Chapter 9

Metric Spaces and Topology

> One of the natural ways we can extend the concepts we have discussed in the previous chapters is by moving from $\mathbb{R}$ to $\mathbb{R}^m$. Some of the notions that generalize nicely to vectors and functions on vectors include **compactness** and **connectedness**.

The symbol $\mathbb{K}$ is sometimes used to denote either $\mathbb{R}$ or $\mathbb{C}$.

9.1 Compact Spaces

Let A be a finite set. A function $f : A \to \mathbb{K}$ is necessarily **bounded** (in the sense that $\exists M \in \mathbb{K}$ such that $|f(a)| \leq M$ for all $a \in A$).

Might this be due to the **finiteness** of A? While finiteness is sufficient, it is not a necessary condition for boundedness: the Dirichlet function $\chi_{\mathbb{Q}} : [0, 1] \to \mathbb{R}$ is bounded, even though its domain is the **uncountable** set $[0, 1]$.

Perhaps it is the **boundedness of the function's domain** that does the trick? Unfortunately, that condition is neither sufficient nor necessary, as can be seen from the functions

$$f : [0, 1] \to \mathbb{R}, \quad f(x) = \frac{1}{x} \quad \text{for } x > 0, \quad \text{and} \quad f(0) = 0,$$

and $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = \exp(-x^2)$.

Could the culprit instead be the **continuous nature** of the function? Not as such, no, as we have examples of continuous functions being bounded, others being unbounded; and non-continuous functions being bounded, others being unbounded.

A condition on the domain of the function alone cannot guarantee boundedness; and neither can one on the nature of the function. However, a **combination** of two conditions, one each on the domain and on the function, can provide such a guarantee.

In this section, we study the appropriate property on the domain, that of **compactness**, which generalizes the property of finiteness. Its definition, which in all honesty is not super intuitive, is due to Borel and Lebesgue, is applicable to metric and general topological spaces alike.

9.1.1 The Borel-Lebesgue Property

A space E is **compact** if any family of open subsets covering E contains a finite sub-family which also covers E. In other words, E is compact if, for any collection $\mathcal{U} = \{U_i\}_{i\in I}$ of open subsets $U_i \subseteq_O E$ with $E \subseteq \bigcup_{i\in I} U_i$, $\exists$ a finite $J \subseteq I$ such that $E \subseteq \bigcup_{j\in J} U_j$.

Examples

1. Every finite metric space (E, d) is compact.

   Proof: let $\mathcal{U}$ be an open cover of $E = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$. Thus, for each $1 \leq i \leq n$, $\exists U_i \in \mathcal{U}$ such that $\mathbf{x}_i \in U_i$. Then $\{U_1, \ldots, U_n\}$ is a finite subcover of E. ■

2. In the standard topology, $\mathbb{R}$ is not compact.

   Proof: consider the open cover $\mathbb{R} = \bigcup_{n\in\mathbb{N}} (-n, n)$.

   Any finite subcollection $\{(-n_1, n_1), \ldots, (-n_m, n_m)\}$ is bounded by $M = \max\{n_j \mid 1 \leq j \leq m\}$, and thus cannot be a cover of $\mathbb{R}$ according to the Archimedean Property. Consequently, no such finite subcover exists and $\mathbb{R}$ is not compact. ■

3. Show that $\mathbb{R}$ is compact in the indiscrete topology.

   Proof: the only open cover of $\mathbb{R}$ in the indiscrete topology is $\{\mathbb{R}\}$, which is already a finite sub-cover of $\mathbb{R}$ (the only other open subset of $\mathbb{R}$ in the indiscrete topology is $\varnothing$). ■

4. Show that any compact metric (E, d) space is bounded.

   Proof: consider the open cover $\mathcal{U} = \{B(\mathbf{x}, 1) \mid \mathbf{x} \in E\}$. Since E is compact, $\exists \mathbf{x}_1, \ldots, \mathbf{x}_n \in E$ such that $E = B(\mathbf{x}_1, 1) \cup \cdots B(\mathbf{x}_n, 1)$. Consequently, E has a finite diameter $\leq n$ and is thus bounded. ■

By abuse of notation, we often write: “let $\bigcup U_i$ be an open cover of E” rather than “let $\{U_i\}$ be an open cover of E,” as in the second example above.

Incidentally, does the fourth example contradict the third one? It doesn’t actually, but what does that imply about the indiscrete topology?

The duality open $\leftrightsquigarrow$ closed/ union $\leftrightsquigarrow$ intersection yields an equivalent definition: a space E is **compact** if any family of closed subsets of E with an empty intersection contains a finite sub-family whose intersection is also empty.

In other words, E is compact if, for any collection $\mathcal{W} = \{V_i\}_{i\in I}$ of closed subsets $V_i \subseteq_C E$ with $\bigcap_{i\in I} V_i = \varnothing$, $\exists$ a finite $J \subseteq I$ such that $\bigcap_{j\in J} V_j = \varnothing$.

Proposition 115
Let $(F_n)_{n\geq 1}$ be a decreasing sequence of non-empty closed subsets of a compact space E. Then $\bigcap_{n\geq 1} F_n \neq \varnothing$.

Proof: if $\bigcap_{n\geq 1} F_n = \varnothing$, then $E = \bigcup_{n\geq 1} E \setminus F_n$, where $E \setminus F \subseteq_O E$. Since E is compact, $\exists$ a finite subsequence of indices $n_1 < \cdots < n_k$ such that

$$E = \bigcup_{i=1}^{k} E \setminus F_{n_i}.$$

Consequently, $\bigcap_{i=1}^k F_{n_i} = \varnothing$. But the original sequence is decreasing, so that

$$\bigcap_{i=1}^{k} F_{n_i} = F_{n_k} = \varnothing,$$

which contradicts the hypothesis that all F_n are non-empty. As a result, we conclude that $\bigcap_{n\geq 1} F_n \neq \varnothing$. ■

Continuous functions on compact domains have quite useful properties.

Proposition 116
Let $f : (E, d) \to (F, \delta)$ be any continuous function over a compact metric space. Then f is uniformly continuous.

Proof: let $\mathbf{x} \in E$. Since f is continuous at $\mathbf{x} \in E$, $\forall \varepsilon > 0$, $\exists M_{\mathbf{x}}(\varepsilon) > 0$ such that

$$f(B(\mathbf{x}, M_{\mathbf{x}})) \subseteq B(f(\mathbf{x}), \varepsilon).$$

Furthermore, $E = \bigcup_{\mathbf{x}\in E} B(\mathbf{x}, M_{\mathbf{x}})$ is an open cover of E, which is compact. Consequently, $\exists \mathbf{x}_1, \ldots, \mathbf{x}_n \in E$ such that $E = \bigcup_{i=1}^n B(\mathbf{x}_i, M_{\mathbf{x}_i})$. Set

$$M = M(\varepsilon) = \tfrac{1}{2} \cdot \min\{M_{\mathbf{x}_1}, \ldots, M_{\mathbf{x}_n}\} > 0.$$

Then, $\forall_{\varepsilon>0}$, $\exists M(\varepsilon) > 0$ such that $f(B(\mathbf{x}, M)) \subseteq B(f(\mathbf{x}), \varepsilon)$ for all $\mathbf{x} \in E$. As M does not depend on $\mathbf{x}$, f is uniformly continuous. ■

A subset $A \subseteq E$ is deemed to be a **compact subset of** E, which we denote by $A \subseteq_K E$, if any family of open subsets of E covering A contains a finite sub-family which also covers A.

Proposition 117
A finite union of compact subsets of E is itself compact.

Proof: let $A_1, \ldots, A_n \subseteq_K E$ and write $A = \bigcup_{k=1}^n A_k$. Let $\{U_i\}_{i \in I} \subseteq \wp(E)$ be an open cover of A. Then $\{U_i\}_{i \in I}$ is also an open cover of A_k for each k.

Since all A_k are compact, $\exists$ finite $J_1, \ldots, J_k \subseteq I$ such that $A_k \subseteq \bigcup_{j \in J_k} U_j$ for each k. Thus, $A \subseteq \bigcup_{k=1}^n \bigcup_{j \in J_k} U_j$. But $\bigcup_{k=1}^n \{U_j\}_{j \in J_k}$ is a finite sub-family of $\{U_i\}_{i \in I}$, from which we conclude that $A \subseteq_K E$. ■

The infinite union of compact subsets could be compact or not, however.

