = f^{-1}(f(B(\mathbf{x}, \delta)) \subseteq f^{-1}(W)$$

(see exercises) and so $f^{-1}(W) \subseteq_O E$.

2. $\implies$ 1.: Let $f(\mathbf{x}) \in W \subseteq_O \tilde{E}$. Set $V = f^{-1}(W) \subseteq_O E$. Then $\mathbf{x} \in V$ and $f(V) \subseteq W$; consequently, f is continuous. ■

Consider a map $f : E \to \tilde{E}$ as above. If $f(W) \subseteq_O \tilde{E}$ for all $W \subseteq_O E$, then we say that f is an **open mapping**; by analogy, if $f(Y) \subseteq_C \tilde{E}$ for all $Y \subseteq_C E$, then we say that f is a **closed mapping**.

Generally speaking, continuous maps are neither open nor closed; the constant function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = a$ provides an example of a continuous function which is not open in the standard topology, as $(0, 1) \subseteq_O \mathbb{R}$, but $f((0, 1)) = \{a\} \subseteq_C \mathbb{R}$, for instance.

Proposition 101
Let $f : (E, d) \to (\tilde{E}, \tilde{d})$ and $g : (\tilde{E}, \tilde{d}) \to (\hat{E}, \hat{d})$ be continuous. Then the composition $g \circ f : (E, d) \to (\hat{E}, \hat{d})$ is continuous.

Proof: let $\mathbf{a} \in E$ and $\varepsilon > 0$. As g is continuous at $f(\mathbf{a}) \in \tilde{E}$, $\exists \delta_\varepsilon > 0$ such that

$$\mathbf{y} \in \tilde{E} \text{ and } \mathbf{y} \in B_{\tilde{d}}(f(\mathbf{a}), \delta_\varepsilon) \implies g(\mathbf{y}) \in B_{\hat{d}}(g(f(\mathbf{a})), \varepsilon).$$

Since f is continuous at $\mathbf{a}$, $\exists \eta_{\delta_\varepsilon} = \eta_\varepsilon > 0$ such that

$$\mathbf{x} \in E \text{ and } \mathbf{x} \in B_d(\mathbf{a}, \eta_{\delta_\varepsilon}) \implies f(\mathbf{x}) \in B_{\tilde{d}}(f(\mathbf{a}), \delta_\varepsilon).$$

Combining these results together, we get

$$\mathbf{x} \in E \text{ and } \mathbf{x} \in B_d(\mathbf{a}, \eta_{\delta_\varepsilon}) \implies g(f(\mathbf{x})) \in B_{\hat{d}}(g(f(\mathbf{a})), \varepsilon),$$

which completes the proof. ■

As we can see, in many instances, the broad strokes of proofs in the multi-dimensional cases follow those of the corresponding one-dimensional proofs.

Corollary 102 *Let $f : (E, d) \to (\tilde{E}, \tilde{d})$ be a continuous function. If $F \subseteq E$, then the restriction $f|_F : (F, d|_F) \to (\tilde{E}, \tilde{d})$ is continuous.*

Proof: it suffices to show that the inclusion $F \hookrightarrow E_1$ is continuous, which is left as an exercise, and then to apply Proposition 101. ■

Some standard examples are shown below.

Examples

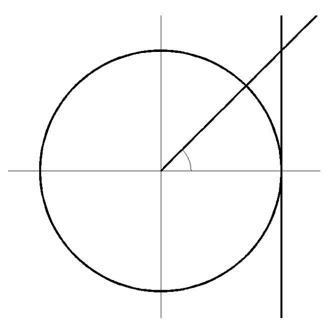
1. The functions $f : (\mathbb{R}, d_2) \to (\mathbb{R}, d_2)$ defined by $f(x) = x^3$ is continuous. □
2. The identity function id : $(\mathbb{R}, d_{\text{discrete}}) \to (\mathbb{R}, d_2)$ is continuous, since $\text{id}^{-1}(V) = V \subseteq_O (\mathbb{R}, d_{\text{discrete}})$ for all $V \subseteq_O (\mathbb{R}, d_2)$. □
3. The identity function $\text{id}^{\text{inv}} : (\mathbb{R}, d_2) \to (\mathbb{R}, d_{\text{discrete}})$ is not continuous, since, for instance,
$$\left(\text{id}^{\text{inv}}\right)^{-1}(\{a\}) = \{a\}$$
is not open in $(\mathbb{R}, d_2)$ even though $\{a\} \subseteq_O (\mathbb{R}, d_{\text{discrete}})$. □
4. Consider the characteristic function $\chi_{\mathbb{R}\setminus\mathbb{Q}} : \mathbb{R} \to \mathbb{R}$. Then $\chi_{\mathbb{R}\setminus\mathbb{Q}}$ is continuous when restricted to $\mathbb{Q}$ (being a constant function), but $\chi_{\mathbb{R}\setminus\mathbb{Q}}$ is nowhere continuous on $\mathbb{R}$. □

A metric d on E gives rise to a topology by **defining the open sets** of E. A natural question to ask is: can two different metrics give rise to the same topology? In order to answer that question, we need to introduce a new concept.

Let $(E, d), (\tilde{E}, \tilde{d})$ be metric spaces. A function $f : E \to \tilde{E}$ is a **homeomorphism** if f is bijective and both f and f^{inv} are continuous.[4]

Examples

1. $f : (\mathbb{R}, d_2) \to (\mathbb{R}, d_2)$, $f(x) = x^3$, is a homeomorphism. □
2. $\text{id} : (\mathbb{R}, d_{\text{discrete}}) \to (\mathbb{R}, d_2)$, $\text{id}(x) = x$, is not a homeomorphism. □
3. The function $g : (\mathbb{R}, d_2) \to \left((-\frac{\pi}{2}, \frac{\pi}{2}), d_2\right)$ defined by $g(x) = \arctan(x)$ is a homeomorphism. □



[4]Alternatively, f is a homeomorphism if it is bijective, continuous and open.

These examples illustrate that the notion of **boundedness** is not necessarily preserved by homeomorphisms: for instance, $\mathbb{R}$ is unbounded while $(-\frac{\pi}{2}, \frac{\pi}{2})$ is bounded, but both spaces are homemorphic to one another *via* arctan.

Furthermore, neither is the notion of **distance** necessarily preserved by homeomorphisms: in general,

$$d(x_1, x_2) \neq \tilde{d}(f(x_1), f(x_2)).$$

For instance, in the first example,

$$d(0,2) = |0-2| = 2 \neq \tilde{d}(0^3, 2^3) = |0^3 - 2^3| = 9.$$

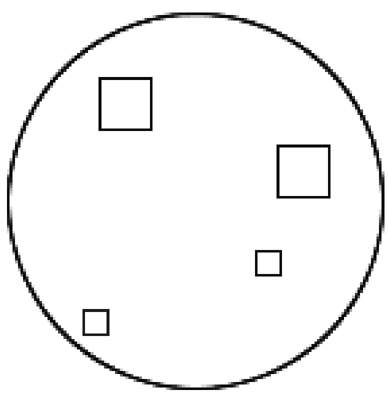
However, homeomorphisms $f : E \to \tilde{E}$ **preserve the topologies of** E **and** $\tilde{E}$**:**

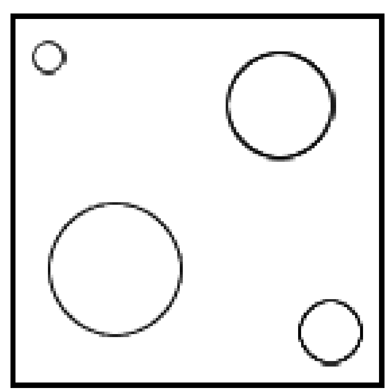
$$W \subseteq_O E \iff f(W) \subseteq_O \tilde{E} = f(E)$$
$$Y \subseteq_C E \iff f(Y) \subseteq_C \tilde{E} = f(E).$$

Two metrics $d, \tilde{d}$ on E are **topologically equivalent** if id $: (E,d) \to (E,\tilde{d})$ is a homeomorphism. In that case, d and $\tilde{d}$ give rise to the same topologies on E.

Example: if $p, q \geq 1$, d_p and d_q induce the same topologies on $\mathbb{R}^n$.

For instance, to show that d_2 and d_∞ are topologically equivalent in $\mathbb{R}^2$, it suffices to show that any point of a $2-$ball has an $\infty-$neighbourhood contained in the $2-$ball, and, conversely, that any point of an $\infty-$ball has a $2-$neighbourhood contained in the $\infty-$ball (see exercises). In the illustration below, we see a $2-$ball filled with $\infty-$balls (left) and an $\infty-$ball filled with with $2-$balls (right). $\square$





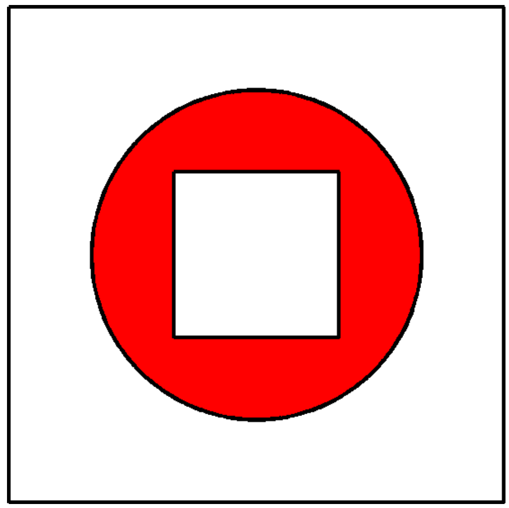
There is an associated notion: two metrics $d, \tilde{d}$ on E are **(strongly) equivalent** if $\exists A, B > 0$ such that

$$Ad(\mathbf{x},\mathbf{y}) \leq \tilde{d}(\mathbf{x},\mathbf{y}) \leq Bd(\mathbf{x},\mathbf{y}) \quad \forall \mathbf{x},\mathbf{y} \in E.$$

Intuitively, two metrics are equivalent if it is always possible to fit a $\tilde{d}-$ball between two $d-$balls, while maintaining the ratios of the balls' radii. Topological equivalence is not an equivalent notion, as we see in exercise 36.

Example: if $p, q \geq 1$, d_p and d_q are equivalent on $\mathbb{R}^n$.

For instance, to show that d_2 and d_∞ are equivalent in $\mathbb{R}^2$, it suffices to show that $\exists A, B > 0$ such that any $2-$ball of radius $R > 0$ contains an $\infty-$ball of radius $\frac{R}{A}$, and is contained in an $\infty-$ball of radius $\frac{R}{B}$.



Given the geometry of squares and circles, what values can A and B take? $\square$

There is also a similar notion for norms. Two norms $\|\cdot\|^*$, $\|\cdot\|^\circ$ on E are **equivalent** if $\exists a, b > 0$ such that

$$a\|\mathbf{x}\|^* \leq \|\mathbf{x}\|^\circ \leq b\|\mathbf{x}\|^*, \quad \forall \mathbf{x} \in E.$$

Clearly, two equivalent norms on E give rise to two equivalent metrics on E. But there is an important difference: over a **finite**$-$dimensional vector space, **any two norms are equivalent**, which we can show using the following proof outline:

1. without loss of generality, assume $\|\cdot\|^* = \|\cdot\|_1$;
2. only the vectors $\mathbf{x} \in S_1(\mathbf{0}, 1)$ need to be considered (why?);
3. show that $\|\cdot\|^\circ$ is continuous with respect to $\|\cdot\|_1$, and
4. use the max/min theorem over $S_1(\mathbf{0}, 1)$ to bound $a \leq \|\mathbf{x}\|^\circ \leq b$.

We end this section on preliminaries with two definitions that generalize the notion of a continuous function.

Let $f : (E, d) \to (\tilde{E}, \tilde{d})$. We say that f is

1. **uniformly continuous** if $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$ such that $\forall \mathbf{x}, \mathbf{y} \in E$, $d(\mathbf{x}, \mathbf{y}) < \delta \implies \tilde{d}(f(\mathbf{x}), f(\mathbf{y})) < \varepsilon$;
2. **Lipschitz continuous** if $\exists K > 0$ such that $\tilde{d}(f(\mathbf{x}), f(\mathbf{y})) \leq K d(\mathbf{x}, \mathbf{y})$ $\forall \mathbf{x}, \mathbf{y} \in E$.

The conceptual difference between continuity and uniform continuity is that δ may depend on $\mathbf{x}$ and $\mathbf{y}$ as well as ε in the former case, but it can only depend on ε in the latter case.

Examples

1. Any polynomial $p : \mathbb{R} \to \mathbb{R}$ is uniformly continuous over a closed, bounded interval. □
2. Any uniformly continuous function is automatically continuous. □
3. Any Lipschitz continuous function is automatically uniformly continuous, hence continuous. □
4. The function $f : (0, 1] \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is continuous but not uniformly continuous. □

This allows us to define another type of equivalence between metrics: two metrics $d, \tilde{d}$ on E are **uniformly equivalent** if $\text{id} : (E, d) \to (E, \tilde{d})$ is uniformly continuous, and so is its inverse.

Uniformly equivalent metrics are topologically equivalent, as uniform continuity also implies continuity, but there are topologically equivalent metrics that are not uniformly equivalent. However, uniform equivalence and strong equivalence of metrics are ... well, equivalent.

Lastly, note that uniform continuity, unlike continuity, is not a **topological notion**: given a function $f : E \to \tilde{E}$, the knowledge of the topologies on E and $\tilde{E}$, respectively, is sufficient to determine if f is continuous. But more must be known in order to determine if f is uniformly continuous. There is something fundamental at play here; we will return to it at a later stage.

8.2 Sequence in a Metric Space

Consider the sequence $(\mathbf{x}_n) \subseteq (E, d)$. The sequence **converges** to $\mathbf{x} \in (E, d)$, which we denote by $\mathbf{x}_n \to \mathbf{x}$, if

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } n > N \Longrightarrow d(\mathbf{x}_n, \mathbf{x}) < \varepsilon.$$

In light of the notions presented in the previous section, this is equivalent to the following definition: $\mathbf{x}_n \to \mathbf{x} \in E$ if

$$\forall V \in \mathcal{V}(\mathbf{x}),\ \exists N \in \mathbb{N} \text{ such that } n > N \Longrightarrow \mathbf{x}_n \in V.$$

Thus a sequence converges to $\mathbf{x}$ if any neighbourhood of $\mathbf{x}$ contains infinitely many terms in the sequence.