Examples

1. Both $[0,1], [2,3] \subseteq_K (\mathbb{R}, d_1)$, so $[0,1] \cup [2,3] \subseteq_K (\mathbb{R}, d_1)$. □
2. For any $x \geq 1$, $[0, \frac{1}{x}] \subseteq_K (\mathbb{R}, d_1)$. The union $\bigcup_{x \geq 1} [0, \frac{1}{x}] = [0,1]$ is also a compact subset of $(\mathbb{R}, d_1)$. □
3. For any $n \in \mathbb{N}$, $[-n, n] \subseteq_K (\mathbb{R}, d_1)$, but the union $\bigcup_{n \in \mathbb{N}} [-n, n] = \mathbb{R}$ is not a compact subset of $(\mathbb{R}, d_1)$. □

9.1.2 The Bolzano-Weierstrass Property

For metric spaces, compactness can also be established via a property of **sequences** which is often easier to ascertain than the Borel-Lebesgue property, but it comes with a warning: **the two properties are not equivalent in general for non-metric spaces**.

Let (E, d) be a metric space. We say that E is **precompact** if $\forall \varepsilon > 0$, $\exists \mathbf{x}_1, \ldots, \mathbf{x}_n \in E$ such that $E = \bigcup_{i=1}^n B(\mathbf{x}_i, \varepsilon)$.

Proposition 118
A compact space is precompact.

Proof: left as an exercise. ■

We now present the section's main result, a "special case" of which we saw in Theorem 20.

Theorem 119 (BOLZANO-WEIERSTRASS COMPACTNESS)
Let (E, d) be a metric space. Then E is compact if and only if any sequence in E has a convergent sub-sequence in E.

Proof: assume E is compact and let $(\mathbf{x}_n) \subseteq E$. If the range of $(\mathbf{x}_n)$ is finite, there is a constant subsequence which would then automatically be convergent. We thus consider sequences with infinite range $A = \{\mathbf{x}_n \mid n \in \mathbb{N}\}$.

We show that such an A has at least one cluster point. Suppose, instead, that there A has no cluster point. Thus for any $\mathbf{x} \in E$, $\exists r_{\mathbf{x}} > 0$ with $B(\mathbf{x}, r_{\mathbf{x}}) \cap A$ is finite. Since E is compact, there exists a finite $J \subseteq E$ such that $E = \bigcup_{\mathbf{x} \in J} B(\mathbf{x}, r_{\mathbf{x}})$.

Then

$$A = \bigcup_{\mathbf{x} \in J} (B(\mathbf{x}, r_{\mathbf{x}}) \cap A)$$

is a finite union of finite sets, hence A is itself finite.

But this contradicts the fact that A is infinite. Hence, A has at least one cluster point $\mathbf{x} \in E$. Such a cluster point is a limit point of $(\mathbf{x}_n)$: consequently, there is a subsequence of $(\mathbf{x}_n)$ which converges to $\mathbf{x} \in E$ (in which case we say that E satisfies the **Bolzano-Weierstrass property**).

Conversely, assume all sequences in E have convergent subsequence in E. First, note that any metric space (E, d) satisfying the Bolzano-Weierstrass property is precompact. Indeed, suppose that $\exists \varepsilon > 0$ such that E can not be covered with a finite number of $\varepsilon-$balls. Let $\mathbf{x}_0 \in E$. By assumption, $B(\mathbf{x}_0, \varepsilon) \neq E$. Thus $\exists \mathbf{x}_1 \in E$ such that $d(\mathbf{x}_0, \mathbf{x}_1) \geq \varepsilon$.

Since $B(\mathbf{x}_0, \varepsilon) \cup B(\mathbf{x}_1, \varepsilon) \neq E$, $\exists \mathbf{x}_2 \in E$ such that $d(\mathbf{x}_0, \mathbf{x}_1), d(\mathbf{x}_0, \mathbf{x}_2) \geq \varepsilon$. Continuing this process, we build a list $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n$ for which $d(\mathbf{x}_i, \mathbf{x}_j) \geq \varepsilon$ for all $i < j \leq n$.

Since $\bigcup_{i=0}^n B(\mathbf{x}_i, \varepsilon) \neq E$, $\exists \mathbf{x}_{n+1} \in E$ such that $d(\mathbf{x}_i, \mathbf{x}_{n+1}) \geq \varepsilon$ for all $0 \leq i \leq n$. By induction, there is a sequence $(\mathbf{x}_n) \subseteq E$ such that $d(\mathbf{x}_i, \mathbf{x}_j) \geq \varepsilon$ whenever $i \neq j$. Consequently, this sequence has no convergent subsequence, since no subsequence is a Cauchy sequence. This contradicts the hypothesis that E satisfies the Bolzano-Weierstrass property, thus E is precompact.

Next, we show that if the metric space (E, d) satisfies the Bolzano-Weierstrass property and if $\{U_i\}_{i \in I}$ is an open cover of E, then

$$\exists \alpha > 0, \forall \mathbf{x} \in E, \exists i \in I \implies B(\mathbf{x}, \alpha) \subseteq U_i. \tag{9.1}$$

Indeed, suppose that

$$\forall \alpha > 0, \exists \mathbf{x} \in E, \forall i \in I \implies B(\mathbf{x}, \alpha) \not\subseteq U_i. \tag{9.2}$$

In particular,

$$\forall n \in \mathbb{N}^{\times}, \exists \mathbf{x}_n \in E, \forall i \in I \implies B(\mathbf{x}, \tfrac{1}{n}) \not\subseteq U_i.$$

Let $(\mathbf{x}_{\varphi(n)})$ be a convergent subsequence of $(\mathbf{x}_n)$ (such a sequence exists since E satisfies the Bolzano-Weierstrass property).

Write $\mathbf{x}_{\varphi(n)} \to \mathbf{x}$. Since $\{U_i\}_{i\in I}$ covers E, $\exists i \in I$ such that $\mathbf{x} \in U_i$. But $U_i \subseteq_O E$, so $\exists r > 0$ such that $B(\mathbf{x}, 2r) \subseteq U_i$.

Accordingly, $\exists N \in \mathbb{N}$ such that $d(\mathbf{x}_{\varphi(n)}, \mathbf{x}) < r$ and $\varphi(n) > \frac{1}{r}$ for all $n > N$. Consequently, $\forall n > N$ and $\forall \mathbf{y} \in B(\mathbf{x}_{\varphi(n)}, \frac{1}{\varphi(n)})$, we have

$$d(\mathbf{x}, \mathbf{y}) \leq d(\mathbf{x}, \mathbf{x}_{\varphi(n)}) + d(\mathbf{x}_{\varphi(n)}, \mathbf{y}) < r + r = 2r.$$

Thus $\forall n > N$, $B(\mathbf{x}_{\varphi(n)}, \frac{1}{\varphi(n)}) \subseteq U_i$, which contradicts (9.2), and so (9.1) holds.

To show E is compact, let $\{U_i\}_{i\in I}$ be an open cover of E. We know from (9.1) that

$$\exists \alpha > 0, \forall \mathbf{x} \in E, \exists i \in I \implies B(\mathbf{x}, \alpha) \subseteq U_i.$$

But E is precompact, so $\exists \mathbf{x}_1, \ldots, \mathbf{x}_n \in E$ such that $E = \bigcup_{j=1}^{n} B(\mathbf{x}_j, \alpha)$.

Let $i_1, \ldots, i_n$ be the indices for which $B(\mathbf{x}_j, \alpha) \subseteq U_{i_j}$, $1 \leq j \leq n$. Then $E = \bigcup_{j=1}^{n} U_{i_j}$ is a finite subcover of E; E is indeed compact. ■

The following result has a similar flavour.

Theorem 120
Let (E, d) be a metric space. Then E is compact if and only if any sequence in E has a limit point if and only if every infinite subset of E has a cluster point.

Proof: left as an exercise. ■

It is usually easier to show that the Bolzano-Weierstrass is violated than to show that it holds.

Example: Show that the set $(0, 1)$ is not a compact subset of $(\mathbb{R}, d_1)$.

Proof: Consider the sequence $(1/n) \subseteq (0, 1)$. Every subsequence of $(1/n)$ converges to $0 \notin (0, 1)$. According to Theorem 119, $(0, 1)$ is not a compact subset of $(\mathbb{R}, d_1)$. ■

Compact sets really have quite useful properties.

Proposition 121
Let (E, d) be a metric space.

1. *If E is compact and $A \subseteq_C E$, then $A \subseteq_K E$.*
2. *If $A \subseteq_K E$, then $A \subseteq_C E$ and A is bounded.*

Proof:

1. Since E is compact, it is precompact (see the proof of Theorem 119) and so is A. The set E is also complete (see exercise 2). Thus A is a closed subset of the complete set E: A is then complete (see Proposition 110). But A is precompact and complete, and so $A \subseteq_K E$ (see exercise 3).
2. Since $A \subseteq_K E$, it is precompact. Hence for $\varepsilon > 0$, $\exists \mathbf{x}_1, \ldots, \mathbf{x}_n \in A$ such that
$$A \subseteq \bigcup_{j=1}^{n} B(\mathbf{x}_j, \varepsilon).$$
Thus, $\delta(A) \leq n\varepsilon < \infty$ and A is bounded.

   To show that $A \subseteq_C E$, it suffices to show that any sequence in A which converges does so in A, according to Proposition 105. So let $(\mathbf{x}_n) \subseteq A$ be such that $\mathbf{x}_n \to \mathbf{x} \in E$. But A is compact, so that $\exists$ a convergent subsequence $(\mathbf{x}_{\varphi(n)})$ which converges in A. Since any subsequence of a sequence converging to $\mathbf{x}$ also converges to $\mathbf{x}$, $\mathbf{x}_{\varphi(n)} \to \mathbf{x} \in A$ and so $A \subseteq_C E$. ■

Unlike completeness, compactness is a **topological notion**.

Proposition 122
Let (E, d) and (F, δ) be metric spaces, together with a continuous function $f : (E, d) \to (F, \delta)$. If $A \subseteq_K E$ then $f(A) \subseteq_K F$.

Proof: let $\{U_\lambda\}_{\lambda \in \Lambda}$ be an open cover of $f(A)$. Since f is continuous, we have that $A \cap f^{-1}(U_\lambda) \subseteq_O A$ for all $\lambda \in \Lambda$. Thus $\{A \cap f^{-1}(U_\lambda)\}_{\lambda \in \Lambda}$ is an open cover of A. But $A \subseteq_K E$ so that $\exists$ a finite $H \subseteq \Lambda$ such that
$$\bigcup_{\lambda \in H} \left(A \cap f^{-1}(U_\lambda)\right) = A.$$
As such, $\{f(U_\lambda)\}_{\lambda \in H}$ is a finite sub-cover of $f(A)$, and so $f(A) \subseteq_K F$. ■

There is also a link with homeomorphisms.

Proposition 123
Let $f : (E,d) \to (F,\delta)$ be a continuous bijection. If (E,d) is compact, then f is a homeomorphism.

Proof: let $Y \subseteq_C E$. We need to show that $f(Y) \subseteq_C F$. According to Proposition 122, $f(Y) \subseteq_K F$. But, according to Proposition 121, part 2, $f(Y) \subseteq_C F$. So f is closed, meaning that f^{inv} is continuous. ■

Perhaps the most famous theorem linking continuous functions and compact spaces is the result to which we were alluding to at the start of this section (we proved a restricted case in Theorem 33).

Proposition 124 (MAX/MIN THEOREM (REPRISE))
Let $f : (E,d) \to \mathbb{R}$ be continuous. If (E,d) is compact, then f is bounded and $\exists \mathbf{a}, \mathbf{b} \in E$ such that $f(\mathbf{a}) = \inf_{\mathbf{x}\in E} f(\mathbf{x})$ and $f(\mathbf{b}) = \sup_{\mathbf{x}\in E} f(\mathbf{x})$.

Proof: since E is compact and f is continuous, then $f(E)$ is compact according to Proposition 122. As such, $f(E)$ is both closed and bounded in $\mathbb{R}$, according to Proposition 121.

Now, set $A = \inf_{\mathbf{x}\in E} f(\mathbf{x})$. By definition, for each $n \geq 1$, $\exists \mathbf{a}_n \in E$ such that $A \leq f(\mathbf{a}_n) < A + \frac{1}{n}$ (otherwise $\inf_{\mathbf{x}\in E} f(\mathbf{x}) \geq A + \frac{1}{n} > A$).

But $(\mathbf{a}_n)$ is a subsequence of the compact space E (hence a subsequence of a closed space) so $\exists$ a subsequence $(\mathbf{a}_{\varphi(n)})$ which converges to some $\mathbf{a} \in A$ according to Proposition 105.

As f is continuous, $f(\mathbf{a}_{\varphi(n)}) \to f(\mathbf{a})$. But $f(\mathbf{a}_{\varphi(n)}) \to A$, since

$$A \leq f(\mathbf{a}_{\varphi(n)}) < A + \frac{1}{\varphi(n)} \to A.$$

The limit of a convergent sequence is unique in a metric space, so $f(\mathbf{a}) = A$.

A similar argument shows $\exists \mathbf{b} \in E$ such that $f(\mathbf{b}) = \sup_{\mathbf{x}\in E} f(\mathbf{x})$. ■

The next result is often used as the definition of a compact set, but it cannot be generalized to infinite dimensional spaces (such as $\ell^2(\mathbb{N})$ or other infinite dimensional Banach spaces).

Proposition 125 (HEINE-BOREL)
Any closed bounded subset of $\mathbb{K}^n$ is compact in the usual topology.

Proof: since $\mathbb{C}^m \simeq \mathbb{R}^{2m}$, we only need to verify that this is the case for $\mathbb{R}^n$. Furthermore, the proposition will be established if we can show it to be valid for any $A = [a_1, b_1] \times \cdots \times [a_n, b_n] \subseteq_C \mathbb{R}^n$ (why is that the case?).

Since $\mathbb{R}^n$ is complete and $A \subseteq \mathbb{R}^n$, then A is a complete subset of $\mathbb{R}^n$, according to Proposition 110. It will then be sufficient to show that A is precompact, according to the proof of Theorem 119.

But that is obvious (see exercise 5). ■

9.2 Connected Spaces

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\exists a, b \in A$ with $f(a)f(b) < 0$. What condition do we need on A in order to guarantee the existence of a solution to $f(x) = 0$ on A?

Whether A is compact or not is irrelevant: for instance, in the standard topology, the function $f : A = [0, 1] \cup [2, 3] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} -1 & x \in [0, 1] \\ 1 & x \in [2, 3] \end{cases}$$

is continuous over the compact set A, there are points $a, b \in A$ such that $f(a)f(b) < 0$, yet $f(x) \neq 0$ for all $x \in A$. On the other hand, $f : A = [-1, 1] \to \mathbb{R}$ defined by $f(x) = x$ is such that $f(-1)f(1) < 0$ and $\exists x \in A$ such that $f(x) = 0$ (namely, $x = 0$).

The key notion is that of **connectedness**. Let (E, d) be a metric space. A **partition** of E is a collection of two disjoint non-empty subsets $U, V \subseteq E$ such that $E = U \cup V$.[1] An **open partition** of E is a partition where $U, V \subseteq_O E$; a **closed partition** of E is a partition where $U, V \subseteq_C E$.

Examples

1. There are many partitions of $\mathbb{R}$ in the usual topology, such as

$$(-\infty, 0] \sqcup (0, \infty) \quad \text{or} \quad [(-\infty, -3] \cup \{0\}] \sqcup [(-3, 0) \cup (0, \infty)],$$

   but no such partition can be an open partition or a closed partition. □

2. The metric space $A = [0, 1] \cup [2, 3]$ is partitioned by $[0, 1]$ and $[2, 3]$. This is both an open partition and a closed partition in the usual **subspace** topology (note that this is not the case in $\mathbb{R}$, but we are only interested in the set A, not the space in which it is embedded). □

3. The singleton set $E = \{*\}$ cannot be partitioned. □

[1] We denote the disjoint union by $E = U \sqcup V$.

The next result establishes an "easy" way to determine if a space has such partitions.

Proposition 126
Let (E, d) be a metric space. The following conditions are equivalent:

1. *E has no open partition;*
2. *E has no closed partition;*
3. *The only subsets of E that are both open and closed are $\varnothing$ and E (such sets are rather unfortunately known as **clopen sets**).*

Proof: we show $1. \implies 2. \implies 3. \implies 1.$

$1. \implies 2.$: Suppose that $\{F_1, F_2\}$ forms a closed partition of E. Then $F_i = E \setminus F_{i-1} \subseteq_O E$ for $i = 1, 2$. Hence $\{F_1, F_2\}$ also forms an open partition of E, which contradicts the hypothesis that no such partition of E exists. Thus E has no closed partition.