A **subsequence of** $(\mathbf{x}_n)$ is a sequence $(\mathbf{y}_n)$ such that $\mathbf{y}_n = \mathbf{x}_{\varphi(n)}$ for some strictly increasing function $\varphi : \mathbb{N} \to \mathbb{N}$. It is easy to show that if $\mathbf{x}_n \to \mathbf{x}$, then any subsequence of $(\mathbf{x}_n)$ also converges to $\mathbf{x}$ (see the exercises).

Let $(\mathbf{x}_n)$ be a sequence in a metric space (E,d). We say that $\mathbf{a} \in E$ is a **limit point of** $(\mathbf{x}_n)$ if $\forall \varepsilon > 0, \forall \rho \in \mathbb{N}, \exists n \geq \rho$ such that $d(\mathbf{x}_n, \mathbf{a}) < \varepsilon$.[5]

Proposition 103
Let $(\mathbf{x}_n) \subseteq (E,d)$, $\mathbf{a} \in E$. *The following are equivalent:*

1. $\mathbf{a}$ *is a limit point of* $(\mathbf{x}_n)$*;*
2. *there is a subsequence of* $(\mathbf{x}_n)$ *which converges to* $\mathbf{a}$*;*
3. $\forall \rho \in \mathbb{N}$, *we have* $\mathbf{a} \in \overline{A_\rho}$, *where* $A_\rho = \{\mathbf{x}_n | n \geq \rho\}$, *and*
4. *either* $\mathbf{a}$ *is a cluster point of* A_1 *or* $\{\mathbf{x}_n \mid \mathbf{x}_n = \mathbf{a}\}$ *is infinite (in the latter case, we say that* $\mathbf{a}$ *is a* ***replicating point of*** $(\mathbf{x}_n)$*).*

Proof: we prove $1. \implies 2. \implies 3. \implies 4. \implies 1.$

$1. \implies 2.$: Set $\varepsilon_n = \frac{1}{n}$. Since $\mathbf{a}$ is a limit point of the sequence $(\mathbf{x}_n)$, there is a smallest integer n for which $d(\mathbf{y}_n, \mathbf{a}) < \frac{1}{n}$, where $\mathbf{y}_n$ is a member of the sequence $(\mathbf{x}_m)_{m \geq n}$. By construction, $(\mathbf{y}_n)$ is a subsequence of $(\mathbf{x}_n)$ and $\mathbf{y}_n \to \mathbf{a}$.

$2. \implies 3.$: If there is a subsequence $(\mathbf{y}_n) \subseteq (\mathbf{x}_n)$ which converges to $\mathbf{a}$, then $\forall \varepsilon > 0, \forall \rho \in \mathbb{N}, \exists N \in \mathbb{N}$ such that $\mathbf{y}_n \in A_\rho \cap B(\mathbf{a}, \varepsilon)$ whenever $n > N$. But according to Proposition 94, $\mathbf{a} \in \overline{A_\rho}$ if and only if $\forall \varepsilon > 0,\ A_\rho \cap B(\mathbf{a}, \varepsilon) \neq \varnothing$. Consequently, $\forall \rho \in \mathbb{N}$, $\mathbf{a} \in \overline{A_\rho}$.

$3. \implies 4.$: If $\forall \rho \in \mathbb{N}$, $\mathbf{a} \in \overline{A_\rho}$, then $\forall \rho \in \mathbb{N}$, $\forall \varepsilon > 0$, $\exists$ a smallest $n_\rho \geq \rho$ such that $d(\mathbf{x}_{n_\rho}, a) < \varepsilon$. As such, $\mathbf{x}_{n_\rho}$ is a subsequence of $(\mathbf{x}_n)$ and

$$\lim_{\rho \to \infty} \mathbf{x}_{n_\rho} = \mathbf{a}.$$

- If $(\mathbf{x}_n)$ converges, it must do so to $\mathbf{a}$, according to exercise 40. Consequently, $\forall \eta > 0$, $A_1 \cap B(\mathbf{a}, \varepsilon)$ is infinite and so must contain at least one point distinct from $\mathbf{a}$. Consequently, $\mathbf{a}$ is a cluster point of A_1.
- If $(\mathbf{x}_n)$ diverges and $\mathbf{a}$ is not a replicating point of $(\mathbf{x}_n)$, then $\mathbf{x}_{n_\rho} \not\to \mathbf{a}$ (why?), which is a contradiction. Consequently, if $(\mathbf{x}_n)$ diverges then $\mathbf{a}$ is a replicating point of $(\mathbf{x}_n)$.

$4. \implies 1.$: Left as an exercise. ∎

8.2.1 Closure, Closed Subsets, and Continuity

We can conclude from Proposition 103 that the set $\bigcap_{\rho \in \mathbb{N}} \overline{A_\rho}$ of limit points of $(\mathbf{x}_n)$ is closed and that if $\mathbf{x}_n \to \mathbf{x}$, then $\mathbf{x}$ is the unique limit point of $(\mathbf{x}_n)$.

[5] Compare with the notion of a cluster point.

There is a nice way to characterize closure, closed subsets and continuity using sequences and convergence, provided by the next three results.

Proposition 104
Let (E,d) be a metric space, $A \subseteq E$ and $\mathbf{x} \in E$. Then,

$$\mathbf{x} \in \overline{A} \Longleftrightarrow \exists(\mathbf{x}_n) \subseteq A \text{ such that } \mathbf{x}_n \to \mathbf{x}.$$

Proof: the direction $\Longleftarrow$ is a clear consequence of the remark at the start of this subsection. For $\Longrightarrow$, consider the following argument. Let $n \in \mathbb{N}$. Since $\mathbf{x} \in \overline{A}$, $\exists \mathbf{x}_n (\neq \mathbf{x}) \in A$ with $d(\mathbf{x}_n, \mathbf{x}) < \frac{1}{n}$. Clearly, $\mathbf{x}_n \to \mathbf{x}$. ■

Proposition 105
Let (E,d) be a metric space, with $F \subseteq E$. Then, $F \subseteq_C E$ if and only if any sequence $(\mathbf{x}_n) \subseteq F$ which converges in E converges to a point in F.

Proof: if $F \subseteq_C E$, then $\overline{F} = F$. Assume that $\mathbf{x}_n \in F$ and $\mathbf{x}_n \to \mathbf{x}$. We must show that $\mathbf{x} \in F = \overline{F}$. If (x_n) is eventually constant, then $\mathbf{x}_n = \mathbf{x} \in F$ for all n greater than some index. Otherwise $\forall \varepsilon > 0$, $B(\mathbf{x}, \varepsilon) \cap F$ contains an infinite subset of $\{\mathbf{x}_n \mid n \geq 1\}$; consequently, $\mathbf{x} \in \overline{F}$.

Conversely, let $\mathbf{x} \in \overline{F}$. According to Proposition 104, there is a subsequence $(\mathbf{x}_n) \subseteq F$ such that $\mathbf{x}_n \to \mathbf{x}$. By hypothesis, any such sequence must converge *in* F. Hence, $\mathbf{x} \in F$. Consequently, $F = \overline{F}$ and $F \subseteq_C E$. ■

Proposition 106
Let $(E,d), (\tilde{E}, \tilde{d})$ be a metric spaces. Then $f : E \to \tilde{E}$ is continuous if and only $f(\mathbf{x}_n) \to f(\mathbf{x})$ whenever $\mathbf{x}_n \to \mathbf{x}$.

Proof: the direction $\Longleftarrow$ is a clear consequence of the definition of a continuous function.

Conversely, let $F \subseteq_C \tilde{E}$. We want to show that $f^{-1}(F) \subseteq_C E$. Let $(\mathbf{x}_n) \subseteq f^{-1}(F)$ with $\mathbf{x}_n \to \mathbf{x}$. By hypothesis, $f(\mathbf{x}_n) \to f(\mathbf{x})$. But $F \subseteq_C \tilde{E}$ so that $f(\mathbf{x}) \in F$, according to Proposition 105.

Consequently, $\mathbf{x} \in f^{-1}(F)$. According to Proposition 105, we must then have $f^{-1}(F) \subseteq_F E$; in other words, f is continuous. ■

We will see in Part IV that these characterizations do not always apply to general (as in, non-metric) topological spaces.

8.2.2 Complete Spaces and Cauchy Sequences

The sequence $(\mathbf{x}_n) \subseteq (E, d)$ is a **Cauchy sequence** if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } n, m > N \Longrightarrow d(\mathbf{x}_n, \mathbf{x}_m) < \varepsilon.$$

Some properties of Cauchy sequences in $\mathbb{R}$ carry over to metric spaces.

Proposition 107
Convergent sequences in (E, d) are Cauchy.

Proof: let $\mathbf{x}_n \to \mathbf{x}$ and $\varepsilon > 0$; thus $\exists N \in \mathbb{N}$ such that $d(\mathbf{x}_n, \mathbf{x}) < \frac{\varepsilon}{2}$ whenever $n > N$. Now, let $m > N$. According to the triangle inequality,

$$d(\mathbf{x}_n, \mathbf{x}_m) \leq d(\mathbf{x}_n, \mathbf{x}) + d(\mathbf{x}, \mathbf{x}_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Consequently, $(\mathbf{x}_n)$ is a Cauchy sequence. ■

In a normed space $(E, \|\cdot\|)$, a sequence $(\mathbf{x}_n)$ is **bounded** if $\exists M \in \mathbb{N}$ such that $\|\mathbf{x}_n\| < M$ for all $n \in \mathbb{N}$.

But a metric space (E, d) is not necessarily a normed vector space, so there might not be a norm available to determine boundedness.

In a general metric space (E, d), a sequence $(\mathbf{x}_n)$ is **bounded** if $\exists M > 0$ s.t. $\mathbf{x}_n \in B(\mathbf{0}, M)$ for all $n \in \mathbb{N}$. Similarly, $A \subseteq E$ is **bounded** if $\delta(A) < \infty$ (using the definition from p. 192).

Proposition 108
Every Cauchy sequence in (E, d) is bounded.

Proof: let $(\mathbf{x}_n)$ be a Cauchy sequence. If $1 > \varepsilon > 0$, then $\exists N \in \mathbb{N}$ such that $d(\mathbf{x}_n, \mathbf{x}_m) < \varepsilon$ whenever $n, m > N$. Now, let

$$M = \max\{d(\mathbf{0}, \mathbf{x}_1), d(\mathbf{0}, \mathbf{x}_2), \ldots, d(\mathbf{0}, \mathbf{x}_N), d(\mathbf{0}, \mathbf{x}_{N+1})\} + 2.$$

Then, for any $n > N$, the triangle inequality yields

$$d(\mathbf{0}, \mathbf{x}_n) \leq d(\mathbf{0}, \mathbf{x}_{N+1}) + d(\mathbf{x}_{N+1}, \mathbf{x}_n) \leq M - 2 + 1,$$

i.e. for any $n > N$, $\mathbf{x}_n \in B(\mathbf{0}, M)$. Since $\mathbf{x}_n \in B(\mathbf{0}, M-2)$ for all $1 \leq n \leq N$, then $\mathbf{x}_n \in B(\mathbf{0}, M)$ for all $n \in \mathbb{N}$. ■

Interestingly, given its link to convergence in the case of complete spaces, the notion of a Cauchy sequence is not topological.

Example: let $A = (0, \infty)$. Consider the following metrics on A:

$$d_1(x, y) = |x - y| \quad \text{and} \quad d_2(x, y) = |\ln x - \ln y|.$$

Show that both metrics induce the same topology on A, but that Cauchy sequences under one are not necessarily Cauchy sequences under the other.

Proof: the mapping $\text{id} : (A, d_1) \to (A, d_2)$ is homeomorphic. Indeed, for $x, z \in A$ and $\varepsilon, \eta > 0$, we have

$$B_{d_1}(x, \varepsilon) = \{y \in A \mid |x - y| < \varepsilon\} = (x - \varepsilon, x + \varepsilon) \cap A,$$

and

$$B_{d_2}(z, \eta) = \{y \in A \mid |\ln z - \ln y| < \eta\} = \{y \in A \mid e^{-\eta} < \frac{y}{z} < e^{\eta}\} = (ze^{-\eta}, ze^{\eta}).$$

It is left as an exercise to show that

$$B_{d_1}(z, \tfrac{1}{2}z(1 - e^{-\eta})) \subseteq B_{d_2}(z, \eta) \quad \text{and} \quad B_{d_2}(x, \ln(\tfrac{2x+\varepsilon}{2x})) \subseteq B_{d_1}(x, \varepsilon)$$

for all $x, z \in A$, $\varepsilon, \eta > 0$. Thus $W \subseteq_O (A, d_1) \iff W \subseteq_O (A, d_2)$. We already know that the sequence $(\frac{1}{n})$ is Cauchy in (A, d_1). But if $m = 2n$, then

$$d_2(\tfrac{1}{m}, \tfrac{1}{n}) = \left|\ln \tfrac{1}{m} - \ln \tfrac{1}{n}\right| = \left|\ln \tfrac{n}{m}\right| = \left|\ln \tfrac{n}{2n}\right| = \ln 2 \geq 1/2$$

for every $n \in \mathbb{N}$, and so $(\frac{1}{n})$ is not a Cauchy sequence in (A, d_2). ■

This could not happen, however, if the metrics are strongly equivalent, which further illustrates the distinctness of the notions of strong equivalence and topological equivalence.

Proposition 109
Let d and $\tilde{d}$ be two equivalent metrics on E. Then, $(\mathbf{x}_n)$ is a Cauchy sequence in (E, d) if and only if $(\mathbf{x}_n)$ is a Cauchy sequence in $(E, \tilde{d})$.

Proof: since d and $\tilde{d}$ are equivalent, $\exists a, b > 0$ such that

$$ad(\mathbf{x}, \mathbf{y}) \leq \tilde{d}(\mathbf{x}, \mathbf{y}) \leq bd(\mathbf{x}, \mathbf{y}) \quad \forall \mathbf{x}, \mathbf{y} \in E.$$

If $(\mathbf{x}_n)$ is a Cauchy sequence in $(E, \tilde{d})$, then, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $m, n > N \implies \tilde{d}(\mathbf{x}_n, \mathbf{x}_m) < \varepsilon$. Thus, it is the case that

$$ad(\mathbf{x}_n, \mathbf{x}_m) \leq \tilde{d}(\mathbf{x}_n, \mathbf{x}_m) < \varepsilon \quad \forall m, n > N \implies d(\mathbf{x}_n, \mathbf{x}_m) < \frac{\varepsilon}{a} \quad \forall m, n > N.$$

Consequently, $(\mathbf{x}_n)$ is also a Cauchy sequence in (E, d). By symmetry, the reverse implication is clearly true. ■

A metric space (E, d) is **complete** if every single one of its Cauchy sequences is convergent. If a complete metric space is also a normed vector space, then it is a **Banach space**. If a Banach space is also an inner product space, then it is a **Hilbert space**.

Examples (COMPLETE, BANACH, AND HILBERT SPACES)

1. We have already seen that $(\mathbb{R}, d_2)$ is a complete space. Since it is a normed space, it is also a Banach space. The inner product $(x \mid y) = xy$ makes it a Hilbert space.
2. The same applies to $(\mathbb{K}^n, d_2)$, with the inner product $(\mathbf{x} \mid \mathbf{y}) = \sum x_i \overline{y_i}$.
3. The space $\mathcal{C} = (C_{\mathbb{K}}([0,1]), \|\cdot\|_\infty)$ is a Hilbert space with the inner product
$$(f \mid g) = \int_{[0,1]} f\overline{g}\, dm, \quad f \sim g \Longleftrightarrow f = g \text{ a.e.}$$
4. It is a bit less obvious that the space
$$\ell^2(\mathbb{N}) = \{\mathcal{X} \mid \mathcal{X} = (x_n)_{n\in\mathbb{N}};\ x_n \in \mathbb{C},\ \sum |x_n|^2 < \infty\}$$
is a Hilbert space, together with
$$(\mathcal{X} \mid \mathcal{Y}) = \sum x_n \overline{y_n} \quad \text{and} \quad \|\mathcal{X}\|_2 = (\mathcal{X} \mid \mathcal{X})^{1/2} = \left(\sum |x_n|^2\right)^{1/2},$$
but it is a classical result (see Chapter 27). □

Closed subsets of complete spaces are especially well-behaved, as we see in the next two results.

Proposition 110
Every closed subset of a complete metric space is complete.

Proof: let $A \subseteq_C E$ and $(\mathbf{x}_n) \subseteq A$ be a Cauchy sequence. Since E is complete, $\mathbf{x}_n \to \mathbf{x}$ converges in E. But A is closed, so $\mathbf{x} \in A$, according to Proposition 105. ■

Proposition 111
Every complete subspace of a metric space is closed.

Proof: let $A \subseteq (E, d)$ be complete. Let $\mathbf{x} \in \overline{A}$. According to Proposition 104, $\exists(\mathbf{x}_n) \subseteq A$ such that $\mathbf{x}_n \to \mathbf{x}$. Therefore, $(\mathbf{x}_n)$ is a convergent sequence in E. In particular, it is a Cauchy sequence of points in A, according to Proposition 107. But A is complete so that $\mathbf{x} \in A$. Hence $\overline{A} \subseteq A$ and so $\overline{A} = A$, which means that $A \subseteq_C E$. ■

The **product** of two metric spaces (E', d') and (E^*, d^*) is the metric space

$$(E, d) = (E' \times E^*, \sup\{d', d^*\});$$

it is easy to see how this definition can be extended to a product of n metric spaces. At any rate, the product of metric spaces is also a metric space.[6]

Proposition 112
Let (E_i, d_i) be metric spaces for $i = 1, \ldots, n$. The product metric space $(E, d) = (E_1 \times \cdots \times E_n, \sup_{i=1,\ldots,n}\{d_i\})$ is complete if and only if (E_i, d_i) for all $i = 1, \ldots, n$.

Proof: left as an exercise. ■

The following result is a generalization of the nested intervals theorem of Chapter 1.

Proposition 113
Let (E, d) be a complete metric space. If (F_n) is a decreasing sequence of non-empty closed subsets of E

$$E \supseteq F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n \supseteq \cdots$$

such that $\lim_{n\to\infty} \delta(F_n) = 0$, then $\bigcap_{n\geq 1} F_n = \{\mathbf{x}\}$ for some $\mathbf{x} \in E$.

Proof: let $\Gamma = \bigcap F_n$. For each $n \in \mathbb{N}$, pick $\mathbf{x}_n \in F_n$.

Let $\varepsilon > 0$. Since $\delta(F_n) \to 0$, $\exists N_\varepsilon \in \mathbb{N}$ such that

$$n > N_\varepsilon \Longrightarrow \delta(F_n) < \sup\{d(\mathbf{w}, \mathbf{z}) \mid \mathbf{w}, \mathbf{z} \in F_n\} < \tfrac{\varepsilon}{2}.$$

Let $m > n > N_\varepsilon$ and pick $\mathbf{y} \in F_m \subseteq F_n$. Then

$$m > n > N_\varepsilon \Longrightarrow d(\mathbf{x}_n, \mathbf{x}_m) \leq d(\mathbf{x}_n, y) + d(\mathbf{y}, \mathbf{x}_m) < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$$

As $(\mathbf{x}_n) \subseteq E$ is Cauchy and E is complete, $\exists \mathbf{x} \in E$ such that $\mathbf{x}_n \to \mathbf{x}$. For all $p \geq 1$, $(\mathbf{x}_n)_{n\geq p} \subseteq F_p$. As $F_p \subseteq_C E$, $(\mathbf{x}_n)_{n\geq p}$ converges in F_p, according to Proposition 105. Hence $\mathbf{x} \in F_p$ for all $p \geq 1$. Consequently, $\mathbf{x} \in \Gamma$.

But if $\mathbf{y} \in \Gamma$, then $\mathbf{y} \in F_n$ for all n, so that $0 \leq d(\mathbf{x}, \mathbf{y}) \leq \delta(F_n) \to 0$ for all n. Thus $d(\mathbf{x}, \mathbf{y}) = 0$, so that $\mathbf{y} = \mathbf{x}$ and $\Gamma = \{\mathbf{x}\}$. ■

If $r \in (0, 1)$, for instance, and if we have $F_n = \overline{B(\mathbf{0}, r^n)} \subseteq (\mathbf{R}^m, d_2)$ for some $m \geq 1$, then $\bigcap F_n = \{\mathbf{0}\}$.

[6] In fact, the definition can be generalized to arbitrary collections $\{E_\alpha\}_{\alpha\in J}$, but we will see in Part IV that there are complications.

The following contraction result is representative of a family of extremely useful theorems.

Theorem 114 (FIXED POINT THEOREM)
*Let (E, d) be a a complete metric space and let $f : E \to E$ be a **contraction** on E, that is,*

$$\exists k \in (0,1) \text{ such that } d(f(\mathbf{x}), f(\mathbf{y})) \leq k d(\mathbf{x}, \mathbf{y}) \text{ for all } \mathbf{x}, \mathbf{y} \in E.$$

*Then $\exists! \mathbf{x}^* \in E$ such that $f(\mathbf{x}^*) = \mathbf{x}^*$; $\mathbf{x}^*$ is a **fixed point of** f.*

Proof: let $\mathbf{x}_0 \in E$. If $f(\mathbf{x}_0) = \mathbf{x}_0$, we are done. Otherwise, consider the sequence $(f^n(\mathbf{x}_0))_n$, where f^n represents n successive compositions of f:

$$\begin{aligned} d(f^n(\mathbf{x}_0), f^{n+1}(\mathbf{x}_0)) &= d(f(f^{n-1}(\mathbf{x}_0)), f(f^n(\mathbf{x}_0))) \leq k d(f^{n-1}(\mathbf{x}_0), f^n(\mathbf{x}_0)) \\ &= k d(f(f^{n-2})(\mathbf{x}_0), f(f^{n-1})(\mathbf{x}_0)) \leq \cdots \leq k^n d(\mathbf{x}_0, f(\mathbf{x}_0)). \end{aligned}$$

Then, for any $m > n$,

$$\begin{aligned} d(f^m(\mathbf{x}_0), f^n(\mathbf{x}_0)) &\leq d(f^m(\mathbf{x}_0), f^{m-1}(\mathbf{x}_0)) + \cdots + d(f^{n+1}(\mathbf{x}_0), f^n(\mathbf{x}_0)) \\ &\leq (k^n + \cdots + k^{m-1}) d(\mathbf{x}_0, f(\mathbf{x}_0)) \leq \frac{k^n}{1-k} d(\mathbf{x}_0, f(\mathbf{x}_0)) \end{aligned}$$

For any ε, let $M_\varepsilon = \left\lceil \ln\left(\frac{\varepsilon}{d(\mathbf{x}_0, f(\mathbf{x}_0))}(1-k)\right) - \ln k \right\rceil$. Then, whenever $m > n > M_\varepsilon$, we have

$$d(f^m(\mathbf{x}_0), f^n(\mathbf{x}_0)) \leq \frac{k^n}{1-k} d(\mathbf{x}_0, f(\mathbf{x}_0)) \leq \frac{k^{M_\varepsilon}}{1-k} d(\mathbf{x}_0, f(\mathbf{x}_0)) < \varepsilon.$$

Consequently, $(f^n(\mathbf{x}_0))$ is a Cauchy sequence in E. But E is complete so that $f^n(\mathbf{x}_0) \to \mathbf{x}$ for some $\mathbf{x} \in E$.

By definition, contraction mappings are Lipschitz continuous, and thus also continuous, and so

$$f(\mathbf{x}) = f\big(\lim_{n\to\infty} f^n(\mathbf{x}_0)\big) = \lim_{n\to\infty} f(f^n(\mathbf{x}_0)) = \lim_{n\to\infty} f^{n+1}(\mathbf{x}_0) = \mathbf{x}.$$

Now, suppose that $\mathbf{x}$ and $\mathbf{y}$ are two fixed points of f. Then,

$$d(\mathbf{x}, \mathbf{y}) = d(f(\mathbf{x}), f(\mathbf{y})) \leq k d(\mathbf{x}, \mathbf{y}).$$

Since $k < 1$, the only way for the inequality to be valid is if $d(\mathbf{x}, \mathbf{y}) = 0$, which implies that $\mathbf{x} = \mathbf{y}$. The fixed point of f is thus unique. Call it $\mathbf{x}^*$ to match with the statement of the theorem. ■

The choice of $\mathbf{x}_0 \in E$ in the proof of Theorem 114 is arbitrary; if f is a contraction, the sequence $(f^n(\mathbf{x}))$ converges to the unique fixed point $\mathbf{x}^*$ for all $\mathbf{x} \in E$. Note that the restriction $k \in (0,1)$ is necessary, as the following example demonstrates.

Example: let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1, & x < 0 \\ x + \frac{1}{x+1}, & x \geq 0 \end{cases}.$$

It is not hard to see that f has no fixed point (see exercise 45), yet

$$d(f(x), f(y)) \leq d(x,y) \quad \text{for all } x, y \in \mathbb{R}. \qquad \square$$

8.3 Solved Problems

1. Let A, B be subsets of a metric space (E, d). Show that

   a) $B \subseteq A \Longrightarrow \text{int}(B) \subseteq \text{int}(A)$
   b) $B \subseteq A \Longrightarrow \overline{B} \subseteq \overline{A}$
   c) $\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$
   d) $\overline{A \cup B} = \overline{A} \cup \overline{B}$
   e) $\text{int}(A) \cup \text{int}(B) \subseteq \text{int}(A \cup B)$
   f) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$

   Proof:

   a) By definition, $\text{int}(B) \subseteq B \subseteq A$, i.e. $\text{int}(B)$ is an open set contained in A. Consequently, $\text{int}(B)$ is contained in the largest open set contained in A, namely $\text{int}(A)$.
   b) By definition, $B \subseteq A \subseteq \overline{A}$, i.e. $\overline{A}$ is a closed set containing B. Consequently, $\overline{A}$ contains the smallest closed set containing B, i.e. $\overline{B}$.
   c) Since $\text{int}(A) \cap \text{int}(B) \subseteq_O E$ and since $\text{int}(A) \subseteq A$ and $\text{int}(B) \subseteq B$, we must have $\text{int}(A) \cap \text{int}(B) \subseteq A \cap B$. As such, $\text{int}(A) \cap \text{int}(B)$ must be contained in the largest open set contained in $A \cap B$, so that $\text{int}(A) \cap \text{int}(B) \subseteq \text{int}(A \cap B)$. On the other hand, since $A \cap B \subseteq A, B$, then we must have $\text{int}(A \cap B) \subseteq \text{int}(A), \text{int}(B)$ and so
   $$\text{int}(A \cap B) \subseteq \text{int}(A) \cap \text{int}(B).$$
   d) Basically the same proof with $\cap \leftrightsquigarrow \cup$, $\subseteq \leftrightsquigarrow \supseteq$, $\text{int}(\cdot) \leftrightsquigarrow \overline{(\cdot)}$.
   e) Since $A, B \subseteq A \cup B$, then $\text{int}(A), \text{int}(B) \subseteq \text{int}(A \cup B)$. Hence $\text{int}(A) \cup \text{int}(B) \subseteq \text{int}(A \cup B)$.
   f) Basically the same proof with $\cap \leftrightsquigarrow \cup$, $\subseteq \leftrightsquigarrow \supseteq$, $\text{int}(\cdot) \leftrightsquigarrow \overline{(\cdot)}$. $\blacksquare$

2. In each instance, give an example showing that, in general,

   a) $\text{int}(A) \cup \text{int}(B) \neq \text{int}(A \cup B)$
   b) $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$