$2. \implies 3.$: Let $A \subseteq E$ be such that $A \subseteq_C E$ and $A \subseteq_O E$. Then $\{A, E \setminus A\}$ is a closed partition of E. By hypothesis, there can be no such partition of E. Hence $A = \varnothing$ or $E \setminus A = \varnothing$.

$3. \implies 1.$: This is clear once one realizes that any open partition is automatically also a closed partition. ■

A metric space (E, d) is said to be **connected** if it satisfies any of the conditions listed in Proposition 126. Similarly, a subset $A \subseteq E$ is **connected** if its only clopen partition is trivial, that is: whenever $A = X \sqcup Y$, $X, Y \subseteq_O E$, either $X = \varnothing$ or $Y = \varnothing$. We will denote such a situation with $A \subseteq_{©} E$ (this is emphatically not a notation you will find anywhere else).

Examples

1. In the usual topology, $\mathbb{R}$ is connected. □
2. In the same topology, $A = [0, 1] \cup [2, 3]$ is not a connected subspace of $\mathbb{R}$. □
3. The singleton set $E = \{*\}$ is vacuously connected. □
4. Is $A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ a connected subset of $\mathbb{R}$ in the usual topology?

   Solution: since $A = \{1\} \sqcup \{\frac{1}{n} \mid n \geq 2\}$ is a non-trivial open partition of A, A is not a connected subset of $\mathbb{R}$ in the usual topology. Indeed, $\{1\} \subseteq_O A$ since $\{1\} = (\frac{1}{2}, \infty) \cap A$, $\{\frac{1}{n} \mid n \geq 2\} \subseteq_O A$ since $\{\frac{1}{n} \mid n \geq 2\} = (0, 1) \cap A$. □

As was the case with compactness, connectedness is a **topological notion**.

Proposition 127
Let $f:(E,d)\to(F,\delta)$ be continuous. If $A\subseteq_{©} E$, then $f(A)\subseteq_{©} F$.

Proof: Let $B\subseteq_{O,C} f(A)$. We will show that $B=\varnothing$ or $B=f(A)$.

Since $B\subseteq_O f(A)$, then $\exists U\subseteq_O F$ such that $B=f(A)\cap U$. Similarly, since $B\subseteq_C f(A)$, then $\exists W\subseteq_C F$ such that $B=f(A)\cap W$. But f is continuous so $f^{-1}(U)\subseteq_O E$ and $f^{-1}(W)\subseteq_C E$. Therefore,

$$f^{-1}(B)=A\cap f^{-1}(U)\subseteq_O A \quad\text{and}\quad f^{-1}(B)=A\cap f^{-1}(W)\subseteq_C A.$$

Thus $f^{-1}(B)\subseteq_{O,C} A$. However A is a connected subset of E, so either $f^{-1}(B)=\varnothing$ or $f^{-1}(B)=A$. Since $B\subseteq f(A)$, that leaves only two possibilities: $B=\varnothing$ or $B=f(A)$, which means $f(A)\subseteq_{©} B$. ■

9.2.1 Characterization of Connected Spaces

We now give a simple necessary and sufficient condition for connectedness. Throughout, we endow the set $\{0,1\}$ with the discrete metric.

Proposition 128
A metric space (E,d) is connected if and only if every continuous function $f:E\to\{0,1\}$ is constant.

Proof: assume (E,d) is connected. If $f:E\to\{0,1\}$ is continuous and not constant, then $f^{-1}(0),f^{-1}(1)\subseteq_{O,C} E$ and $E=f^{-1}(0)\sqcup f^{-1}(1)$.

Since f is not constant, neither $f^{-1}(0)$ nor $f^{-1}(1)$ is $\varnothing$ or all of E. Hence E is not connected, as it contains non-trivial clopens, which contradicts our starting assumption. Thus f is constant.

Conversely, if E is not connected, $\exists$ non-trivial clopens X,Y such that $E=X\sqcup Y$. Consider the characteristic function $\chi_X:E\to\{0,1\}$: we have $f^{-1}(0)=Y\subseteq_O E$ and $f^{-1}(1)=X\subseteq_O E$. Consequently, f is continuous and clearly not constant. ■

In practice, Proposition 128 is typically easier to use to show that a space is **not connected**.

Proposition 129
Let (E,d) be a metric space and $A\subseteq_{©} E$. If $B\subseteq E$ is such that $A\subseteq B\subseteq\overline{A}$, then $B\subseteq_{©} E$.

Proof: if such a B is not connected, then $\exists$ a non-trivial open partition $\{X, Y\}$ of B. In particular, $\{A \cap X, A \cap Y\}$ is an open (in A) partition of A. But A is dense in B: if $\mathbf{x} \in B$, every neighbourhood around $\mathbf{x}$ contains at least a point of A.

In particular, if $\mathbf{x} \in B \cap X$, then any neighbourhood around $\mathbf{x}$ must contain at least a point of $A \cap X$. Consequently, $A \cap X \neq \varnothing$. Similarly, $A \cap Y \neq \varnothing$.

Thus, $\{A \cap X, A \cap Y\}$ is a non-trivial open partition of A, which contradicts the fact that A is connected. So B must be connected. ∎

There is a series of other useful propositions about connected spaces.

Proposition 130
If $(B_i)_{i\in I}$ is a family of connected subsets of a metric space (E, d) such that $\bigcap_{i\in I} B_i \neq \varnothing$, then $B = \bigcup_{i\in I} B_i \subseteq_{©} E$.

Proof: if $\{X, Y\}$ is a non-trivial open partition of B and if $\mathbf{b} \in \bigcap_{i\in I} B_i$, we may assume $\mathbf{b} \in X$ without loss of generality. But $B = \bigcup_{i\in I} = X \sqcup Y$ and $Y \neq \varnothing$; hence $\exists i_0 \in I$ such that $Y \cap B_{i_0} \neq \varnothing$.

Since $\mathbf{b} \in \bigcap_{i\in I} B_i$, then $\mathbf{b} \in X \cap B_{i_0} \neq \varnothing$ and so $\{X \cap B_{i_0}, Y \cap B_{i_0}\}$ is a non-trivial open partition of B_{i_0}, which contradicts the hypothesis that $B_{i_0} \subseteq_{©} E$. Consequently, $B \subseteq_{©} E$. ∎

Proposition 131
If $(C_n)_{n\in\mathbb{N}}$ is a sequence of connected subsets of a metric space (E, d) such that $C_{n-1} \cap C_n \neq \varnothing$, then $C = \bigcup_{n\in\mathbb{N}} C_n \subseteq_{©} E$.

Proof: left as an exercise. ∎

Proposition 132
Let $(E_1, d_1), \ldots, (E_n, d_n)$ be metric spaces. Then

$$(E, d) = (E_1 \times \cdots \times E_n, \sup\{d_i \mid 1 \leq i \leq n\})$$

is connected if and only if (E_i, d_i) is connected for all i.

Proof: left as an exercise. ∎

Let (E, d) be a metric space once more. We define an equivalence relation on E as follows:

$$\mathbf{x} R \mathbf{y} \iff \exists C \subseteq_{©} E \text{ such that } \mathbf{x}, \mathbf{y} \in C. \tag{9.3}$$

The equivalence class

$$[\mathbf{x}] = \{\mathbf{y} \in E \mid \mathbf{y}R\mathbf{x}\} = \bigcup_{\substack{C \subseteq_{©} E \\ \mathbf{x} \in C}} C$$

is a connected subset of E, which we call the **connected component** of **x**. It is not difficult to show that $[\mathbf{x}] \subseteq_C E$ and that if a metric space only has a finite number of connected components, then each of those components is a clopen subset of E (see exercises 10 and 11).

Proposition 133
Consider $\mathbb{R}$ with the usual topology. Then, $A \subseteq_{©} \mathbb{R}$ if and only if A is an interval.

Proof: let $A \subseteq_{©} \mathbb{R}$. If A is not an interval, $\exists a, b \in A$ for which $\exists c \in (a, b)$ with $c \notin A$. Thus, $A \subseteq (-\infty, c) \cup (c, \infty)$.