   Solution:

   a) Let $E = \mathbb{R}$ with the Euclidean metric, and let $A = [a, b]$ and $B = [b, c]$ with $c > b > a$, for instance. Then

$$\begin{aligned} \text{int}(A) &= (a, b), \quad \text{int}(B) = (b, c), \quad A \cup B = [a, c], \\ \text{int}(A \cup B) &= (a, c), \quad \text{int}(A) \cup \text{int}(B) = (a, b) \cup (b, c) = (a, c) \setminus \{b\}. \end{aligned}$$

   b) Let $E = \mathbb{R}$ with the Euclidean metric, and $A = (a, b)$ and $B = (b, c)$ with $c > b > a$, for instance. Then

$$\overline{A} = [a, b], \quad \overline{B} = [b, c], \quad A \cap B = \varnothing, \quad \overline{A \cap B} = \varnothing, \quad \overline{A} \cap \overline{B} = \{b\}. \qquad \square$$

3. Let A be subset of a metric space (E, d). Show that

   a) $E \setminus \text{int}(A) = \overline{E \setminus A}$
   b) $E \setminus \overline{A} = \text{int}(E \setminus A)$
   c) $\partial(\text{int}(A)) \subseteq \partial A$
   d) $\partial \overline{A} \subseteq \partial A$

   Proof:

   a) We have

$$\begin{aligned} \text{int}(A) &\subseteq A, \quad \text{by definition} \\ E \setminus A &\subseteq E \setminus \text{int}(A), \quad \text{again by definition} \\ \overline{E \setminus A} &\subseteq \overline{E \setminus \text{int}(A)} = E \setminus \text{int}(A), \quad \text{as } E \setminus \text{int}(A) \subseteq_C E \end{aligned}$$

   On the other hand, we have

$$\begin{aligned} E \setminus A &\subseteq \overline{E \setminus A}, \quad \text{by definition} \\ E \setminus \overline{E \setminus A} &\subseteq E \setminus (E \setminus A) = A, \quad \text{again by definition} \\ E \setminus \overline{E \setminus A} &= \text{int}(E \setminus \overline{E \setminus A}) \subseteq \text{int}(A) = E \setminus \text{int}(A), \quad \text{as } E \setminus \overline{E \setminus A} \subseteq_O E \\ E \setminus \text{int}(A) &\subseteq \overline{E \setminus A} \end{aligned}$$

   b) We have

$$\begin{aligned} A &\subseteq \overline{A}, \quad \text{by definition} \\ E \setminus \overline{A} &\subseteq E \setminus A, \quad \text{again by definition} \\ E \setminus \overline{A} &= \text{int}(E \setminus \overline{A}) \subseteq \text{int}(E \setminus A), \quad \text{as } E \setminus \overline{A} \subseteq_O E \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \text{int}(E \setminus A) &\subseteq E \setminus A, \quad \text{by definition} \\ A = E \setminus (E \setminus A) &\subseteq E \setminus \text{int}(E \setminus A), \quad \text{again by definition} \\ \overline{A} \subseteq \overline{E \setminus \text{int}(E \setminus A)} &= E \setminus \text{int}(E \setminus A) \quad \text{as } E \setminus \text{int}(E \setminus A) \subseteq_C E \\ \text{int}(E \setminus A) &\subseteq E \setminus \overline{A} \end{aligned}$$

c) Since $\text{int}(A) \subseteq A$, we have $\overline{\text{int}(A)} \subseteq \overline{A}$ and so

$$\partial \, \text{int}(A) = \overline{\text{int}(A)} \setminus \text{int}(A) \subseteq \overline{A} \setminus \text{int}(A) = \partial A.$$

d) Basically the same idea, as above, but with $X \setminus \text{int}(\overline{A}) \subseteq X \setminus \text{int}(A)$. ■

4. Find an example of a subset A of a metric space (E, d) for which $\partial(\text{int}(A))$, ∂A and $\partial \overline{A}$ are all different.

Solution: let $E = \mathbb{R}$ with the Euclidean metric, and let $A = \mathbb{Q} \cup (0, 1)$, for instance. Then

$$\begin{aligned} \overline{A} &= \overline{\mathbb{Q} \cup (0,1)} = \overline{\mathbb{Q}} \cup \overline{(0,1)} = \mathbb{R} \\ \text{int}(A) &= \{x \in \mathbb{R} \mid \exists r > 0 \text{ s.t. } B(x, r) \subseteq A\} = (0, 1) \\ \partial(\text{int}(A)) &= \overline{\text{int}(A)} \setminus \text{int}(A) = \overline{(0,1)} \setminus (0,1) = [0,1] \setminus (0,1) = \{0, 1\} \\ \partial A &= \overline{A} \setminus \text{int}(A) = \mathbb{R} \setminus (0,1) \\ \partial \overline{A} &= \overline{A} \setminus \text{int}(\overline{A}) = \mathbb{R} \setminus \text{int}(\mathbb{R}) = \mathbb{R} \setminus \mathbb{R} = \varnothing \end{aligned}$$

which are all distinct. □

5. Find two subsets $A, B \subseteq (R, d_2)$ for which $A \cup B$, $\text{int}(A) \cup B$, $A \cup \text{int}(B)$, $\text{int}(A) \cup \text{int}(B)$, and $\text{int}(A \cup B)$ are all distinct.

Solution: let $E = \mathbb{R}$ with the Euclidean metric, and let

$$A = [\sqrt{2}, \varphi) \cup (\varphi, e) \cup \{\pi\} \cup (\mathbb{Q} \cap (8, 9)), \quad \varphi = \frac{1 + \sqrt{5}}{2}$$

for instance. Then

$$\begin{aligned} \text{int}(A) &= (\sqrt{2}, \varphi) \cup (\varphi, e), \quad \overline{A} = [\sqrt{2}, e] \cup \{\pi\} \cup [8, 9] \\ \text{int}(\overline{A}) &= (\sqrt{2}, e) \cup (8, 9) \\ \overline{\text{int}(A)} &= [\sqrt{2}, e] \\ \overline{\text{int}(\overline{A})} &= [\sqrt{2}, e] \cup [8, 9] \\ \text{int}\left(\overline{\text{int}(A)}\right) &= (\sqrt{2}, e) \end{aligned}$$

are all distinct. □

6. Find a subset $A \subseteq (R, d_2)$ for which A, $\text{int}(A)$, $\overline{A}$, $\text{int}(\overline{A})$, $\overline{\text{int}(A)}$, $\overline{\text{int}(\overline{A})}$ and $\text{int}\left(\overline{\text{int}(A)}\right)$ are all distinct.

Solution: let $E = \mathbb{R}$ with the Euclidean metric, and let $A = [\sqrt{2}, e]$ and $B = [e, \pi]$, for instance. Then

$$\begin{aligned} A \cup B &= [\sqrt{2}, \pi] \\ \text{int}(A) \cup B &= (\sqrt{2}, \pi] \\ A \cup \text{int}(B) &= [\sqrt{2}, \pi) \\ \text{int}(A) \cup \text{int}(B) &= (\sqrt{2}, \pi) \setminus \{e\} \\ \text{int}(A \cup B) &= (\sqrt{2}, \pi) \end{aligned}$$

which are all distinct. □

7. For any subset $A \subseteq (R, d_2)$, show that $\text{int}\left(\overline{\text{int}(\overline{A})}\right) = \text{int}(\overline{A})$.

Proof: By definition,

$$\text{int}(\overline{A}) \subseteq \overline{A} \Longrightarrow \overline{\text{int}(\overline{A})} \subseteq \overline{\overline{A}} = \overline{A} \Longrightarrow \text{int}\left(\overline{\text{int}(\overline{A})}\right) \subseteq \text{int}(\overline{A}).$$

On the other hand, whenever B is open we have

$$B \subseteq \overline{B} \Longrightarrow B = \text{int}\, B \subseteq \text{int}(\overline{B}).$$

Set $B = \text{int}(\overline{A})$. Then B is open and

$$\text{int}(\overline{A}) \subseteq \text{int}(\overline{B}) = \text{int}\left(\overline{\text{int}(\overline{A})}\right),$$

which completes the proof. ■

(Could we replace $(\mathbb{R}, d_2)$ by any metric space? Any topological space?)

8. We say that $A \subseteq E$ is **meagre** (or nowhere dense) if and only if $\text{int}(\overline{A}) = \varnothing$. Show that
   a) A is meagre if and only if $\text{int}(E \setminus A)$ is dense in E (A is **dense** in B if $A \subseteq B \subseteq \overline{A}$);
   b) A is meagre if and only if A is contained in a closed subset of E whose interior is empty;
   c) A is closed and meagre if and only if $A = \partial A$, and
   d) A is meagre $\Longrightarrow \overline{A} = \partial A$.

Proof:

a) $\boxed{\Longrightarrow}$ If $\text{int}(\overline{A}) = \varnothing$, then

$$E = E \setminus \varnothing = E \setminus \text{int}(\overline{A}) = \overline{E \setminus \overline{A}} = \overline{\text{int}\, E \setminus A}.$$

Hence $\text{int}(E \setminus A)$ is dense in E.

$\boxed{\Longleftarrow}$ It's pretty much the same thing: if $\overline{\text{int}(E \setminus A)} = E$, then

$$E = \overline{\text{int}\, E \setminus A} = \overline{E \setminus \overline{A}} = E \setminus \text{int}(\overline{A}).$$

Hence $\text{int}(\overline{A}) = \varnothing$.

b) $\boxed{\Longrightarrow}$ If int$(\overline{A}) = \varnothing$, then $\overline{A}$ does not have interior points. Since $\overline{A} \subseteq_C E$ and since $A \subseteq \overline{A}$, then A is contained in a closed set whose interior is empty.

$\boxed{\Longleftarrow}$ Let $A \subseteq B$, where $B \subseteq_C E$ and int$(B) = \varnothing$. By definition, $\overline{A} \subseteq B$ and so int$(\overline{A}) \subseteq$ int$(B) = \varnothing$.

c) $\boxed{\Longrightarrow}$ If $\overline{A} = A$ and int$(\overline{A}) = \varnothing$, then int$(A) =$ int$(\overline{A}) = \varnothing$. Then

$$\partial A = \overline{A} \setminus \text{int}(A) = \overline{A} \setminus \varnothing = \overline{A} = A.$$

$\boxed{\Longleftarrow}$ We have $A = \partial A \iff A = \overline{A} \setminus A \implies A \subseteq \overline{A} \setminus \text{int}(A)$. However int$(A) \subseteq A$ so that int$(A) \neq \varnothing \implies A \not\subseteq \overline{A} \setminus \text{int}(A)$. Consequently, int$(A) = \varnothing$, which means that $A = \partial A = \overline{A}$ and so $A \subseteq_C E$. Then int$(\overline{A}) =$ int$(A) = \varnothing$.

d) If int$(\overline{A}) = \varnothing$, we have $A \subseteq \overline{A} \implies \text{int}(A) \subseteq \text{int}(\overline{A}) = \varnothing$. Hence

$$\partial A = \overline{A} \setminus \text{int}(A) = \overline{A} \setminus \varnothing = \overline{A}.$$

(What condition must hold for the converse to be satisfied?) ∎

9. Show that d_∞, d_1 and d_2 are equivalent on $\mathbb{R}^2$.

Proof: we could do it directly, but notice that these metrics are all derived from norms on $\mathbb{R}^2$. Since $\mathbb{R}^2$ is a finite-dimensional vector space, all norms on $\mathbb{R}^2$ are equivalent. Hence the three metrics are equivalent. That is all there is to it. ∎

10. For $i = 1, \ldots, n$, let (E_i, d_i) be metric spaces and $U_i \subseteq_O E_i$. Show that $U_1 \times \cdots \times U_n$ is an open subset of

$$(E, d) = (E_1 \times \cdots \times E_n, \sup\{d_i \mid i = 1, \ldots, n\}).$$

Proof: consider the subset $U = U_1 \times \cdots \times U_n \subseteq E$, where $U_i \subseteq_O E_i$ for all i. Let $\mathbf{x} \in U$. Then $\pi_i(\mathbf{x}) = \mathbf{x}_i \in U_i$ for all i. But $U_i \subseteq_O E_i$ so that $\exists \eta_i > 0$ with $B_{d_i}(\mathbf{x}_i, \eta_i) \subseteq U_i$. Set $\eta = \min\{\eta_i\}_{i=1}^n > 0$. Then

$$\begin{aligned} B(\mathbf{x}, \eta) &= \{\mathbf{y} | d(\mathbf{x}, \mathbf{y}) < \eta\} = \{\mathbf{y} | \sup\{d_i(\mathbf{x}_i, \mathbf{y}_i)\}_{i=1}^n < \eta\} \\ &= \{\mathbf{y} | d_i(\mathbf{x}_i, \mathbf{y}_i) < \eta \; \forall i = 1, \ldots, n\} = \prod_{i=1}^n B_{d_i}(\mathbf{x}_i, \eta) \subseteq \prod_{i=1}^n U_i = U \end{aligned}$$

Consequently, $U \subseteq_O E$. ∎

11. For $i = 1, \ldots, n$, let (E_i, d_i) be metric spaces and let $\pi_i : E_1 \times \cdots \times E_n \to E_i$ be defined by $\pi_i(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \mathbf{x}_i$. Show that π_i is open and continuous.

Proof: let $i \in \{1, \ldots, n\}$ and $U \subseteq_O E_i$. Since

$$\pi_i^{-1}(U) = E_1 \times \cdots E_{i-1} \times U \times E_{i+1} \times \cdots E_n,$$

then $\pi_i^{-1}(U) \subseteq_0 E_1 \times \cdots \times E_n$ according to the previous problem, and so π_i is continuous.