Hence $\{A \cap (-\infty, c), A \cap (c, \infty)\}$ is a non-trivial open partition of A, which implies that A is not a connected subset of $\mathbb{R}$, a contradiction as $A \subseteq_{©} E$, and so A is an interval.

Conversely, if $A = \{*\}$, we have already shown that $A \subseteq_{©} \mathbb{R}$. According to Proposition 129, it is sufficient to verify that $A = (a, b) \subseteq_{©} \mathbb{R}$ for any $a < b$. We will show that any continuous map $f : (a, b) \to \{0, 1\}$ is constant.

Suppose otherwise that $\exists x, y \in (a, b)$ such that $x < y$ and $f(x) \neq f(y)$. Without loss of generality, let $f(x) = 0$ and $f(y) = 1$. Set

$$\Gamma = \{z \mid z \geq x \text{ and } f(t) = 0 \, \forall t \in [x, z]\}.$$

Clearly, $\Gamma \neq \varnothing$ since $x \in \Gamma$. Furthermore Γ is bounded above by y. Thus, since $\mathbb{R}$ is complete, $\exists c \in [x, y] \subseteq (a, b)$ such that $c = \sup \Gamma$.

By continuity of f at c, $f(c) = 0$ and $\exists \delta > 0$ such that

$$s \in (c - \delta, c + \delta) \implies |f(s)| = |f(s) - f(c)| < \tfrac{1}{2}.$$

As such, $f(s) < \frac{1}{2}$ for all $s \in (c - \delta, c + \delta)$. But f can only take two values: 0 or 1. Consequently, $f(s) = 0$ for all $s \in (c - \delta, c + \delta)$.

This in turn implies that $c + \frac{\delta}{2} \in \Gamma$, which contradicts the fact that $c = \sup \Gamma$. Thus, f is constant, and $(a, b) \subseteq_{©} \mathbb{R}$. ■

We can now give a proof of the remark made after Theorem 36.

Corollary 134 (BOLZANO'S THEOREM)
Consider $\mathbb{R}$ with the usual topology and a continuous function $f : \mathbb{R} \to \mathbb{R}$. The image of any interval by f is an interval.

Proof: let $A \subseteq_{©} \mathbb{R}$. By the preceding proposition, A is an interval. Since f is continuous, $f(A) \subseteq_{©} \mathbb{R}$. But the only connected subsets of $\mathbb{R}$ are the intervals. Consequently, $f(A)$ is an interval. ■

9.2.2 Path-Connected Spaces

We can also define other types of connectedness.

Let (E, d) be a metric space. We say that E is **path-connected** if for any two points $\mathbf{x}, \mathbf{y} \in E$, there is a continuous function $\gamma : [0, 1] \to E$ such that $\gamma(0) = \mathbf{x}$ and $\gamma(1) = \mathbf{y}$. The **segment between x and y** is

$$[\mathbf{x}, \mathbf{y}] = \{t\mathbf{x} + (1 - t)\mathbf{y} \mid t \in [0, 1]\}.$$

The continuous function associated to this segment is the function

$$f_{\mathbf{x},\mathbf{y}} : [0, 1] \to E, \quad \text{defined by} \quad f_{\mathbf{x},\mathbf{y}}(t) = t\mathbf{x} + (1 - t)\mathbf{y}.$$

If $[\mathbf{x}, \mathbf{y}]$ and $[\mathbf{z}, \mathbf{w}]$ are two segments, define their **sum** (concatenation) to be

$$[\mathbf{x}, \mathbf{y}] + [\mathbf{z}, \mathbf{w}] = \{2t\mathbf{x} + (1 - 2t)\mathbf{y} \mid t \in [0, \tfrac{1}{2}]\} \cup \{(2t - 1)\mathbf{z} + (2 - 2t)\mathbf{w} \mid t \in [\tfrac{1}{2}, 1]\}.$$

If $\mathbf{y} = \mathbf{z}$, the continuous function associated to this sum is the function

$$g_{\mathbf{x},\mathbf{y},\mathbf{w}} : [0, 1] \to E, \quad \text{defined by} \quad g_{\mathbf{x},\mathbf{y},\mathbf{w}}(t) = \begin{cases} 2t\mathbf{x} + (1 - 2t)\mathbf{y} & \text{if } t \in [0, \frac{1}{2}] \\ (2t - 1)\mathbf{y} + (2 - 2t)\mathbf{w} & \text{if } t \in [\frac{1}{2}, 1] \end{cases}$$

Examples

1. Show that $B(\mathbf{0}, 1)$ is path-connected in $(\mathbb{R}^2, d_2)$.

   Proof: Let $\mathbf{a} \neq \mathbf{b} \in B(\mathbf{0}, 1)$. Then $[\mathbf{a}, \mathbf{0}], [\mathbf{0}, \mathbf{b}] \subseteq B(\mathbf{0}, 1)$. Indeed, if $\mathbf{x} \in [\mathbf{a}, \mathbf{0}]$, then $\mathbf{x} = t\mathbf{a}$ for $t \in [0, 1]$. But $\|\mathbf{x}\| = |t|\|\mathbf{a}\| \leq \|\mathbf{a}\| < 1$, so that $\mathbf{x} \in B(\mathbf{0}, 1)$. Then $g_{\mathbf{a},\mathbf{0},\mathbf{b}} \in C_{B(\mathbf{0},1)}([0, 1])$ is such that $g_{\mathbf{a},\mathbf{0},\mathbf{b}}(0) = \mathbf{a}$ and $g_{\mathbf{a},\mathbf{0},\mathbf{b}}(1) = \mathbf{b}$. ■

2. In any normed vector space $(E, \|\cdot\|)$ over $\mathbb{K}$, any open ball $B(\mathbf{x}, \rho)$ is path-connected (see exercise 13). □

There is clearly a link between the two connectedness definitions.

Proposition 135
If (E, d) is path-connected, then it is also connected.

Proof: let $f : E \to \{0,1\}$ be a continuous function and $\mathbf{a}, \mathbf{b} \in E$. Since E is path-connected, $\exists$ a continuous path $\gamma : [0,1] \to \mathbb{R}$ such that $\gamma(0) = \mathbf{a}$ and $\gamma(1) = \mathbf{b}$.

Since the composition $f \circ \gamma : [0,1] \to \{0,1\}$ is continuous and since $[0,1] \subseteq_{©} \mathbb{R}$, then $f \circ \gamma$ is constant: in particular,

$$f(\mathbf{a}) = f(\gamma(0)) = f(\gamma(1)) = f(\mathbf{b}),$$

so that f itself is constant. Consequently, E is connected. ■

If $E = (\mathbb{K}^n, d_{\text{Euclidean}})$, the converse is also true.

Proposition 136
If $A \subseteq_{©} \mathbb{K}^n$ in the usual topology, then A is path-connected.

Proof: left as an exercise. ■

But connected spaces are not path-connected, in general (see exercise 22, for instance). The following result will allow us to segue gently into Chapter 10.

Theorem 137
Let $(E, \|\cdot\|)$ be a normed vector space over $\mathbb{K}$. Then any $A \subseteq_{O,©} E$ is path-connected.

Proof: Let $\mathbf{x}_0 \in A$ and set

$$F_{\mathbf{x}_0} = \{\mathbf{x} \in A \mid \exists \gamma \in C_E([0,1]) \text{ such that } \gamma(0) = \mathbf{x}_0, \gamma(1) = \mathbf{x}\}.$$

We need to show that $F_{\mathbf{x}_0} = A$. In order to do so, note that $F_{\mathbf{x}_0} \neq \varnothing$ as $\mathbf{x}_0 \in F_{\mathbf{x}_0}$. If we can show that $F_{\mathbf{x}_0} \subseteq_{O,C} A$, then we are done as $A \subseteq_{©} E$.