Now, suppose that $V \subseteq_O E_1 \times \cdots \times E_n$. We need to show that

$$\pi_i(V) = \{\mathbf{x} \in E_i | \mathbf{x} = \pi_i(\mathbf{y}), \mathbf{y} \in V\} \subseteq_O E_i.$$

Let $\mathbf{u} \in \pi_i(V)$ and consider $\mathbf{x} \in \pi_i^{-1}(\mathbf{u})$. Since $V \subseteq_O E_1 \times \cdots \times E_n$, $\exists r_{\mathbf{x}} > 0$ such that $B_d(\mathbf{x}, r_{\mathbf{x}}) \subseteq V$. We will show that $B_{d_i}(\mathbf{u}, r_{\mathbf{x}}) \subseteq \pi_i(V)$. Let $\mathbf{z} \in B_{d_i}(\mathbf{u}, r_{\mathbf{x}})$. Then $d_i(\mathbf{u}, \mathbf{z}) < r_{\mathbf{x}}$. Set $\mathbf{w} = \mathbf{x}$, except in the ith position, where $\mathbf{w}_i = \mathbf{z}$. Then $\pi_i(\mathbf{w}) = \mathbf{z}$ and

$$d(\mathbf{w}, \mathbf{x}) = \sup\{d_i(\mathbf{w}_i, \mathbf{x}_i)\} = \sup\{0, \ldots, d_i(\mathbf{z}, \mathbf{u}), \ldots, 0\} = d_i(\mathbf{z}, \mathbf{u}) < r_{\mathbf{x}},$$

that is, $\mathbf{w} \in B_d(\mathbf{x}, r_{\mathbf{x}}) \subseteq V$. Thus $\mathbf{z} = \pi_i(\mathbf{w}) \in \pi_i(V)$, and so π_i is open. ■

12. Show that a map $f : (F, \delta) \to (E_1, d_1) \times \cdots \times (E_n, d_n)$ is continuous at $\mathbf{a} \in F$ if and only if $\pi_i \circ f$ is continuous at $\mathbf{a} \in F$ for all i.

    Proof: if f is continuous at $\mathbf{a}$, then $\pi \circ f$ is continuous at $\mathbf{a}$ for all i, since π_i is continuous and the composition of continuous functions is continuous.

    Now, if $\pi_i \circ f$ is continuous at $\mathbf{a} \in F$ for all i, then, for all $\varepsilon > 0$, $\exists \eta_1, \ldots, \eta_n > 0$ such that $d_i(\pi_i(f(\mathbf{x})), \pi_i(f(\mathbf{a}))) < \varepsilon$ whenever $\delta(\mathbf{x}, \mathbf{a}) < \eta_i$ for all $i = 1, \ldots, n$.

    Set $\eta = \sup\{\eta_i\} > 0$. Then, for all $\varepsilon > 0$,

    $$d(f(\mathbf{x}), f(\mathbf{a})) = \sup\{d_i(\pi_i(f(\mathbf{x})), \pi_i(f(\mathbf{a})))\} < \varepsilon$$

    whenever $\delta(\mathbf{x}, \mathbf{a}) < \eta$; as such, f is continuous at $\mathbf{a}$. ■

13. Let $f : (E_1, d_1) \times \cdots \times (E_n, d_n) \to (F, \delta)$ and $\mathbf{a} = (\mathbf{a}_1, \ldots, \mathbf{a}_n) \in E$. For all i, define $f_i : (E_i, d_i) \to (F, \delta)$ by $f_i(\mathbf{x}) = f(\mathbf{a}_1, \ldots, \mathbf{a}_{i-1}, \mathbf{x}, \mathbf{a}_{i+1}, \ldots, \mathbf{a}_n)$. Show that if f is continuous at $\mathbf{a}$, then f_i is continuous at $\mathbf{a}$ for all i.

    Proof: by continuity of f, for all $\varepsilon > 0$, $\exists \eta > 0$ such that

    $$d(\mathbf{x}, \mathbf{a}) < \eta \Longrightarrow \delta(f(\mathbf{x}), f(\mathbf{a})) < \varepsilon.$$

    For any $\mathbf{x} \in E_i$, write $\tilde{\mathbf{x}} = (\mathbf{a}_1, \ldots, \mathbf{a}_{i-1}, \mathbf{x}, \mathbf{a}_{i+1}, \ldots, \mathbf{a}_n)$. Then, if $d(\tilde{\mathbf{x}}, \mathbf{a}) < \eta$, we have

    $$\delta(f_i(\mathbf{x}), f_i(\mathbf{a})) = \delta(f(\tilde{\mathbf{x}}), f(\mathbf{a})) < \varepsilon.$$

    Since $d_i(\mathbf{x}, \mathbf{a}_i) \leq d(\tilde{\mathbf{x}}, \mathbf{a}) < \eta$, f_i is continuous at $\mathbf{a}$. ■

14. Show that $d = \sup\{d_i \mid i = 1, \ldots, n\}$ defines a metric on $E = \prod_{i=1}^n (E_i, d_i)$.

    Proof: the only property which is not immediately obvious is the triangle inequality (and even at that, it is pretty obvious). Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in E$. Then

    $$\begin{aligned} d(\mathbf{x}, \mathbf{y}) &= \sup\{d_i(\mathbf{x}_i, \mathbf{y}_i)\} \leq \sup\{d_i(\mathbf{x}_i, \mathbf{z}_i) + d_i(\mathbf{z}_i, \mathbf{y}_i)\} \\ &\leq \sup\{d_i(\mathbf{x}_i, \mathbf{z}_i)\} + \sup\{d_i(\mathbf{z}_i, \mathbf{y}_i)\} = d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y}) \end{aligned}$$

    So we've got that going for us, which is nice. ■

15. Let (E_i, d_i) be metric spaces for $i = 1, \ldots, n$. Show that the metric product space $(E, d) = (\prod E_i, \sup\{d_i\})$ is complete if and only if (E_i, d_i) is complete for each i.

 Proof: Assume (E, d) is complete, and let $(\mathbf{x}_n)$ be a Cauchy sequence in (E_i, d_i) for some i. Then for all $\varepsilon > 0$, $\exists M \in \mathbb{N}$ such that $d_i(\mathbf{x}_n, \mathbf{x}_m) < \varepsilon$ whenever $n, m > M$.

 For each $j \neq i$, pick $\mathbf{a}_j \in E_j$.

 Write $\mathbf{w}_n = (\mathbf{a}_1, \ldots, \mathbf{a}_{i-1}, \mathbf{x}_n, \mathbf{a}_{i+1}, \ldots, \mathbf{a}_n)$. Then $(\mathbf{w}_n)$ is a Cauchy sequence in E: indeed for all $\varepsilon > 0$, we have

$$\begin{aligned} d(\mathbf{w}_n, \mathbf{w}_m) &= \sup\{d_i(\pi_i(\mathbf{w}_n), \pi_i(\mathbf{w}_m))\} \\ &= \sup\{d_1(\mathbf{a}_1, \mathbf{a}_1), \ldots, d_i(\mathbf{x}_n, \mathbf{x}_m), \ldots, d_n(\mathbf{a}_n, \mathbf{a}_n)\} \\ &= \sup\{0, \ldots, 0, d_i(\mathbf{x}_n, \mathbf{x}_m), 0, \ldots, 0\} = d_i(\mathbf{x}_n, \mathbf{x}_m) < \varepsilon \end{aligned}$$

 whenever $n, m > M$.

 Since (E, d) is complete, $\exists \mathbf{w} \in E$ for which $\mathbf{w}_n \to \mathbf{w}$. Furthermore, π_i is continuous, so that $\mathbf{x}_n = \pi_i(\mathbf{w}_n) \to \pi_i(\mathbf{w}) \in E_i$, and so $(\mathbf{x}_n)$ converges in (E_i, d_i). Consequently, (E_i, d_i) is complete for all i.

 On the other hand, suppose that (E_i, d_i) is complete for all i, and let $(\mathbf{w}_n)$ be a Cauchy sequence in (E, d).

 Since $d_i(\pi_i(\mathbf{w}_n), \pi_i(\mathbf{w}_m)) \leq d(\mathbf{w}_n, \mathbf{w}_m)$ for all i, $(\pi_i(\mathbf{w}_n))$ is a Cauchy sequence in (E_i, d_i) for all i. As all (E_i, d_i) are complete, $\exists \mathbf{x}_1, \ldots, \mathbf{x}_n$, $\mathbf{x}_i \in E_i$, such that $\pi_i(\mathbf{w}_n) \to \mathbf{x}_i$ for all i, i.e. for all $\varepsilon > 0$, $\exists M_1, \ldots, M_n \in \mathbb{N}$ such that

$$\forall i,\ d_i(\pi_i(\mathbf{w}_n), \mathbf{x}_i) < \varepsilon \quad \text{whenever } n > M_i.$$

 Set $M = \max\{M_i | i = 1, \ldots, n\} < \infty$ and $\mathbf{w} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$. Let $\varepsilon > 0$.

 Then

$$d(\mathbf{w}_n, \mathbf{w}) = \sup\{d_i(\pi_i(\mathbf{w}_n), \pi_i(\mathbf{w}))\} = \sup\{d_i(\pi_i(\mathbf{w}_n), \mathbf{x}_i)\} < \varepsilon$$

 whenever $n > M$.

 As we have shown that $\mathbf{w}_n \to \mathbf{w} \in E$, we conclude that (E, d) is complete. ■

16. Show that the converse of the previous result does not hold in general, for instance for $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0, & \text{else} \end{cases}$$

 Solution: the problem is that $f(x, 0)$ is continuous at $x = 0$, $f(0, y)$ is continuous at $y = 0$, but $f(x, y)$ is not continuous at $(x, y) = (0, 0)$ since, among other things, $\lim_{z \to 0} f(z, z) = \frac{1}{2} \neq 0$. ■

17. Let $d_1, d_2 : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ be defined according to

$$d_1(m,n) = \begin{cases} 0, & \text{if } m = n \\ 1 + \frac{1}{m+n}, & \text{otherwise} \end{cases} \qquad d_2(m,n) = \frac{|m-n|}{mn}.$$

a) Show that d_1 and d_2 are metrics on $\mathbb{N}$.

b) Show that the topologies of $(\mathbb{N}, d_1)$ and $(\mathbb{N}, d_2)$ are both discrete.

c) Show that $(\mathbb{N}, d_1)$ is complete but that $(\mathbb{N}, d_2)$ is not.

d) What does this say about completeness as a topological property of a space?

Proof:

a) The only property which is not immediately obvious is the triangle inequality.

- If $d_1(m,n) = 0$, then $0 = d_1(m,n) \leq d_1(m,k) + d_1(k,n)$ for all k.

  If $d_1(m,n) \neq 0$ and $d_1(m,k) = 0$, then $d_1(m,n) \leq d_1(m,k) + d_1(k,n)$.

  If $d_1(m,n), d_1(m,k), d_1(k,n) \neq 0$, then

  $$d_1(m,n) = 1 + \frac{1}{m+n} \leq 2 + \frac{1}{m+k} + \frac{1}{k+n} = d_1(m,k) + d_1(k,n)$$

  since $\frac{1}{m+n} < 1$.
- For d_2, notice that

  $$\begin{aligned} d_2(m,k) + d_2(k,n) &= \frac{|m-k|}{mk} + \frac{|k-n|}{kn} = \frac{n|m-k| + m|k-n|}{mkn} \\ &= \frac{|nm-nk| + |mk-mn|}{mkn} \\ &\geq \frac{|mk-nk|}{mkn} = \frac{|m-n|k}{mkn} = \frac{|m-n|}{mn} = d_2(m,n) \end{aligned}$$

b) For all $n \in \mathbb{N}$, we need to show that $\{n\}$ is open in both $(\mathbb{N}, d_1)$ and $(\mathbb{N}, d_2)$, that is, we must show $\exists r_1, r_2 > 0$ such that $B_{d_i}(n, r_i) \subseteq \{n\}$.

- Pick any $r_1 < 1$. Then

  $$B_{d_1}(n, r_1) = \{y \in \mathbb{N} \mid_1 (y,n) < r_1\} = \left\{y \in \mathbb{N} \mid y = n \text{ or } \tfrac{1}{n+y} < 1\right\} = \{n\}.$$

- Simple algebraic manipulations show that $d_2(n,m) \geq \frac{1}{n(n+1)}$ whenever $n \neq m \in \mathbb{N}$. Set $r_2 = \frac{1}{n(n+1)} > 0$. Then

  $$B_{d_2}(n, r_2) = \left\{y \in \mathbb{N} \mid_2 (n,y) < \frac{1}{n(n+1)}\right\} = \{n\}$$

c) For completeness:

- Let (k_n) be a Cauchy sequence in $(\mathbb{N}, d_1)$. Then, for all $1 > \varepsilon > 0$, $\exists M \in \mathbb{N}$ such that $d_1(k_n, k_m) < \varepsilon$ whenever $n, m > M$.

  Since $d_1(x, y) > 1$ for all $x \neq y$, we must have $k_n = k_m$ for all $n, m > M$. Then (k_n) is constant for all $n > M$, and as such, it is a convergent sequence in $(\mathbb{N}, d_1)$.
- Consider the sequence (n) in $(\mathbb{N}, d_2)$. To show that (n) is a Cauchy sequence, let $\varepsilon > 0$ and $M > \frac{2}{\varepsilon}$. Then

$$d_2(m,n) = \frac{|m-n|}{mn} \leq \frac{m+n}{mn} = \frac{1}{m} + \frac{1}{n} \leq \frac{2}{\min\{m,n\}} < \frac{2}{M} < \varepsilon$$

  whenever $m, n > M$.

  Now, if $n \to K$ in $(\mathbb{N}, d_2)$, then, for $\varepsilon = \frac{1}{K(K+1)}$, $\exists M \in \mathbb{N}$ such that $d(K, n) < \frac{1}{K(K+1)}$ whenever $n > M$ (except for possibly $K = n$).

  But this contradicts the fact that $d(K, n) \geq \frac{1}{K(K+1)}$ whenever $K \neq n$. Hence (n) cannot converge in $(\mathbb{N}, d_2)$.

d) This is yet another example that completeness is not a topological property... ■

18. Let (E, d) be a metric space. Define $d_1, d_2 : E \times E \to \mathbb{R}$ by $d_1(\mathbf{x}, \mathbf{y}) = \frac{d(\mathbf{x},\mathbf{y})}{1+d(\mathbf{x},\mathbf{y})}$ and $d_2(\mathbf{x}, \mathbf{y}) = \min\{d(\mathbf{x}, \mathbf{y}), 1\}$.

a) Show that d_1 and d_2 are metrics on E.

b) Show that d is topologically equivalent to d_2.

c) Show that d_1 is topologically equivalent to d_2.

Proof:

a) The only property which is not immediately obvious is the triangle inequality.

- Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in E$.