- Let $\mathbf{x} \in F_{\mathbf{x}_0} \subseteq A$. Since $A \subseteq_O E$, $\exists \rho > 0$ such that $B(\mathbf{x}, \rho) \subseteq A$. For any $\mathbf{y} \in B(\mathbf{x}, \rho)$, $[\mathbf{y}, \mathbf{x}] \in B(x, \rho)$ (modify the proof of exercise 13). Since $\mathbf{x}_0 \in F_{\mathbf{x}_0}$, $B(\mathbf{x}, \rho) \subseteq F_{\mathbf{x}_0}$. Consequently, $F_{\mathbf{x}_0} \subseteq_O A$.
- If $\mathbf{x} \in \overline{F_{\mathbf{x}_0}} \cap A$, then for any $\rho > 0$ we have $B(\mathbf{x}, \rho) \cap F_{\mathbf{x}_0} \neq \varnothing$. Since $A \subseteq_O E$, $\exists \rho_0 > 0$ such that $B(\mathbf{x}, \rho_0) \subseteq A$; in particular $\varnothing \neq B(\mathbf{x}, \rho_0) \cap F_{\mathbf{x}_0} \subseteq A$. Now, let $\mathbf{y} \in B(\mathbf{x}, \rho_0) \cap F_{\mathbf{x}_0}$. Since $[\mathbf{y}, \mathbf{x}] \subseteq B(\mathbf{x}, \rho_0)$, there is a continuous path in A from $\mathbf{y}$ to $\mathbf{x}$. Since $\mathbf{y} \in F_{\mathbf{x}_0}$, there is a continuous path in A from $\mathbf{x}_0$ to $\mathbf{y}$. Combining these paths, there is a continuous path in A from $\mathbf{x}_0$ to $\mathbf{x}$. Hence, $\mathbf{x} \in F_{\mathbf{x}_0}$. Consequently, $F_{\mathbf{x}_0} \subseteq_C A$.

This concludes the proof. ■

Finally, we note that path-connectedness is a **topological notion**.

> **Proposition 138** Let $f : (E, d) \to (F, \delta)$ be a continuous map. If E is path-connected, then $f(E)$ is path-connected. **Proof:** left as an exercise. ■

9.3 Solved Problems

1. Let (E, d) be a metric space.

   a) If $W_1, W_2 \subseteq_K E$, show that $\exists \mathbf{x}_i \in W_i$ such that $d(\mathbf{x}_1, \mathbf{x}_2) = d(W_1, W_2)$.

   b) If $W \subseteq_K E$ and $F \subseteq_C E$ are such that $W \subseteq F = \varnothing$, show that $d(W, F) \neq 0$. Is the conclusion still valid when $W \subseteq_C E$ is not necessarily compact?

   Proof:

   a) The mapping $\varphi : K_1 \to \mathbb{R}$ defined by $\varphi(\mathbf{x}) = d(\mathbf{x}, K_2)$ is continuous. Since K_1 is compact, the Max/Min Theorem applies: $\exists \mathbf{x}_1 \in K_1$ such that

   $$\varphi(\mathbf{x}_1) = d(\mathbf{x}_1, K_2) = \inf_{\mathbf{x} \in K_1} \{d(\mathbf{x}, K_2)\} = d(K_1, K_2).$$

   Similarly, the mapping $\eta : K_2 \to \mathbb{R}$ defined by $\eta(\mathbf{y}) = d(\mathbf{x}_1, \mathbf{y})$ is continuous on a compact set: as such, $\exists x_2 \in K_2$ such that

   $$\eta(\mathbf{x}_2) = d(\mathbf{x}_1, \mathbf{x}_2) = \inf_{\mathbf{y} \in K_2} \{d(\mathbf{x}_1, K_2)\} = d(K_1, K_2).$$

   b) The mapping $\theta : K \to \mathbb{R}$ defined by $\theta(\mathbf{x}) = d(\mathbf{x}, F)$ is continuous on the compact K so that $\exists \mathbf{x}_0 \in K$ such that

   $$\theta(\mathbf{x}_0) = d(\mathbf{x}_0, F) = \inf_{\mathbf{x} \in K} \{d(\mathbf{x}, F)\} = d(K, F).$$

   If $d(\mathbf{x}_0, F) = 0$ then $\mathbf{x}_0 \in F$ since F is closed. But that is impossible as $K \cap F = \varnothing$ and so $d(\mathbf{x}_0, F) \neq 0$.

   If K is only assumed closed, the conclusion may not hold. For instance in $\mathbb{R}^2$, the sets $K = \{(x, y) \mid y \leq 0\}$ and $F = \{(x, y) \mid y \geq e^x\}$ are closed and disjoints, yet $d(K, F) = 0$. ■

2. Let $(E, d) = (\mathbb{R}^n, d_2)$.

   a) If $F \subseteq_C E$ is unbounded and $f : F \to \mathbb{R}$ is a continuous map such that

   $$\lim_{\|\mathbf{x}\| \to \infty} f(\mathbf{x}) = +\infty, \qquad \mathbf{x} \in F,$$

   show $\exists \mathbf{x} \in F$ such that $f(\mathbf{x}) = \inf_{\mathbf{y} \in F} f(\mathbf{y})$.

   b) If $W \subseteq_K E$ and $F \subseteq_C E$, show $\exists \mathbf{x} \in W, \mathbf{y} \in F$ such that $d(\mathbf{x}, \mathbf{y}) = d(W, F)$. Is the conclusion still valid when E is an infinite-dimensional vector space over $\mathbb{R}$?

Proof:

a) Fix $\mathbf{a} \in F$ and consider the set $\Gamma = \{\mathbf{x} \in F \mid f(\mathbf{x}) \leq f(\mathbf{a})\}$. Since f is continuous, $\Gamma = f^{-1}((-\infty, f(a)]) \subseteq_C F$ and so $\Gamma \subseteq_C E$. It is also bounded since

$$\lim_{\|\mathbf{x}\|\to\infty} f(\mathbf{x}) = +\infty, \qquad \mathbf{x} \in F.$$

Thus $\Gamma \subseteq_K \mathbb{R}^n$ by the Heine-Borel Theorem. Furthermore, $\Gamma \neq \varnothing$ since $\mathbf{a} \in \Gamma$. According to the Max/Min Theorem, $\exists \mathbf{x} \in \Gamma$ such that $f(\mathbf{x}) = \inf_{\mathbf{y}\in\Gamma}\{f(\mathbf{y})\}$. By construction,

$$\inf_{\mathbf{y}\in\Gamma}\{f(\mathbf{y})\} = \inf_{\mathbf{y}\in F}\{f(\mathbf{y})\},$$

whence $f(\mathbf{x}) = \inf_{\mathbf{y}\in F}\{f(\mathbf{y})\}$ for some $\mathbf{x} \in F$.

b) Since the mapping $\varphi : K \to \mathbb{R}$ defined by $\varphi(\mathbf{x}) = d(\mathbf{x}, F)$ is continuous, $\exists \mathbf{x} \in K$ such that

$$d(\mathbf{x}, F) = \inf_{\mathbf{y}\in K}\{d(\mathbf{y}, F)\} = d(K, F).$$

Note that the mapping $\psi_{\mathbf{x}} : F \to \mathbb{R}$ defined by $\psi_{\mathbf{x}}(\mathbf{y}) = d(\mathbf{x}, \mathbf{y})$ is also continuous. If F is bounded, then $F \subseteq_K \mathbb{R}^n$ and the desired result is derived from the result in (a).

Otherwise, if F is unbounded we have

$$\lim_{\|\mathbf{y}\|\to\infty} \psi_{\mathbf{x}}(\mathbf{y}) = \infty, \quad \mathbf{y} \in F$$

so that $\exists \mathbf{y} \in F$ such that

$$\psi_{\mathbf{x}}(\mathbf{y}) = \inf_{\mathbf{z}\in F}\{\psi_{\mathbf{x}}(\mathbf{z})\} = d(\mathbf{x}, F) = d(K, F),$$

which proves the desired result.

The result is false in general if E is infinite-dimensional: consider for instance the vector space of bounded sequences in $\mathbb{R}$, with the norm $\|(u_n)\| = \sup_{n\in\mathbb{N}}\{|u_n|\}$.

For any $n \in \mathbb{N}$, let $\mathcal{X}_n$ be the sequence where the n^{th} term is $1 + 2^{-n}$ and all the other terms are 0. The set $F = \{\mathcal{X}_n \mid n \in \mathbb{N}\}$ is closed in E since all its points are isolated points. If $K = \{\mathbf{0}\}$, it is obvious that $d(K, F) = 1$, yet $d(K, \mathcal{X}_n) = 1 + 2^{-n} > 1$ for all $n \in \mathbb{N}$. ■

3. Let (E, d) be a compact metric space with a map $f : E \to E$ such that $\forall \mathbf{x} \neq \mathbf{y} \in E$, $d(f(\mathbf{x}), f(\mathbf{y})) < d(\mathbf{x}, \mathbf{y})$.

   a) Show that f admits a unique fixed point $\alpha \in E$.
   b) Let $\mathbf{x}_0 \in E$. For each $n \in \mathbb{N}$, set $\mathbf{x}_{n+1} = f(\mathbf{x}_n)$. Show that $\mathbf{x}_n \to \alpha$.
   c) Are these results still valid if E is complete but not compact?