  Write $t = d(\mathbf{x}, \mathbf{y}) \geq 0$, $k = d(\mathbf{x}, \mathbf{z}) \geq 0$, $\ell = d(\mathbf{z}, \mathbf{y}) \geq 0$. Since d is a metric, $t \leq k + \ell$. Since the function $f(w) = \frac{w}{1+w}$ is increasing over $[0, \infty)$,

$$\begin{aligned} d_1(\mathbf{x}, \mathbf{y}) &= \frac{t}{1+t} \leq \frac{k+\ell}{1+k+\ell} = \frac{k}{1+k+\ell} + \frac{\ell}{1+k+\ell} \\ &\leq \frac{k}{1+k} + \frac{\ell}{1+\ell} = d_1(\mathbf{x}, \mathbf{z}) + d_1(\mathbf{z}, \mathbf{w}). \end{aligned}$$

- Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in E$. If $d_2(\mathbf{x}, \mathbf{z}) \geq 1$ or $d_2(\mathbf{z}, \mathbf{y}) \geq 1$, then

$$d_2(\mathbf{x}, \mathbf{z}) + d_2(\mathbf{z}, \mathbf{y}) \geq 1 \geq d_2(\mathbf{x}, \mathbf{y}).$$

  If $d_2(\mathbf{x}, \mathbf{z}) < 1$ and $d_2(\mathbf{z}, \mathbf{y}) < 1$, then

$$d_2(\mathbf{x}, \mathbf{z}) \leq d(\mathbf{x}, \mathbf{y}) \leq d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y}) = d_2(\mathbf{x}, \mathbf{z}) + d_2(\mathbf{z}, \mathbf{y}).$$

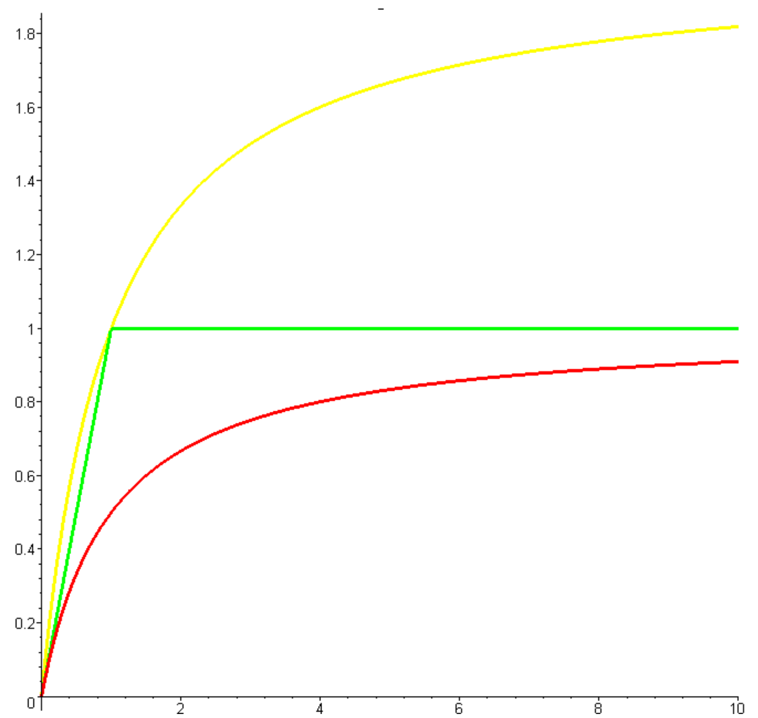
b) Since $d_2 \leq d$, $B_d(\mathbf{x}, r) \subseteq B_{d_2}(\mathbf{x}, r)$ for all $\mathbf{x} \in E$ and $r > 0$. That is, $B_{d_2}(\mathbf{x}, r)$ is open in the $d-$topology.

Similarly, $B_{d_2}(\mathbf{x}, \min\{r, 1\}) \subseteq B_d(\mathbf{x}, r)$ for all $\mathbf{x} \in E$. That is, $B_d(\mathbf{x}, r)$ is open in the d_2-topology. Hence d and d_2 are equivalent.

c) Lengthy but simple manipulations show that

$$\underbrace{d_1}_{\text{red}} \leq \underbrace{d_2}_{\text{green}} \leq \underbrace{2d_1}_{\text{yellow}}$$

and so the metrics are equivalent. □



19. Let (E, d) and $(F, \hat{d})$ be two metric spaces, and let $A \subseteq E$ be dense in E.

a) If $f : (A, d) \to (F, \hat{d})$ is continuous and if $\lim_{\mathbf{y}\to\mathbf{x}, \mathbf{y}\in A} f(\mathbf{y})$ exists for all $\mathbf{x} \in E \setminus A$, show that there exists a unique continuous function $g : E \to F$ with $g|_A = f$.

b) Assume further that $(F, \hat{d})$ is complete. If $f : (A, d) \to (F, \hat{d})$ is uniformly continuous, show that there exists a unique function $g : E \to F$, uniformly continuous, with $g|_A = f$.

Proof:

a) The function $g : E \to F$ that does the trick is given by

$$g(\mathbf{x}) = \begin{cases} f(\mathbf{x}), & \mathbf{x} \in A \\ \lim_{\mathbf{y}\to\mathbf{x}, \mathbf{y}\in A} f(\mathbf{y}), & \mathbf{x} \in E \setminus A \end{cases} \tag{8.4}$$

In order to show that g is continuous, let $\mathbf{x} \in E$ and $(\mathbf{x}_n) \subseteq E$ be such that $\mathbf{x}_n \to \mathbf{x}$. For all $n \in \mathbb{N}$, $g(\mathbf{x}_n) = \lim_{\mathbf{y}\to\mathbf{x}_n, \mathbf{y}\in A} f(\mathbf{y})$. Consequently, for any $n \in \mathbb{N}$, $\exists \mathbf{y}_n \in A$ such that

$$d(\mathbf{x}_n, \mathbf{y}_n) \leq \frac{1}{n} \quad \text{and} \quad \hat{d}(g(\mathbf{x}_n), f(\mathbf{y}_n)) < \frac{1}{n}.$$

From the triangle inequality

$$d(\mathbf{x}, \mathbf{y}_n) \leq d(\mathbf{x}, \mathbf{x}_n) + d(\mathbf{x}_n, \mathbf{y}_n) \leq \frac{1}{n} + d(\mathbf{x}, \mathbf{x}_n)$$

we conclude that $\mathbf{y}_n \to \mathbf{x}$ and so that $f(\mathbf{y}_n) \to g(\mathbf{x})$. Combining this result with

$$\hat{d}(g(\mathbf{x}_n), g(\mathbf{x})) \leq \hat{d}(g(\mathbf{x}_n), f(\mathbf{y}_n)) + \hat{d}(f(\mathbf{y}_n), g(\mathbf{x})) \leq \frac{1}{n} + \hat{d}(f(\mathbf{y}_n), g(\mathbf{x})),$$

we conclude that $g(\mathbf{x}_n) \to g(\mathbf{x})$. By the Sequential Criterion, g is thus continuous at $\mathbf{x}$ for all $\mathbf{x} \in E$, and so it is continuous on E.

It remains only to show that g is the unique function satisfying the conditions outlined in the statement of the problem.

Let $g, h : E \to F$ be two continuous functions with $g|_A = h|_A = f|_A$. Then $g(\mathbf{x}) = h(\mathbf{x})$ for all $\mathbf{x} \in A$.

Now, let $\mathbf{x} \in E \setminus A$. Since A is dense in E, there is a sequence $(\mathbf{x}_n) \subseteq A$ such that $\mathbf{x}_n \to \mathbf{x}$. Since g and h are continuous,

$$g(\mathbf{x}) = \lim_{n\to\infty} g(\mathbf{x}_n) = \lim_{n\to\infty} f(\mathbf{x}_n) = \lim_{n\to\infty} h(\mathbf{x}_n) = h(\mathbf{x}).$$

Hence $g(\mathbf{x}) = h(\mathbf{x})$ for all $\mathbf{x} \in E$. Consequently, $g = h$ on E.

b) Let $\mathbf{x}_0 \in E \setminus A$ and $\varepsilon > 0$. Since f is uniformly continuous on A, $\exists \alpha > 0$ such that $\hat{d}(f(\mathbf{x}), f(\mathbf{y})) < \varepsilon$ whenever $\mathbf{x}, \mathbf{y} \in A$ and $d(\mathbf{x}, \mathbf{y}) < \alpha$.

In particular, if $\mathbf{x}, \mathbf{y} \in A$ are such that $d(\mathbf{x}, \mathbf{x}_0), d(\mathbf{y}, \mathbf{x}_0) < \frac{\alpha}{2}$, then $d(\mathbf{x}, \mathbf{y}) < \alpha$ and $\hat{d}(f(\mathbf{x}), f(\mathbf{y})) < \varepsilon$.

Since $(F, \hat{d})$ is complete, the Cauchy Criterion for Functions (we will discuss this one later) applies and we conclude that $\lim_{\mathbf{y}\to\mathbf{x}_0, \mathbf{y}\in A} f(\mathbf{y})$ exists. According to the result of part (a), the function $g : E \to F$ defined by (8.4) is continuous on E.

It remains only to show that g is uniformly continuous on E.

Let $\varepsilon > 0$. By hypothesis, f is uniformly continuous on A. As a result, $\exists \alpha > 0$ such that $\hat{d}(f(\mathbf{x}), f(\mathbf{y})) < \varepsilon$ whenever $\mathbf{x}, \mathbf{y} \in A$ and $d(\mathbf{x}, \mathbf{y}) < \alpha$.

Let $\mathbf{x}, \mathbf{y} \in E$ satisfy $d(\mathbf{x}, \mathbf{y}) < \alpha$. Since A is dense in E, two sequences $(\mathbf{x}_n), (\mathbf{y}_n) \subseteq$

A can be found such that $\mathbf{x}_n \to \mathbf{x}$ and $\mathbf{y}_n \to \mathbf{y}$. Since d is a continuous mapping, $d(\mathbf{x}_n, \mathbf{y}_n) \to d(\mathbf{x}, \mathbf{y}) < \alpha$ which shows the existence of an index $N \in \mathbb{N}$ such that $d(\mathbf{x}_n, \mathbf{y}_n) < \alpha$ for all $n > N$.

Hence, $\hat{d}(f(\mathbf{x}_n), f(\mathbf{y}_n)) < \varepsilon$ for all $n > N$. By continuity,

$$\hat{d}(f(\mathbf{x}_n), f(\mathbf{y}_n)) \to \hat{d}(g(\mathbf{x}), g(\mathbf{y})) \leq \varepsilon,$$

which shows that g is uniformly continuous on E. ■

20. Let (E, d) be a metric space. Let $\mathcal{C}$ denote the set of Cauchy sequences in E.

a) i. Let $U = (\mathbf{u}_n), V = (\mathbf{v}_n) \in \mathcal{C}$. Show that $(d(\mathbf{u}_n, \mathbf{v}_n))$ converges, and denote its limit by $\delta(U, V)$.

ii. Show that δ is symmetric and satisfies the triangle inequality.

b) Consider the equivalence relation $\sim$ on $\mathcal{C}$ defined by

$$U \sim V \iff \delta(U, V) = 0.$$

Write $\hat{E} = \mathcal{C}/\sim$ and denote the equivalence class of $U \in \mathcal{C}$ in $\hat{E}$ by $\hat{U}$.

i. What is the equivalence class of a sequence which converges in E?

ii. If $U \sim U'$ and $V \sim V'$, show that $\delta(U, V) = \delta(U', V')$. Thus, for $\hat{U}, \hat{V} \in \hat{E}$, the real number $\delta(\hat{U}, \hat{V}) = \delta(U, V)$ is well-defined, not being dependent on the choice of class representatives.

iii. Show that δ is a metric on $\hat{E}$.

iv. Let $\iota : E \to \hat{E}$ be defined by $\iota(\alpha) = \widehat{(\alpha)}$, where (α) is the constant sequence. Show that ι is an isometry (and so also $1-1$). Furthermore, show that $\iota(E)$ is dense in $\hat{E}$.

c) Show that $(\hat{E}, \delta)$ is complete.

d) Let (E_1, d_1) and (E_2, d_2) be complete metric spaces, and suppose that there are isometries $\iota_k : E \to E_k$ with $\iota_k(E)$ dense in E_k, for $k = 1, 2$. Show that there is a unique bijective isometry $\varphi : E_1 \to E_2$ such that $\varphi(\iota_1(\mathbf{x})) = \iota_2(\mathbf{x})$ for all $\mathbf{x} \in E$.

Proof:

a) i. Since $\mathbb{R}$ is complete, it will suffice to show that $(d(\mathbf{u}_n, \mathbf{v}_n))$ is a Cauchy sequence. For all $p, q \in \mathbb{N}$,

$$d(\mathbf{u}_p, \mathbf{v}_p) \leq d(\mathbf{u}_p, \mathbf{u}_q) + d(\mathbf{u}_q, \mathbf{v}_q) + d(\mathbf{v}_p, \mathbf{v}_q)$$
$$d(\mathbf{u}_q, \mathbf{v}_q) \leq d(\mathbf{u}_p, \mathbf{u}_q) + d(\mathbf{u}_p, \mathbf{v}_p) + d(\mathbf{v}_p, \mathbf{v}_q)$$

whence

$$d(\mathbf{u}_p, \mathbf{v}_p) - d(\mathbf{u}_q, \mathbf{v}_q) \leq d(\mathbf{u}_p, \mathbf{u}_q) + d(\mathbf{v}_p, \mathbf{v}_q)$$
$$d(\mathbf{u}_q, \mathbf{v}_q) - d(\mathbf{u}_p, \mathbf{v}_p) \leq d(\mathbf{u}_p, \mathbf{u}_q) + d(\mathbf{v}_p, \mathbf{v}_q)$$

and so $|d(\mathbf{u}_p, \mathbf{v}_p) - d(\mathbf{u}_q, \mathbf{v}_q)| \leq d(\mathbf{u}_p, \mathbf{u}_q) + d(\mathbf{v}_p, \mathbf{v}_q) \to 0$, since both U and V are Cauchy sequences. Consequently, $(d(\mathbf{u}_n, \mathbf{v}_n))$ is a Cauchy sequence.

ii. Symmetry is clear, since the limit of a convergent sequence is unique in a metric space and

$$\delta(V,U) \leftarrow d(\mathbf{v}_n,\mathbf{u}_n) = d(\mathbf{u}_n,\mathbf{v}_n) \to \delta(U,V).$$

The triangle inequality is also obvious since

$$\delta(U,V) \leftarrow d(\mathbf{u}_n,\mathbf{v}_n) \leq d(\mathbf{u}_n,\mathbf{w}_n) + d(\mathbf{w}_n,\mathbf{v}_n) \to \delta(U,W) + \delta(W,V)$$

implies that $\delta(U,V) \leq \delta(U,W) + \delta(W,V)$.

b) i. Let $U = (\mathbf{u}_n)$ be a convergent sequence in E which converges to $\alpha \in E$. Since any convergent sequence is a Cauchy sequence, $U \in \mathcal{C}$. Let $V = (\mathbf{v}_n) \in \mathcal{C}$. Then

$$U \sim V \iff \delta(U,V) = 0 \iff d(\mathbf{u}_n,\mathbf{v}_n) \to 0.$$

Thanks to the inequalities

$$d(\alpha,\mathbf{v}_n) \leq d(\alpha,\mathbf{u}_n) + d(\mathbf{u}_n,\mathbf{v}_n) \quad \text{and} \quad d(\mathbf{u}_n,\mathbf{v}_n) \leq d(\alpha,\mathbf{u}_n) + d(\alpha,\mathbf{v}_n),$$

we see that $U \sim V$ if and only if $d(\alpha,\mathbf{v}_n) \to 0$ (since we already have $d(\alpha,\mathbf{u}_n) \to 0$). Then, $\hat{U} = \{V = (\mathbf{v}_n) \in \mathcal{C} \mid \mathbf{v}_n \to \alpha\}$.

ii. If $U \sim U'$ and $V \sim V'$, then, according to the triangle inequality, we have

$$\delta(U,V) \leq \delta(U,U') + \delta(U',V') + \delta(V,V') = \delta(U',V').$$

Similarly, $\delta(U',V') \leq \delta(U,V)$ so that $\delta(U,V) = \delta(U',V')$.

iii. It remains only to show that $\delta(\hat{U},\hat{V}) = 0$ if and only if $\hat{U} = \hat{V}$. But that is exactly how the equivalence relation was built in the first place.

iv. For any $\alpha \in E$, let $(\alpha) \in \mathcal{C}$ be the constant sequence. Then

$$\delta(\iota(\alpha),\iota(\beta)) = \delta((\alpha),(\beta)) = d(\alpha,\beta)$$

and so ι is an isometry.

Let $\hat{U} \in \hat{E}$, with $U = (\mathbf{u}_n) \in \mathcal{C}$, and $\varepsilon > 0$. Since U is a Cauchy sequence, $\exists N \in \mathbb{N}$ such that for all $p,q > N$ we have $d(\mathbf{u}_p,\mathbf{u}_q) < \varepsilon$. Now, fix $p > N$. Then

$$\delta(\hat{U},\iota(\mathbf{u}_p)) = \delta(U,(\mathbf{u}_p)) = \lim_{n\to\infty} d(\mathbf{u}_n,\mathbf{u}_p) \leq \varepsilon.$$

Since this holds for all $p > N$, we conclude that $\iota(\mathbf{u}_n) \to \hat{U}$. Hence any element of $\hat{E}$ is the limit of a sequence of elements of $\iota(E)$, i.e. $\iota(E)$ is dense in $\hat{E}$.

c) Let (α_n) be a Cauchy sequence in $\hat{E}$. Since $\iota(E)$ is dense in $\hat{E}$, $\forall n \in \mathbb{N}$, $\exists \mathbf{x}_n \in E$ with $\delta(\alpha_n, \iota(\mathbf{x}_n)) < \frac{1}{n}$. Then

$$\begin{aligned} d(\mathbf{x}_p, \mathbf{x}_q) &= \delta(\iota(\mathbf{x}_p), \iota(\mathbf{x}_q)) \leq \delta(\iota(\mathbf{x}_p), \alpha_p) + \delta(\alpha_p, \alpha_q) + \delta(\alpha_q, \iota(\mathbf{x}_q)) \\ &\leq \delta(\alpha_p, \alpha_q) + \frac{1}{p} + \frac{1}{q} \end{aligned}$$

so that $d(\mathbf{x}_p, \mathbf{x}_q) \to 0$ as $p, q \to \infty$, which is to say that $(\mathbf{x}_n) \in \mathcal{C}$. Denote $\alpha = (\hat{\mathbf{x}}_n) \in \hat{E}$.

We will show that $\alpha_n \to \alpha$. Since

$$\delta(\alpha_n, \alpha) \leq \delta(\alpha_n, \iota(\mathbf{x}_n)) + \delta(\iota(\mathbf{x}_n), \alpha) < \frac{1}{n} + \delta(\iota(\mathbf{x}_n), \alpha),$$

it suffices to show that $\delta(\iota(\mathbf{x}_n), \alpha) \to 0$.

Let $\varepsilon > 0$. The sequence $(\mathbf{x}_n)$ being Cauchy in E, $\exists N \in \mathbb{N}$ such that $d(\mathbf{x}_p, \mathbf{x}_q) < \varepsilon$ whenever $p, q \geq N$. Thus, fixing n and letting $p \to \infty$, we have

$$\delta(\iota(\mathbf{x}_n), \alpha) = \lim_{p \to \infty} d(\mathbf{x}_n, \mathbf{x}_p) \leq \varepsilon$$

for all $n > N$, whence we have the desired result.

d) Define φ on $\iota_1(E)$ by setting $\varphi(\iota_1(\mathbf{x})) = \iota_2(\mathbf{x})$ for all $\mathbf{x} \in E$. Restricted to $\iota_1(E)$, the mapping φ is an isometry since

$$d_2(\varphi(\iota_1(\mathbf{x})), \varphi(\iota_1(\mathbf{y}))) = d_2(\iota_2(\mathbf{x}), \iota_2(\mathbf{y})) = d(\mathbf{x}, \mathbf{y}) = d_1(\iota_1(\mathbf{x}), \iota_1(\mathbf{y}))$$

for all $\mathbf{x}, \mathbf{y} \in E$. Thus, φ is uniformly continuous on $\iota_1(E)$. Since $\iota_1(E)$ is dense in E_1 and since E_2 is complete, we can apply the result of a previous problem to show that φ can be extended to a unique uniformly continuous function on E_1.

Furthermore, φ is an isometry on $\iota_1(E)$; since $\iota_1(E)$ is dense in E_1 and since φ is continuous on E_1, φ is an isometry on E_1 in its entirety. In particular φ is $1-1$.

It remains only to show that φ is onto. Let $\beta \in E_2$. As $\iota_2(E)$ is dense in E_2, $\exists(\beta_n) = (\iota_2(\mathbf{x}_n)) \subseteq \iota_2(E)$ such that $\beta_n \to \beta$. Since

$$d_1(\iota_1(\mathbf{x}_p), \iota_1(\mathbf{x}_q)) = d(\mathbf{x}_p, \mathbf{x}_q) = d_2(\iota_2(\mathbf{x}_p), \iota_2(\mathbf{x}_q)) = d_2(\beta_p, \beta_q)$$

for all $p, q \in \mathbb{N}$, the sequence $(\iota_1(\mathbf{x}_n))$ is a Cauchy sequence in E_1. But E_1 is complete so that $\iota_1(\mathbf{x}_n) \to \alpha \in E_1$. Since φ is continuous, we have

$$\varphi(\alpha) = \lim_{n \to \infty} \varphi(\iota_1(\mathbf{x}_n)) = \lim_{n \to \infty} \iota_2(\mathbf{x}_n) = \lim_{n \to \infty} \beta_n = \beta,$$

that is, φ is onto. ■

21. Let $A, B \subseteq E$, where E is endowed with any metric you care to imagine. Show that

    a) $A \subseteq \overline{A}$
    b) $\overline{(\overline{A})} = \overline{A}$
    c) $\overline{A \cup B} = \overline{A} \cup \overline{B}$
    d) $\overline{\varnothing} = \varnothing$
    e) in general, $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$

    Proof:

    a) This one is clear by definition.
    b) By part (a), $\overline{A} \subseteq \overline{(\overline{A})}$. Conversely, since $\overline{(\overline{A})}$ is the smallest closed set containing $\overline{A}$ and since $\overline{A}$ is also a closed set containing $\overline{A}$, then $\overline{(\overline{A})} \subseteq \overline{A}$. Hence, $\overline{A} = \overline{(\overline{A})}$.
    c) Since the union of two closed sets is closed, $\overline{A} \cup \overline{B}$ is a closed set containing $A \cup B$ and so $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$. Conversely, $\overline{A \cup B}$ is a closed set containing both A and B, so both $\overline{A}, \overline{B} \subseteq \overline{A \cup B}$; therefore $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$. Thus $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
    d) Since $\varnothing$ is always a closed set, $\overline{\varnothing} = \varnothing$.
    e) Consider the following example in $(\mathbb{R}, d_2)$: let $A = (-1, 0)$ and $B = (0, 1)$. Then $\overline{A} = [-1, 0]$, $\overline{B} = [0, 1]$, $A \cap B = \varnothing$, $\overline{A \cap B} = \varnothing$ while $\overline{A} \cap \overline{B} = \{0\}$. ■

22. Let A be a subset of (E, d). Show that $\overline{A} = \text{int}(A) \cup \partial A$.

    Proof: suppose that $\mathbf{x} \in \text{int}(A)$. Then $\mathbf{x} \in A \subseteq \overline{A}$. Now suppose that $\mathbf{x} \in \partial A$. We proceed by contradiction. If $\mathbf{x} \notin \overline{A}$ then, since $E \setminus \overline{A} \subseteq_O E$, $\exists r > 0$ such that $B(\mathbf{x}, r) \subseteq E \setminus \overline{A} \subseteq E \setminus A$. This contradicts the fact that $\mathbf{x} \in \partial A$ (how?) and so we must have $\mathbf{x} \in \overline{A}$. Thus $\text{int}(A) \cup \partial A \subseteq \overline{A}$.

    Conversely, suppose that $\mathbf{x} \in \overline{A}$. There are only three possibilities: $\mathbf{x} \in \text{int}(A)$, $\mathbf{x} \in \partial A$ or $\mathbf{x} \in \text{int}(E \setminus A)$ (why?). If $\mathbf{x} \in \text{int}(E \setminus A)$, then $\exists r > 0$ such that $B(\mathbf{x}, r) \subseteq E \setminus A$. This implies that $A \subseteq E \setminus B(\mathbf{x}, r)$. Therefore $\overline{A} \subseteq E \setminus B(\mathbf{x}, r)$, since $E \setminus B(\mathbf{x}, r) \subseteq_C E$, which in turns implies that $\mathbf{x} \notin \overline{A}$, a contradiction.

    Thus $\mathbf{x} \in \text{int}(A) \cup \partial A$ and so $\overline{A} \subseteq \text{int}(A) \cup \partial A$. ■

23. Let $A = \{\frac{1}{n} \mid n \in \mathbb{N}^{\times}\}$. Under the usual topology on $\mathbb{R}$, show that every point of A is a boundary point and that the only cluster point of A is 0.

    Proof: To show that every point of $x \in A$ is a boundary point, note that any neighbourhood V of x contains an open interval $I_r = (x - r, x + r)$, for some $r > 0$. But $x \in I_r \cap A$ and since any open interval contains an irrational number $I_r \cap (\mathbb{R} \setminus A) \neq \varnothing$. Consequently, any neighbourhood of x contains both points in A and points not in A, which is another definition of $x \in \partial A$.

    To show that 0 is a cluster point of A, note that any neighbourhood of 0 in $(\mathbb{R}, d_2)$ contains an interval of the form $(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$. By the Archimedean property, $\exists N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$. Hence $0 \neq \frac{1}{N} \in B(0, \varepsilon)$ and so 0 is a cluster point of A.

In order to show that there are no other cluster points, first observe that any $x < 0$ cannot be a cluster point of A since the neighbourhood $(-2x, 0)$ contains no points in A. Likewise, any $x > 1$ cannot be a cluster point of A since the neighbourhood $(1, 2x)$ contains no point of A.

If $x \in (0, 1]$, then either $x \in A$ or $x \notin A$. If $x = \frac{1}{n} \in A$, then the open neighbourhood $(x-r, x+r)$ contains no other point of A as long as $r < \frac{1}{n(n-1)}$, and so x is not a cluster point of A. If $x \notin A$, choose $k \in \mathbb{N}$ such that $x \in (\frac{1}{k}, \frac{1}{k-1})$. Then the open neighbourhood $(x-r, x+r)$ contains no other point of A if $r < \min\{x - \frac{1}{k}, \frac{1}{k-1} - x\}$ and so x cannot be a cluster point of A. ■

24. Let $\tau_1 = \{U \subseteq \mathbb{R} \mid \mathbb{R} \setminus U \text{ is finite or } U = \varnothing\}$, $\tau_2 = \{U \subseteq \mathbb{R} \mid \mathbb{R} \setminus U \text{ is countable or } U = \varnothing\}$.
    a) Show that τ_1 and τ_2 define topologies on $\mathbb{R}$ (the **co-finite topology** and **countable complement** topology, respectively).
    b) What is the boundary of the set $A = \{\frac{1}{n} \mid n \in \mathbb{N}^\times\}$ under these two topologies?

Proof:

a) It suffices to verify that the three properties hold for τ_1:
   i. $\varnothing \in \tau_1$ by definition; $\mathbb{R} \in \tau_1$ since $\mathbb{R} \setminus \mathbb{R} = \varnothing$ is finite.
   ii. Let $\{X_\alpha\} \subseteq \tau_1$. Then $\mathbb{R} \setminus X_\alpha$ is finite for all α. According to the de Morgan's Laws, the set
   $$\mathbb{R} \setminus \bigcup_\alpha X_\alpha = \bigcap_\alpha (\mathbb{R} \setminus X_\alpha)$$
   is a finite set as it is the intersection of an arbitrary collection of finite sets. Hence, $\bigcup X_\alpha \in \tau_1$.
   iii. Let $\{X_i\}_{i=1}^n \subseteq \tau_1$. Then $\mathbb{R} \setminus X_i$ is finite for all $i = 1, \dots, n$.