Proof:

a) First note that, being Lipschitz, f is continuous. Then, the mapping $\varphi_f : E \to \mathbb{R}$ defined by $\varphi_f(\mathbf{x}) = d(\mathbf{x}, f(\mathbf{x}))$ is continuous as it is a composition of continuous functions. But E is compact so that $\exists \alpha \in E$ such that $d(\alpha, f(\alpha)) = \inf_{\mathbf{x} \in E}\{d(\mathbf{x}, f(\mathbf{x}))\}$. If $\alpha \neq f(\alpha) = \beta$, then

$$d(\beta, f(\beta)) = d(f(\alpha), f(\beta)) < d(\alpha, \beta) = d(\alpha, f(\alpha))$$

by hypothesis, which contradicts the definition of α. Thus $\alpha = f(\alpha)$.

Now, suppose $\beta = f(\beta)$ with $\beta \neq \alpha$. Then we have

$$d(f(\alpha), f(\beta)) = d(\alpha, \beta),$$

which contradicts the hypothesis. Thus $\alpha = \beta$.

b) Write $\mathbf{u}_n = d(\alpha, \mathbf{x}_n)$. If $\exists n_0 \in \mathbb{N}$ such that $\mathbf{u}_{n_0} = 0$, then $\mathbf{u}_n = \mathbf{u}_{n_0} = 0$ for all $n \geq n_0$ and the result follows. Otherwise, for all $n \in \mathbb{N}$ we have

$$\mathbf{u}_{n+1} = d(f(\alpha), f(\mathbf{x}_n)) < d(\alpha, \mathbf{x}_n) = \mathbf{u}_n,$$

i.e. $(\mathbf{u}_n)$ is a strictly decreasing sequence. As it is bounded below by 0, it is necessarily convergent. Let $\mathbf{u}_n \to \ell \geq 0$. We need to show $\ell = 0$.

Assume that $\ell > 0$. Since $(\mathbf{u}_n)$ is decreasing, $\mathbf{u}_n \geq \ell$ for all n. Since $(\mathbf{x}_n)$ is a sequence in the compact set E, there is a convergent subsequence $(\mathbf{x}_{\varphi(n)})$, with $\varphi : \mathbb{N} \to \mathbb{N}$ strictly increasing. Let $\beta = \lim \mathbf{x}_{\varphi(n)}$. Then

$$\ell = \lim_{n\to\infty} d(\alpha, \mathbf{x}_{\varphi(n)}) = d(\alpha, \beta).$$

Since f is continuous, we have

$$\lim_{n\to\infty} d(\alpha, f(\mathbf{x}_{\varphi(n)})) = d(\alpha, f(\beta)).$$

But that is impossible since

$$d(\alpha, f(\beta)) = d(f(\alpha), f(\beta)) < d(\alpha, \beta) = \ell$$

and

$$d(\alpha, f(\mathbf{x}_{\varphi(n)})) = d(\alpha, \mathbf{x}_{\varphi(n)+1}) \geq \ell \quad \forall n.$$

The only remaining possibility is thus that $\ell = 0$.

c) Completeness of E is not sufficient. For instance, the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x < 0 \\ x + \frac{1}{1+x} & \text{if } x \geq 0 \end{cases}$$

satisfies the hypothesis, but it admits no fixed point. ■

4. Let (E, d) and (F, δ) be two metric spaces, together with a injective map $f : E \to F$. Show that f is continuous if and only if $f(W) \subseteq_K F$ for all $W \subseteq_K E$.

Proof: we already know that if f is continuous and $W \subseteq_K E$, then $f(W) \subseteq_K F$.

Now assume that $f(W) \subseteq_K F$ for all $W \subseteq_K E$. Let $\mathbf{x} \in E$ and $(\mathbf{x}_n) \subseteq E$ be such that $\mathbf{x}_n \to \mathbf{x}$. The set $V = \{\mathbf{x}_n \mid n \in \mathbb{N}\} \cup \{\mathbf{x}\}$ is compact in E, according to the Borel-Lebesgue property. Thus, we have $V' = f(V) \subseteq_K F$.

Let $g : V \to F$ be such that $g = f|_V$. Since f is injective, g is a bijection from V to V'. The map $g^{-1} : V' \to V$ is continuous since any closed subset $W \subseteq_C V$ is automatically compact in V.

As such $(g^{-1})^{-1}(W) = g(W) \subseteq_K V'$ is automatically closed in V'. Since V' is compact, $(g^{-1})^{-1} = g$ is continuous. Thus

$$f(\mathbf{x}_n) = g(\mathbf{x}_n) \to g(\mathbf{x}) = f(\mathbf{x}) \implies f \text{ is continuous.}$$

ote that if f is not injective, the result does not hold in general. For instance, the Heaviside function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 0$ if $x < 0$ and $f(x) = 1$ if $x \geq 0$ sends any compact set to a compact set, but it is not continuous. ∎

5. Let (E, d) be a metric space. If $\varepsilon > 0$, we say that E is $\varepsilon-$**chained** if for all $\mathbf{a}, \mathbf{b} \in E$, $\exists n \in \mathbb{N}^\times$ and $\mathbf{x}_0, \ldots, \mathbf{x}_n \in E$ such that $\mathbf{x}_0 = \mathbf{a}$, $\mathbf{x}_n = \mathbf{b}$ and $d(\mathbf{x}_i, \mathbf{x}_{i-1}) < \varepsilon$ for all $i = 1, \ldots, n$. We say that E is **well-chained** if it is $\varepsilon-$chained for all $\varepsilon > 0$.

   a) If E is connected, show that E is well-chained.

   b) If E is compact and well-chained, show that E is connected. Is the result still true if E is not necessarily compact?

   Proof:

   a) Let $\varepsilon > 0$. We define an equivalence relation $\mathcal{R}_\varepsilon$ on E according to the following: $\mathbf{x}\mathcal{R}_\varepsilon\mathbf{y}$ if and only if $\exists n \in \mathbb{N}^\times$ and $\mathbf{x}_0, \ldots, \mathbf{x}_n \in E$ such that $\mathbf{x}_0 = \mathbf{x}$, $\mathbf{x}_n = \mathbf{y}$ and $d(\mathbf{x}_i, \mathbf{x}_{i-1}) < \varepsilon$ for all $i = 1, \ldots, n$.

      Let $\mathbf{x} \in E$ and $\mathbf{y} \in [\mathbf{x}]$. Then, for all $\mathbf{z} \in B(\mathbf{y}, \varepsilon)$ we have $\mathbf{z} \in [\mathbf{y}] = [\mathbf{x}]$. Thus $B(\mathbf{y}, \varepsilon) \subseteq [\mathbf{x}]$ and so $[\mathbf{x}] \subseteq_O E$.

      Since
      $$[\mathbf{x}] = E \setminus \bigcup_{\mathbf{y} \notin [\mathbf{x}]} [\mathbf{y}]$$
      is the complement of an open set, $[\mathbf{x}] \subseteq_C E$. Consequently, $[\mathbf{x}]$ is a clopen subset of E. But E is connected; we must then have $[\mathbf{x}] = E$ since $[\mathbf{x}] \neq \varnothing$. Hence, every pair of point of E can be joined by an $\varepsilon-$ chain. As ε is arbitrary, E is **well-chained**.

   b) Suppose that E is not connected. Then we can write $E = F_1 \sqcup F_2$, where $\varnothing \neq F_1, F_2 \subseteq_C E$. Since E is compact, $F_1, F_2 \subseteq_K E$.It is left as an exercise to show that $\exists \mathbf{a}_1 \in F_1$ and $\mathbf{a}_2 \in F_2$ such that $d(\mathbf{a}_1, \mathbf{a}_2) = d(F_1, F_2)$.