   According to the de Morgan's Laws, the set
   $$\mathbb{R} \setminus \bigcap_{i=1}^n X_i = \bigcup_{i=1}^n (\mathbb{R} \setminus X_i)$$
   is a finite set as it is the union of a finite collection of finite sets. Hence, $\bigcap_{i=1}^n X_i \in \tau_1$.

   Now for τ_2:
   i. $\varnothing \in \tau_2$ by definition; $\mathbb{R} \in \tau_2$ since $\mathbb{R} \setminus \mathbb{R} = \varnothing$ is countable.
   ii. Let $\{X_\alpha\} \subseteq \tau_2$. Then $\mathbb{R} \setminus X_\alpha$ is countable for all α.

   According to the de Morgan's Laws, the set
   $$\mathbb{R} \setminus \bigcup_\alpha X_\alpha = \bigcap_\alpha (\mathbb{R} \setminus X_\alpha)$$
   is a countable set as it is the intersection of an arbitrary collection of countable sets. Hence, $\bigcup X_\alpha \in \tau_2$.

iii. Let $\{X_i\}_{i=1}^n \subseteq \tau_2$. Then $\mathbb{R} \setminus X_i$ is countable for all $i = 1, \ldots, n$. According to the de Morgan's Laws, the set

$$\mathbb{R} \setminus \bigcap_{i=1}^{n} X_i = \bigcup_{i=1}^{n} (\mathbb{R} \setminus X_i)$$

is a countable set as it is the union of a finite collection of countable sets. Hence, $\bigcap_{i=1}^n X_i \in \tau_2$.

b) In the countable complement topology, $A \subseteq_C \mathbb{R}$, because $\mathbb{R} \setminus (\mathbb{R} \setminus A) = A$ is countable and so $\mathbb{R} \setminus A \subseteq_O \mathbb{R}$. Consequently, $\overline{A} = A$. Furthermore, the only open set of $\mathbb{R}$ contained in A is the empty set, as any other open set is uncountable. Hence int$(A) = \varnothing$ and $\partial A = \overline{A} \setminus \text{int}(A) = A$.

In the co-finite topology, the only closed set containing A is $\mathbb{R}$, as any other closed set is finite. Consequently, $\overline{A} = \mathbb{R}$. Furthermore, the only open set of $\mathbb{R}$ contained in A is the empty set, as any other open set is infinite. Hence int$(A) = \varnothing$ and $\partial A = \mathbb{R}$. ■

25. Let $A, B \subseteq (E, d)$. If $\mathbf{x} \in E$ is a cluster point of $A \cap B$, show that $\mathbf{x}$ is a cluster point of both A and B.

Proof: let $\mathbf{x}$ be a cluster point of $A \cap B$. Then any neighbourhood V of $\mathbf{x}$ contains a point $\mathbf{y} \in A \cap B \subseteq A$ such that $\mathbf{y} \neq \mathbf{x}$. Thus $\mathbf{y}$ is a cluster point of A. The argument for B is identical. ■

26. Show that $B \subseteq (\mathbb{R}^p, d_2)$ is closed if and only if every convergent sequence in B converges to a point in B.

Proof: first, assume that B is closed. Let $\mathbf{x} = \lim \mathbf{x}_n$. Then, for any $\varepsilon > 0$, $\exists n_\varepsilon > 0$ such that $\mathbf{x}_n \in B(\mathbf{x}, \varepsilon)$ for all $n \geq n_\varepsilon$. Consequently, $B \cap B(\mathbf{x}, \varepsilon) \neq \varnothing$ for all $\varepsilon > 0$. Since $\mathbb{R}^p \setminus B \subseteq_O \mathbb{R}^p$, it follows that $\mathbf{x} \in B$ (why?).

Conversely, assume that for every convergent sequence $(\mathbf{x}_k) \subseteq \mathbb{R}^p$, we have $\mathbf{x} = \lim \mathbf{x}_k \in B$. If $\mathbb{R}^p \setminus B$ is not open in $\mathbb{R}^p$, $\exists \mathbf{x} \in \mathbb{R}^p \setminus B$ such that $B(\mathbf{x}, \frac{1}{n}) \cap B \neq \varnothing$ for all $n \in \mathbb{N}$. Then $\exists \mathbf{x}_n \in B(\mathbf{x}, \frac{1}{n}) \cap B$; the sequence $(\mathbf{x}_n) \subseteq B$ converges to $\mathbf{x} \notin B$, which contradicts the hypothesis. Hence $\mathbb{R}^p \setminus B \subseteq_O \mathbb{R}^p$. ■

27. Let $(\mathbf{x}_n) \subseteq (\mathbb{R}^p, \|\cdot\|)$ such that

$$\|\mathbf{x}_{n+1} - \mathbf{x}_n\| \leq r \|\mathbf{x}_n - \mathbf{x}_{n-1}\|$$

where $r < 1$. Show that $(\mathbf{x}_n)$ converges.

Proof: we have $\|\mathbf{x}_3 - \mathbf{x}_2\| \leq r\|\mathbf{x}_2 - \mathbf{x}_1\|$ and it is easily seen by induction that if

$$\|\mathbf{x}_{n+1} - \mathbf{x}_n\| \leq r^{n-1} \|\mathbf{x}_2 - \mathbf{x}_1\|$$

then

$$\|\mathbf{x}_{n+2} - \mathbf{x}_{n+1}\| \leq r\|\mathbf{x}_{n+1} - \mathbf{x}_n\| \leq r^n \|\mathbf{x}_2 - \mathbf{x}_1\|.$$

Therefore, if $m > n$,

$$\|\mathbf{x}_m - \mathbf{x}_n\| = \left\|\sum_{k=n}^{m-1}(\mathbf{x}_{k+1} - \mathbf{x}_k)\right\| \leq \sum_{k=n}^{m-1}\|\mathbf{x}_{k+1} - \mathbf{x}_k\|$$
$$\leq \sum_{k=n}^{\infty}\|\mathbf{x}_{k+1} - \mathbf{x}_k\| \leq \sum_{k=n}^{\infty} r^{k-1}\|\mathbf{x}_2 - \mathbf{x}_1\| \leq \frac{r^{n-1}}{1-r}\|\mathbf{x}_2 - \mathbf{x}_1\|.$$

Let $\varepsilon > 0$. Since $r < 1$, $\exists N_\varepsilon$ so that

$$r^{n-1} < \varepsilon\frac{1-r}{\|\mathbf{x}_2 - \mathbf{x}_1\|} \quad \text{for all } n \geq N,$$

and so $\|\mathbf{x}_m - \mathbf{x}_n\| < \varepsilon$ for all $m \geq n \geq N_\varepsilon$. It follows that $(\mathbf{x}_n)$ is Cauchy and that it is convergent, since $(\mathbb{R}^p, \|\cdot\|)$ is a Banach space. ■

8.4 Exercises

1. Prepare a 2-page summary of this chapter, with important definitions and results.
2. Show that the absolute value defines a norm on $\mathbb{R}$.
3. Show that the modulus defines a norm on $\mathbb{C}$.
4. Show that the sup norm $\|\cdot\|_\infty$ is indeed a norm on $\mathcal{C}_\mathbb{R}([0,1])$.
5. Let $\infty \geq p \geq 1$. Show that the $p-$norm $\|\cdot\|_\infty$ is indeed a norm on $\mathbb{R}^n$.
6. Let $p \geq 1$. Show that (8.1, p. 189), defines a norm on $\mathcal{L}^p([0,1])$.
7. Prove Lemma 8.1.1, p. 189.
8. Let E be any set. Show that (8.2, p. 190) defines a metric on E.
9. Let $E = \mathbb{R}^n$. Show that d_2 is a metric on E.
10. Let $E = \mathbb{R}$, $d(x,y) = |x-y|$, $A = \mathbb{N}$ and $B = \{\frac{n-1}{n} \mid n \in \mathbb{N}\}$. Compute $d(A,B)$, where d is as in (8.3, p. 191)). Can you use this result to show that (8.3, p. 191) does not define a metric on $\wp(E) \setminus \varnothing$?
11. In a metric space, show that $\delta(A) \in [0,\infty]$. Also, show that $\delta(A) = 0 \iff A$ is a singleton.
12. Prove or disprove: In any metric space (E,d), $\delta_d(B(\mathbf{a},r)) = 2r$.
13. Prove or disprove: Let d, d' be metrics on E. Then, A is bounded in (E,d) if and only if A is bounded in (E,d').
14. Where does the proof that a finite intersection of open subsets is open fail for arbitrary intersections?

15. Show that the metric space topology on a discrete metric space is the discrete topology.

16. Show that the intersection of an arbitrary family $\{A_i\}_{i\in I}$ of closed subsets of E is a closed subset of E.

17. Show that the union of a finite family $\{A_i\}_{i=1}^{\ell}$ of closed subsets of E is a closed subset of E.

18. Show that the union of an arbitrary family of closed subsets of E need not be closed in E.

19. Let A be a subset of a metric space (E, d). Show that $\overline{A}$ is the intersection of all closed subsets of E containing A.

20. Let A be a subset of a metric space (E, d). Show that $A \subseteq \overline{A}$.

21. Prove Lemma 92, p. 197.

22. In Proposition 94, p. 198, show that $2. \iff 3 \iff 4.$

23. Let A be a subset of a metric space (E, d). Show that int(A) is the union of all open subsets of E contained in A.

24. Let A be a subset of a metric space (E, d). Show that int$(A) \subseteq A$.

25. Let A be a subset of a metric space (E, d). Show that $A \subseteq_O E \iff A = \text{int}(A)$.

26. Complete the proof of Lemma 98, p. 202.

27. Prove Proposition 99, p. 202.

28. Show that the three definitions of continuity are equivalent.

29. Let $f : C \to D$, $A \subseteq C$ and $B \subseteq D$. Show that $f^{-1}(f(A)) = A$ and that in general, the best we can say is that $f(f^{-1}(B)) \subseteq B$.

30. Can you find a function $f : E \to \tilde{E}$ which is continuous but not closed?

31. Can you find a function $f : E \to \tilde{E}$ which is open and closed but not continuous?

32. Can you find a function $f : E \to \tilde{E}$ which is open and continuous but not closed?

33. Complete the proof of Proposition 101, p. 204.

34. Complete the proof of Corollary 102, p. 204.

35. Provide the details showing that d_2 and d_∞ are topologically equivalent on $\mathbb{R}^2$.

36. Consider the metric space $(\mathbb{R}, d_2)$. Define a new function $\tilde{d} : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \to \mathbb{R}$ by

$$\tilde{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Show that $\tilde{d}$ defines a metric on $\mathbb{R}$, that d and $\tilde{d}$ are topologically equivalent but that they are not equivalent.

37. Let (E, d) be a metric space. Show that $d : E \times E \to \mathbb{R}$ is Lipschitz continuous (with $k = 2$) and so that it is a continuous map.

38. Find a function which is uniformly continuous but not Lipschitz continuous.

39. Show that the two definitions of convergence of a sequence are equivalent.

40. Show that if $\mathbf{x}_n \to \mathbf{x}$, then any subsequence of $(\mathbf{x}_n)$ also converges to $\mathbf{x}$.

41. Show that the set of limit points of a sequence is closed.

42. Complete the proof of Proposition 103, p. 209.

43. Prove Proposition 8.2.2, p. 214.

44. Show that the space $\ell^2(\mathbb{N})$ is a Hilbert space as follows.

    a) Show that $\ell^2(\mathbb{N})$ is a vector space over $\mathbb{C}$.
    b) Show that $(\cdot|\cdot)$ defined in the text is indeed an inner product over $\ell^2(\mathbb{N})$.
    c) Show that $(\cdot|\cdot)$ defines a norm $\|\cdot\|$ over $\ell^2(\mathbb{N})$.
    d) Show that $\ell^2(\mathbb{N})$ is complete under $\|\cdot\|$.

45. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1, & x < 0 \\ x + \frac{1}{x+1}, & x \geq 0 \end{cases}.$$

Show that f has no fixed point but that $d(f(x), f(y)) \leq d(x, y)$ for all $x, y \in \mathbb{R}$.

46. Let X be a compact metric space. Define

$$C_{\mathbb{R}}(X) = \{f | f : X \to \mathbb{R}, f \text{ continuous}\}.$$

Show that $(C_{\mathbb{R}}(X), \|\cdot\|_\infty)$ is a Banach space, but that neither $(C_{\mathbb{R}}(X), \|\cdot\|_1)$ nor $(C_{\mathbb{R}}(X), \|\cdot\|_2)$ is complete.

47. Let $E = \{f \in C_B(\mathbb{R}, \mathbb{R}) | f \text{ uniformly continuous}\}$. Show that E is a complete subalgebra of $C_B(\mathbb{R}, \mathbb{R})$.

48. Let (E,d) be a complete metric space and $f : E \to E$. If there exists a positive integer r and $k \in (0,1)$ such that
$$f^r = \underbrace{f \circ f \circ \cdots \circ f}_{r \text{ times}}$$
and $d(f^r(x), f^r(y)) \leq k d(x,y)$ for all $x, y \in E$, show that f has a unique fixed point.

49. Let $X = (0,\infty)$. Consider the function $\tilde{d} : X \times X \to \mathbb{R}_0^+$ defined by
$$\tilde{d}(x,y) = \left| \frac{1}{x} - \frac{1}{y} \right|.$$
   a) Prepare a 2-page summary of this chapter; identify the important definitions and results.
   b) Show that $\tilde{d}$ is a metric on X.
   c) Show that $\tilde{d}$ and d_2 induce the same topology on X (*i.e.* the open sets of X are exactly the same under both metrics).
   d) Show that $(X, \tilde{d})$ is not a complete metric space.
   e) Show that $((0,1], \tilde{d})$ is a complete metric space.

50. Let $\mathcal{B}(X,\mathbb{R})$ denote the set of bounded functions from X to $\mathbb{R}$. It is easy to see that $\mathcal{B}(X,\mathbb{R})$ is a vector space over $\mathbb{R}$. The norm of $f \in \mathcal{B}(X,\mathbb{R})$ is defined by
$$\|f\| = \sup_{x \in X} |f(x)|.$$
Show that $\mathcal{B}(X,\mathbb{R})$ is a Banach space with this norm.

51. Are the co-finite topologies and the countable complement topologies derived from a metric?