Since $F_1 \cap F_2 \neq \varnothing$, $\mathbf{a}_1 \neq \mathbf{a}_2$ and so $\varepsilon = d(\mathbf{a}_1, \mathbf{a}_2) > 0$; as such, $d(\mathbf{x}, \mathbf{y}) \geq \varepsilon$ for all $(\mathbf{x}, \mathbf{y}) \in F_1 \times F_2$.

Let $(\mathbf{x}, \mathbf{y})$ be such a point. Since E is well-chained, $\exists$ an $\varepsilon-$chain $(\mathbf{x}_0, \ldots, \mathbf{x}_n) \in E^{n+1}$ such that

$$\mathbf{x}_0 = \mathbf{x},\ \mathbf{x}_n = \mathbf{y} \quad \text{and} \quad d(\mathbf{x}_i, \mathbf{x}_{i-1}) < \varepsilon \quad \text{for all } i = 1, \ldots, n.$$

Since $\mathbf{x}_0 \in F_1$ and $\mathbf{x}_n \in F_2$, $\exists i$ such that $\mathbf{x}_{i-1} \in F_1$ and $\mathbf{x}_i \in F_2$.

But this would imply that $\varepsilon > d(\mathbf{x}_{i-1}, \mathbf{x}_i) \geq d(F_1, F_2) = \varepsilon$, which is a contradiction. Consequently, E is connected.

If E is not compact, the result is not valid in general: Q is well-chained when endowed with the usual metric because it is dense in $\mathbb{R}$, but it is not connected. ■

6. Let (E, d) be a metric space, with two disjoint sets $A, B \subseteq_C E$. Show that there exists a continuous function $f : E \to [0, 1]$ such that $A = f^{-1}(\{0\})$ and $B = f^{-1}(\{1\})$, as well as two disjoint sets $U, V \subseteq_O E$ such that $A \subseteq U$ and $B \subseteq V$.

Proof: Let $F \subseteq_C E$. Define $g_F : (E, d) \to (\mathbb{R}, |\cdot|)$ by

$$g_F(\mathbf{x}) = d(\mathbf{x}, F) = \inf_{\mathbf{y} \in F} \{d(\mathbf{x}, \mathbf{y})\}$$

According to the Triangle Inequality, for all $\mathbf{y} \in F$ we have

$$g_F(\mathbf{x}) = d(\mathbf{x}, F) \leq d(\mathbf{x}, \mathbf{y}) \leq d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y}) \quad \forall \mathbf{x}, \mathbf{z} \in E,$$

thus we must have $g_F(\mathbf{x}) \leq d(\mathbf{x}, \mathbf{z}) + g_F(\mathbf{z})$ for all $\mathbf{x}, \mathbf{z} \in E$, that is, $g_F(\mathbf{x}) - g_F(\mathbf{z}) \leq d(\mathbf{x}, \mathbf{z})$ for all $\mathbf{x}, \mathbf{z} \in E$. In a similar fashion, $g_F(\mathbf{z}) - g_F(\mathbf{x}) \leq d(\mathbf{x}, \mathbf{z})$ for all $\mathbf{x}, \mathbf{z} \in E$. Thus,

$$|g_F(\mathbf{x}) - g_F(\mathbf{z})| \leq d(\mathbf{x}, \mathbf{z}) \quad \text{for all } \mathbf{x}, \mathbf{z} \in E,$$

i.e. g_F is Lipschitz (and so continuous).

Since $F \subseteq_C E$, $g_F(\mathbf{x}) = 0$ if and only if $\mathbf{x} \in F$. Let $f : (E, d) \to (\mathbb{R}, |\cdot|)$ be defined by

$$f(\mathbf{x}) = \frac{g_A(\mathbf{x})}{g_A(\mathbf{x}) + g_B(\mathbf{x})} = \frac{d(\mathbf{x}, A)}{d(\mathbf{x}, A) + d(\mathbf{x}, B)};$$

it is well-defined since whenever $d(\mathbf{x}, A) + d(\mathbf{x}, B) = 0$, we must have $d(\mathbf{x}, A) = d(\mathbf{x}, B) = 0$, i.e. $\mathbf{x} \in A$ and $\mathbf{x} \in B$. But $A \cap B = \varnothing$ and so for all $\mathbf{x} \in E$, we have $d(\mathbf{x}, A) + d(\mathbf{x}, B) \neq 0$.

Furthermore, $f(\mathbf{x}) = 0$ if and only if $d(\mathbf{x}, A) = 0$, i.e. $\mathbf{x} \in A$; $f(\mathbf{x}) = 1$ if and only if $d(\mathbf{x}, B) = 0$, i.e. $\mathbf{x} \in B$.

The function f is continuous since it is the composition of continuous functions. It is clear that $0 \leq f(\mathbf{x}) \leq 1$, so that $f : E \to [0, 1]$.

Finally, let

$$A \subseteq U = f^{-1}([0,1/2)) \subseteq_O [0,1] \quad \text{and} \quad B \subseteq V = f^{-1}((1/2,1]) \subseteq_O [0,1].$$

Then $U \cap V = \varnothing$ by construction and we are done. ■

9.4 Exercises

1. Prepare a 2-page summary of this chapter, with important definitions and results.
2. Show that any compact metric space is precompact and complete.
3. Show that any complete precompact metric space is compact.
4. Prove Theorem 120.
5. With the usual metric, show that $A \subseteq \mathbb{R}^n$ is precompact if and only if $\overline{A} \subseteq_K \mathbb{R}^n$.
6. Prove Proposition 131.
7. Prove Proposition 132.
8. Let $(E_1, d_1), \ldots, (E_n, d_n)$ be metric spaces. Show that

   $$(E, d) = (E_1 \times \cdots \times E_n, \sup\{d_i \mid 1 \leq i \leq n\})$$

   is compact if and only if (E_i, d_i) is compact for all $i = 1, \ldots, n$.[2]
9. Show that (9.3) defines an equivalence relation on a metric space (E, d).
10. Let (E, d) be a metric space and let $\mathbf{x} \in E$. Show that $[\mathbf{x}] \subseteq_C E$.
11. Let (E, d) be a metric space with finitely many connected components. Show that each of those components is a clopen subset of E.
12. Prove Proposition 136.
13. Show that if $(E, \|\cdot\|)$ is a normed vector space over $\mathbb{K}$, then any open ball $B(\mathbf{x}, \rho)$ is path-connected.
14. Let (E, d) be a metric space, $B \subseteq_{©} E$ and $A \subseteq E$ such that

    $$B \cap \text{int}(A) \neq \varnothing \quad \text{and} \quad B \cap \text{int}(E \setminus A) \neq \varnothing.$$

    Show that $B \cap \partial A \neq \varnothing$.

[2] This result cannot be generalized to infinite products (**Tychonoff's Theorem**) without calling upon the **Axiom of Choice**, a.k.a **Zorn's Lemma**, a.k.a. the **Existence of Non-Measurable Sets**, a.k.a. the **Banach-Tarksi Paradox**.

15. Let (A, d_1) and (B, d_2) be two metric spaces. Let $X \subsetneq A$ and $Y \subsetneq B$. Show that

$$(A \times B) \setminus (X \times Y) \subseteq_{©} A \times B.$$

16. Prove Proposition 9.2.2.

17. In the usual topology, give an example of a subset $A \subseteq_{©} \mathbb{R}^2$ for which int(A) is not connected.

18. In the usual topology, give an example of a subset $A \subseteq \mathbb{R}^2$ for which $\overline{A} \subseteq_{©} \mathbb{R}^2$ but A is not connected.

19. Show that if the connected components of a compact set are open, then there are finitely many of them.

20. Let (E, d) and (F, δ) be metric spaces, together with a continuous map $f : E \to F$ such that $f_{-1}(W) \subseteq_K E$ for all $W \subseteq_K F$. Show that f is a closed map.

21. Let (E, d) be a connected metric space and let $F \subseteq_C E$, with $\partial F \subseteq_{©} E$. Show that $F \subseteq_{©} E$. Is the result still true if F is not necessarily closed?

22. Let $\Gamma = \left[\bigcup_{x \in \mathbb{Q}} (\{x\} \times (0, \infty))\right] \cup \left[\bigcup_{x \in \mathbb{R} \setminus \mathbb{Q}} (\{x\} \times (-\infty, 0))\right] \subseteq \mathbb{R}^2$.

    a) Show that $\Gamma \subseteq_{©} \mathbb{R}^2$.
    b) Show that Γ is not path-connected.